A formula for the number of spanning trees in circulant graphs with non-fixed generators

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Abstract

We consider the number of spanning trees in circulant graphs of \( \beta n \) vertices with generators depending linearly on \( n \). The matrix tree theorem gives a closed formula of \( \beta n \) factors; while we derive a formula of \( \beta - 1 \) factors. The spanning tree entropy of these graphs is then compared to the one of fixed generated circulant graphs.

1 Introduction

A spanning tree of a connected graph \( G \) is a connected subgraph of \( G \) without cycles with the same vertex set as \( G \). The number of spanning trees in a graph \( G \), \( \tau(G) \), is an important graph invariant and is widely studied. It can be computed from the well-known matrix tree theorem due to Kirchhoff (e.g. see [1]). It states that the number of spanning trees of a graph \( G \) with \( n \) vertices is given by

\[
\tau(G) = \frac{1}{n} \prod_{k=1}^{n-1} \lambda_k
\]

where \( \lambda_k \), \( k = 1, \ldots, n - 1 \), are the non-zero eigenvalues of the combinatorial laplacian on \( G \).

Let \( 1 \leq \gamma_1 \leq \cdots \leq \gamma_d \leq \lfloor n/2 \rfloor \) be positive integers. A circulant graph \( C_{\gamma_1,\ldots,\gamma_d}^n \) is the \( 2d \)-regular graph with \( n \) vertices labelled \( 0, 1, \ldots, n - 1 \) such that each vertex \( v \in \mathbb{Z}/n\mathbb{Z} \) is connected to \( v \pm \gamma_i \mod n \) for all \( i \in \{1, \ldots, d\} \). In this paper we consider circulant graphs with the first generator equal to one and the \( d - 1 \) others linearly depending on the number of vertices, that is \( C_{1,\gamma_1,\ldots,\gamma_{d-1}}^{\beta n} \), where \( 1 \leq \gamma_1 \leq \cdots \leq \gamma_{d-1} \leq \lfloor \beta/2 \rfloor \) and \( \beta \) are integers. Two examples are illustrated in Figure 1 below. It is known that the number of spanning trees in circulant graphs with \( n \) vertices satisfies a linear recurrence relation with constant coefficients in \( n \), this has been shown by Golin, Leung and Wang in [4]. For \( \beta \in \{2, 3, 4, 6, 12\} \), closed formulas have been obtained by Zhang, Yong and Golin in [8] where the authors used techniques inspired from Boesch and Prodinger [2] using Chebyshev polynomials. As noted in [8] this method does not work for other values of \( \beta \). Our theorem in section 2 is derived in a simple way and gives a closed formula for all integer values of \( \beta \). This gives an answer to an open question in [3] and [8] and proves the conjecture stated in [6]. In the last section, we deduce the tree entropy for a sequence of non-fixed generated circulant graphs and compare it to the one with fixed generators.

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2 Spanning trees in circulant graphs with non-fixed generators

Let $G$ be a graph, $V(G)$ its set of vertices and $f : V(G) \to \mathbb{R}$ a function. The combinatorial laplacian on $G$ is defined by

$$\Delta_G f(x) = \sum_{y \sim x} (f(x) - f(y))$$

where the sum is over the vertices adjacent to $x$. Since the circulant graph $C_{\beta n}^{1, \gamma_1 n, \ldots, \gamma_{d-1} n}$ is the Cayley graph of the group $\mathbb{Z}/\beta n \mathbb{Z}$, the eigenvectors of the laplacian are given by the characters

$$\chi_k(x) = e^{\frac{2\pi ikx}{\beta n}}, \quad k = 0, 1, \ldots, \beta n - 1.$$ 

Therefore the eigenvalues are

$$\lambda_k = 2d - 2 \cos(2\pi k/\beta n) - 2 \sum_{m=1}^{d-1} \cos(2\pi k \gamma_m/\beta), \quad k = 0, 1, \ldots, \beta n - 1. \quad (2)$$

Theorem. Given integers $1 \leq \gamma_1 \leq \ldots \leq \gamma_{d-1} \leq \lfloor \beta/2 \rfloor$, the number of spanning trees in the circulant graph $C_{\beta n}^{1, \gamma_1 n, \ldots, \gamma_{d-1} n}$ for all $n \geq 1$ is given by

$$\tau(C_{\beta n}^{1, \gamma_1 n, \ldots, \gamma_{d-1} n}) = \frac{n \beta^{d-1}}{\beta} \prod_{k=1}^{\beta n-1} \left( 2 \cosh(n \operatorname{Argcosh}(d - \sum_{m=1}^{d-1} \cos(2\pi k \gamma_m/\beta))) - 2 \cos(2\pi k/\beta) \right).$$

Proof. Applying the matrix tree theorem [1] to the graph $C_{\beta n}^{1, \gamma_1 n, \ldots, \gamma_{d-1} n}$, with eigenvalues given by (2), leads to

$$\tau(C_{\beta n}^{1, \gamma_1 n, \ldots, \gamma_{d-1} n}) = \frac{1}{\beta n} \prod_{k=1}^{\beta n - 1} \left( 2d - 2 \cos(2\pi k/\beta n) - 2 \sum_{m=1}^{d-1} \cos(2\pi k \gamma_m/\beta) \right).$$

Since there are $n$ spanning trees in the cycle $C_n^1$, that is,

$$n = \tau(C_n^1) = \frac{1}{n} \prod_{k=1}^{n-1} (2 - 2 \cos(2\pi k/n))$$,
it follows that

$$\tau(C^{1,\gamma_1,\ldots,\gamma_{d-1}}_{\beta n}) = \frac{n}{\beta} \prod_{k=1 \atop k \neq |k|}^{\beta n-1} \left(2d - 2\cos(2\pi k/\beta n) - 2 \sum_{m=1}^{d-1} \cos(2\pi k\gamma_m/\beta)\right)$$

Therefore the product (4) is equal to

$$\frac{n}{\beta} \prod_{k=1 \atop k \neq |k|}^{\beta n-1} \prod_{l=0}^{\beta-1 n - 1} \left(2d - 2\cos(2\pi (k + l)/\beta n) - 2 \sum_{m=1}^{d-1} \cos(2\pi (k + l)\gamma_m/\beta)\right)$$

$$= \frac{n}{\beta} \prod_{k=1 \atop k \neq |k|}^{\beta n-1} \prod_{l=0}^{\beta-1 n - 1} \left(2 \cosh(\text{Argcosh}(d - \sum_{m=1}^{d-1} \cos(2\pi k\gamma_m/\beta))) - 2 \cos(2\pi k/\beta n + 2\pi l/n)\right).$$  (3)

We now evaluate the product over \(l\) by the following calculation

$$\prod_{l=0}^{n-1} \left(2 \cosh(\theta - 2 \cos((\omega + 2\pi l)/n))\right) = e^{-n\theta} \prod_{l=0}^{n-1} (e^{2\theta} - 2 \cos((\omega + 2\pi l)/n))$$

$$= e^{-n\theta} \prod_{l=0}^{n-1} (e^{\theta} - e^{i(\omega + 2\pi l)/n})(e^{\theta} - e^{-i(\omega + 2\pi l)/n}).$$  (4)

The complex numbers \(e^{i(\omega + 2\pi l)/n}\) and \(e^{-i(\omega + 2\pi l)/n}\) for \(l = 0, 1, \ldots, n - 1\) are the 2\(n\) roots of the following polynomial in \(e^\theta\)

$$e^{2n\theta} - 2e^{n\theta} \cos \omega + 1 = 0.$$

Therefore the product (4) is equal to

$$e^{-n\theta}(e^{2n\theta} - 2e^{n\theta} \cos \omega + 1) = 2\cosh(n\theta) - 2\cos \omega.$$

Using this relation in (3) with \(\theta = \text{Argcosh}(d - \sum_{m=1}^{d-1} \cos(2\pi k\gamma_m/\beta))\) and \(\omega = 2\pi k/\beta\), the theorem follows. \(\square\)

This formula reproves Theorems 4, 5, 6, 8 and corrects a typographical error in Theorem 7 in [8]. For example, [8] Theorem 5] states that

$$\tau(C^{1,\gamma_1,\ldots,\gamma_{d-1}}_{3n}) = \frac{n}{3} \left[(\sqrt{7}/4 + \sqrt{3}/4)^{2n} + (\sqrt{7}/4 - \sqrt{3}/4)^{2n} + 1\right]^2$$

and [8] Theorem 6] states that

$$\tau(C^{1,\gamma_1,\ldots,\gamma_{d-1}}_{4n}) = \frac{n}{4} \left[(\sqrt{3}/2 + \sqrt{1}/2)^{2n} + (\sqrt{3}/2 - \sqrt{1}/2)^{2n}\right]^2 \left[(\sqrt{2} + 1)^{n} + (\sqrt{2} - 1)^{n}\right]^2$$

which are particular cases of our theorem with \(d = 2\), \(\gamma_1 = 1\), \(\beta = 3\) and \(\beta = 4\) respectively.

### 3 Spanning tree entropy of circulant graphs

For a sequence of regular graphs \(G_n\) with vertex set \(V(G_n)\), one can consider the number of spanning trees as a function of \(n\). Assuming that the following limit exists

$$z = \lim_{n \to \infty} \frac{\log \tau(G_n)}{|V(G_n)|},$$

it is sometimes called the associated tree entropy [7]. From the theorem in section 2 the tree entropy for the non-fixed generated circulant graph \(C^{1,\gamma_1,\ldots,\gamma_{d-1}}_{\beta n}\) as \(n \to \infty\), denoted by \(z_{NF(\beta; \gamma_1, \ldots, \gamma_{d-1})}\), is given in the following corollary.
Corollary. Let $1 \leq \gamma_1 \leq \ldots \leq \gamma_{d-1} \leq [\beta/2]$ and $\beta$ be positive integers. The tree entropy of the circulant graph $C_{\beta n}^{1,\gamma_1,\ldots,\gamma_{d-1}}$ as $n \to \infty$ is given by

$$z_{NF}(\beta; \gamma_1, \ldots, \gamma_{d-1}) = \frac{1}{\beta} \sum_{k=1}^{\beta-1} \text{Argcosh}(d - \sum_{m=1}^{d-1} \cos(2\pi k \gamma_m / \beta))$$

$$= \int_0^\infty (e^{-t} - \frac{1}{\beta} \sum_{k=0}^{\beta-1} e^{-\lambda_k t} e^{-2t(I_0(2t))} \frac{dt}{t}$$

where $\lambda_k = 2(d-1) - 2 \sum_{m=1}^{d-1} \cos(2\pi k \gamma_m / \beta)$, $k = 0, 1, \ldots, \beta - 1$, are the eigenvalues of the combinatorial laplacian on the circulant graph $C_{\beta}^{\gamma_1,\ldots,\gamma_{d-1}}$, and $I_0$ the modified 1-Bessel function of order zero.

Proof. From the theorem, the asymptotic number of spanning trees in $C_{\beta n}^{1,\gamma_1,\ldots,\gamma_{d-1}}$ is given by

$$\tau(C_{\beta n}^{1,\gamma_1,\ldots,\gamma_{d-1}}) = \frac{n}{\beta} e^{n \sum_{k=1}^{\beta-1} \text{Argcosh}(d - \sum_{m=1}^{d-1} \cos(2\pi k \gamma_m / \beta)) + o(1)}$$

as $n \to \infty$. This shows the first equality. The second equality comes from Proposition 2.4 in [6] which expresses the Argcosh in terms of an integral of modified 1-Bessel function: for all $x \geq 2$,

$$\int_0^\infty (e^{-t} - e^{-xt}I_0(2t)) \frac{dt}{t} = \text{Argcosh}(x/2).$$

Recall that for $x \geq 1$,

$$\text{Argcosh} x = \log(x + \sqrt{x^2 - 1}).$$

We emphasize that the circulant graph $C_{\beta n}^{1,\gamma_1,\ldots,\gamma_{d-1}}$ consists of $n$ copies of $C_{\beta}^{\gamma_1,\ldots,\gamma_{d-1}}$ which are embedded in the cycle $C_{\beta n}^{1}$. This structure is reflected by the appearance of the term $\theta_{C_{\beta}^{\gamma_1,\ldots,\gamma_{d-1}}}(t)e^{-2tI_0(2t)}$ in the asymptotic formula, where $\theta_{C_{\beta}^{\gamma_1,\ldots,\gamma_{d-1}}}(t) = \sum_{k=1}^{\beta-1} e^{-\lambda_k t}$ is the theta function on $C_{\beta}^{\gamma_1,\ldots,\gamma_{d-1}}$ and $e^{-2tI_0(2t)}$ is the typical term appearing in the asymptotics of the number of spanning trees in the cycle. Indeed, the tree entropy on the cycle is (see section 3.2 in [6])

$$z_{\text{cycle}} = \int_0^\infty (e^{-t} - e^{-2tI_0(2t)}) \frac{dt}{t} = 0.$$  

Consider the sequence of circulant graphs $C_{\beta n}^{1,\gamma_1,\ldots,\gamma_{d-1}}$ when $n \to \infty$ with $z_{NF}(\beta; 1, \gamma_1, \ldots, \gamma_{d-1})$ the corresponding tree entropy. In the following proposition we show that it is greater than the one of fixed generated circulant graphs.

Proposition. For all positive integers $\gamma_1, \ldots, \gamma_d$, there exists an integer $B \geq 2$ such that for all $\beta \geq B$,

$$z_{NF}(\beta; 1, \gamma_1, \ldots, \gamma_{d-1}) > z_F(1, \gamma_1, \ldots, \gamma_d)$$

where $z_F(1, \gamma_1, \ldots, \gamma_d)$ is the tree entropy of the fixed generated circulant graph $C_{\beta}^{1,\gamma_1,\ldots,\gamma_d}$.

Proof. By letting $\beta \to \infty$ in the corollary, the sum over the laplacian eigenvalues converges to a Riemann integral, so that

$$\lim_{\beta \to \infty} z_{NF}(\beta; 1, \gamma_1, \ldots, \gamma_{d-1}) = \int_0^\infty (e^{-t} - e^{-2(d+1)t}I_0(2t)I_0^{1,\gamma_1,\ldots,\gamma_{d-1}}(2t, \ldots, 2t)) \frac{dt}{t}$$

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where $I_{0}^{1,\ldots,d-1}(2t,\ldots,2t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{2t(cos w + \sum_{m=1}^{d-1} \cos(\gamma_m w))} \, dw$.

It can be expressed in terms of a series of modified $I$-Bessel functions

$$I_{0}^{1,\ldots,d-1}(2t,\ldots,2t) = \sum_{(k_1,\ldots,k_{d-1}) \in \mathbb{Z}^{d-1}} I_{\sum_{i=1}^{d-1} \gamma_i k_i}(2t) \prod_{i=1}^{d-1} I_{k_i}(2t).$$

On the other hand, from Theorem 1.1 in [6], the tree entropy of the fixed generated circulant graph $C_n^{1,\gamma_1,\ldots,\gamma_{d-1}}$ as $n \to \infty$ is given by

$$z_{F}(1,\gamma_1,\ldots,\gamma_d) = \int_{0}^{\infty} \left(e^{-t} - e^{-2(d+1)t} I_{0}^{1,\gamma_1,\ldots,\gamma_{d-1}}(2t,\ldots,2t)\right) \frac{dt}{t},$$

where

$$I_{0}^{1,\gamma_1,\ldots,\gamma_{d-1}}(2t,\ldots,2t) = \sum_{(k_1,\ldots,k_{d-1}) \in \mathbb{Z}^{d}} I_{\sum_{i=1}^{d} \gamma_i k_i}(2t) \prod_{i=1}^{d} I_{k_i}(2t)$$

$$> I_{0}(2t) \sum_{(k_1,\ldots,k_{d-1}) \in \mathbb{Z}^{d-1}} I_{\sum_{i=1}^{d-1} \gamma_i k_i}(2t) \prod_{i=1}^{d-1} I_{k_i}(2t)$$

$$= I_{0}(2t) I_{0}^{1,\gamma_1,\ldots,\gamma_{d-1}}(2t,\ldots,2t), \quad \forall t > 0.$$

Therefore

$$\lim_{\beta \to \infty} \lim_{n \to \infty} \log \tau(C_{n}^{1,\gamma_1,\ldots,\gamma_{d-1}}) = \lim_{\beta \to \infty} \lim_{n \to \infty} \log \tau(C_{\beta n}^{1,\gamma_1,\ldots,\gamma_{d-1}})$$

which by definition is

$$\lim_{\beta \to \infty} z_{F}(\gamma_1,\ldots,\gamma_d) = \lim_{\beta \to \infty} z_{NF}(\beta; \gamma_1,\ldots,\gamma_{d-1}).$$

In the particular case of $d = 2$ it shows that the limits over $n$ and $\beta$ commute, that is,

$$\lim_{\beta \to \infty} \lim_{n \to \infty} \frac{\log \tau(C_{n}^{1,\gamma_1,\gamma_2})}{\beta n} = \lim_{n \to \infty} \lim_{\beta \to \infty} \frac{\log \tau(C_{\beta n}^{1,\gamma_1,\gamma_2})}{\beta n}$$

which does not seem obvious a priori.

5
References


