The gap approximation for Gaussian multiple access channels

Dimitris Toupakaris* and Jungwon Lee†
*Wireless Telecommunications Laboratory, Department of Electrical and Computer Engineering
University of Patras, Rio, Greece 265 00, Email: dtouba@upatras.gr
†Marvell Semiconductor Inc., 5488 Marvell Ln, Santa Clara, CA 95054, Email: jungwon@stanfordalumni.org

Abstract—The applicability of the gap approximation is investigated in the context of Gaussian multi-user channels where suboptimal transmission schemes are employed. In this case, the assumption of all users being subject to Gaussian interference may not always hold. Considering the two-user Multiple Access Channel as an example, and assuming uncoded Pulse Amplitude Modulation (PAM), it is shown that the gap approximation can be used, albeit in a modified fashion compared to the single-user Gaussian channel. It is also demonstrated that successive interference cancellation at the receiver does not cause performance degradation when PAM is used by all transmitters, similar to the case of capacity-achieving transmit waveforms.

I. INTRODUCTION

The gap approximation, first introduced by D. Forney, is a convenient way of expressing the achievable rates in the Additive White Gaussian Noise (AWGN) channel when a suboptimal transmission scheme is used that aims at a given probability of symbol error, $P_e$ [1]. It is accurate when the number of bits per dimension exceeds 1/2, and expresses the signal-to-noise ratio (SNR) “penalty” that has to be paid when using a specific suboptimal coding scheme instead of a capacity-achieving code. For example, when uncoded Pulse Amplitude Modulation (PAM) or Quadrature Amplitude Modulation (QAM) is used, the gap corresponding to a symbol error rate per dimension equal to $P_e = 10^{-6}$ is approximately equal to $\Gamma = 9$ dB for large PAM constellations. This means that, in order to achieve a given rate through the AWGN channel with symbol error rate not exceeding $P_e$, a transmitter using $M$-PAM needs to boost its power by a factor $\Gamma(P_e, M)$ compared to when a capacity-achieving scheme is employed. The gap approximation can also be used when channel coding is applied. In this case, the gap is smaller than the gap for uncoded PAM/QAM. It has also been shown that the gap can be used to describe the performance of variable-rate and variable-power schemes over fading channels [2].

Because of the simple description of the achievable rates that can be obtained when using the gap, and its being equal to the SNR “penalty” associated with a constellation and/or a code, it is frequently possible to quickly apply theoretical results for the AWGN channel to practical scenarios. The results can also be applied to channels with inter-symbol interference by decomposing them to parallel AWGN channels. As an example, in Orthogonal Frequency Division Multiplexing (OFDM) systems employing QAM, the achievable rate can be found easily by waterfiling over a noise floor penalized by $\Gamma$. Even in systems using Orthogonal Frequency Division Multiple Access (OFDMA), the gap approximation is accurate if the interference on each subcarrier is sufficiently close to Gaussian. This can be achieved via the use of hopping patterns that ensure that the streams using a particular subcarrier (or, in general, a frequency band) are rotated [3]. However, this can only be done if interference is to be treated as noise and is not decoded.

In order to improve the efficiency of the usage of the available resources, communications systems have started resorting to multiuser transmission and detection. For example, in the uplink of a cellular system, which is a Multiple Access Channel (MAC) from the point of view of Information Theory, more than one user may be communicating with the base station in a given frequency, in general. It is well known that the capacity of the Gaussian MAC, i.e., the MAC with Gaussian noise at the receiver, is achieved when the transmitted signals follow the Gaussian distribution [4]. The receiver needs to employ successive interference cancellation (SIC). However, in practical systems, the distribution of the sources may not be Gaussian. In order to simplify the design, transmitters typically use signals belonging to a discrete constellation. Moreover, delay and coherence time constraints may limit the length of the code that can be used.

The purpose of this paper is to explore if and how the gap approximation can be used for the case of multi-user channels when the sources do not employ capacity-achieving distributions. Moreover, it is examined whether successive interference cancellation at the receiver is optimal when suboptimal transmit waveforms are employed. The goal is to characterize the rate region with expressions similar to those for the Gaussian case by inserting an appropriate value for the gap. This is done in the context of the two-user Gaussian MAC, and it is shown that it is possible to simplify the calculation of the rate region using the gap approximation. However, this should be done with care because, as shown in the following, the gap should only be taken into account once. Regarding successive interference cancellation, it is demonstrated that no performance loss occurs when PAM is used by all transmitters. Being able to use expressions and receiver architectures similar to the ones that correspond to capacity-achieving distributions can help apply results for multiuser channels, such as [5]–[7], to systems where the sources employ suboptimal modulation schemes. Some related work can be found in [8], [9]. However,
in both references, the gap is used by extending the single-user gap analysis to the multi-user case in a simple way without examining what its value should be for multiuser channels.

The remainder of the paper is organized as follows. In Section II the gap approximation for the AWGN channel is reviewed, and it is examined if and how it can be used for the Gaussian MAC when both users employ 2-PAM. In Section III the more general case is studied where each user employs a PAM constellation or arbitrary size. The analytic results are verified in Section IV via simulation. Finally, Section V contains concluding remarks.

II. THE GAP APPROXIMATION AND ITS APPLICATION ON THE TWO-USER MULTIPLE ACCESS CHANNEL: THE 2-PAM CASE

A. The gap approximation for uncoded PAM/QAM over the AWGN channel

In this section, the gap approximation for uncoded PAM/QAM over the AWGN channel is reviewed. For simplicity, $M$-PAM is considered, although the analysis for QAM is similar. When the average power of the transmitter is equal to $P$, the minimum distance of the constellation is equal to [1]

$$d_{\text{min}} = \sqrt{12P/(M^2 - 1)},$$

and the probability of symbol error can be approximated by

$$P_e = 2\left(1 - \frac{1}{M}\right)Q\left(\sqrt{\frac{3}{M^2 - 1}\text{SNR}}\right), \quad (1)$$

where $Q(x) = \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2} du$ is the Q-function (sometimes denoted by $\Phi(x)$) [1], and $\text{SNR}$ is the signal-to-noise ratio at the receiver.

For a given $P_e$, by rearranging (1),

$$\log_2 M = b = \frac{1}{2} \log_2 \left(1 + \frac{3\text{SNR}}{Q^{-1}\left(M/(M-1)\right)^2}\right) = \frac{1}{2} \log_2 \left(1 + \frac{\text{SNR}}{\Gamma(P_e, M)}\right),$$

where $\Gamma(P_e, M) = \left[Q^{-1}\left(M/(M-1)\right)^2\right]/3$ is the gap. As $M$ increases, the gap converges to $\Gamma_{\infty}(P_e) = \left[Q^{-1}(P_e/2)\right]/3$.

The gap approximation simplifies the calculation of the achievable rates over the AWGN channel for practical transmission and encoding schemes. For coded transmission, the gap becomes $\Gamma/\gamma$, where $\gamma$ is the coding gain. Clearly, $\gamma \leq \Gamma$. However, it should be noted that the derivation of the gap relies on the assumption that the unknown and/or undesired signal that is superimposed on the data destined to the user is Gaussian. Although this holds for single-user Gaussian channels, it may not be true for the multi-user case. This motivates the investigation of if and how the gap should be used that is the topic of this paper.

B. Uncoded 2-PAM over the two-user Gaussian Multiple Access Channel

The two-user real-valued Gaussian MAC of Fig. 1 is considered. It is initially assumed that both users employ 2-PAM. The power of the received signal from user $i$ is equal to $P_i$. This can also account for the (known) attenuation of each link.

![Two-user MAC diagram](image)

If the variance of the real Gaussian noise at the receiver is equal to $\sigma^2$, the capacity region of the Gaussian MAC is given by [4]

$$R_1 \leq \frac{1}{2} \log \left(1 + \frac{P_1}{\sigma^2}\right) = \frac{1}{2} \log (1 + \text{SNR}_1),$$

$$R_2 \leq \frac{1}{2} \log \left(1 + \frac{P_2}{\sigma^2}\right) = \frac{1}{2} \log (1 + \text{SNR}_2)$$

and

$$R_1 + R_2 \leq \frac{1}{2} \log \left(1 + \frac{P_1 + P_2}{\sigma^2}\right) = \frac{1}{2} \log (1 + \text{SNR}_1 + \text{SNR}_2),$$

where $\text{SNR}_i \triangleq P_i/\sigma^2$. In order for each user to be able to transmit 1 bit, if a capacity-achieving scheme is used, $\text{SNR}_1$ and $\text{SNR}_2$ need to satisfy

$$1 \leq \frac{1}{2} \log (1 + \text{SNR}_1),$$

$$1 \leq \frac{1}{2} \log (1 + \text{SNR}_2)$$

and

$$2 \leq \frac{1}{2} \log (1 + \text{SNR}_1 + \text{SNR}_2).$$

Therefore, the $\text{SNR}_i$ need to satisfy the equations

$$\text{SNR}_1 \geq 3 \quad \text{and} \quad \text{SNR}_1 + \text{SNR}_2 \geq 15.$$ 

Moreover, for equal rates, assuming that user 1 is the stronger (in terms of receiver $\text{SNR}$) and is decoded first,

$$\frac{\text{SNR}_1}{1 + \text{SNR}_2} = \text{SNR}_2.$$ 

Solving the equations leads to $\text{SNR}_{1,\text{min}} = 12 \approx 10.8 \text{ dB}$ and $\text{SNR}_{2,\text{min}} = 3 \approx 4.77 \text{ dB}$.

Assume, now, that 2-PAM is used, that the target symbol error rate for each user is equal to $P_e$ and that $P_1 \geq P_2$, 

978-1-4244-4148-8/09/$25.00 ©2009
This full text paper was peer reviewed at the direction of IEEE Communications Society subject matter experts for publication in the IEEE "GLOBECOM" 2009 proceedings.
without loss of generality. The probability density function of the received signal \( y = x_1 + x_2 + z \) is plotted in Fig. 2.

![PDF of received signal when both users employ 2-PAM modulation.](image)

As can be seen, the probability of symbol error for (the stronger) user 1 can be approximated by

\[
P_{e,1} \approx \frac{1}{2} Q \left( \frac{\sqrt{P_1} - \sqrt{P_2}}{\sigma} \right) = \frac{1}{2} Q \left( \sqrt{\text{SNR}_1} - \sqrt{\text{SNR}_2} \right). \tag{2}
\]

Assuming that user 1 has been decoded correctly, the probability of symbol error of user 2 (whose signal is subject to Gaussian noise only) is equal to

\[
P_{e,2} = Q \left( \sqrt{\text{SNR}_2} \right).
\]

Therefore, in order to achieve a rate \( R_2 \) equal to 1 bit/channel use, user 2 needs to employ \( \text{SNR}_2 \geq \left[ Q^{-1}(P_e) \right]^2 \). For \( P_e = 10^{-7}, \text{SNR}_2 \geq 14.3 \) dB (as opposed to 4.77 dB when using a capacity-achieving scheme). If the signal of user 2 were Gaussian, the SNR of user 1 that would allow transmission of 1 bit using 2-PAM would be given by

\[
b_1 = 1 \leq \frac{1}{2} \log_2 \left( 1 + \frac{P_1}{\Gamma(\text{SNR}_1)} \right) = \frac{1}{2} \log_2 \left( 1 + \frac{\text{SNR}_1}{\Gamma(1+\text{SNR}_2)} \right) \Rightarrow \text{SNR}_1 \geq \frac{3}{2} \text{SNR}_2 \approx 4.77 + 14.47 + \Gamma(2) \approx 28.8\text{ dB}.
\]

However, by taking into account that user 2 employs 2-PAM and not a Gaussian codebook, and substituting in (2),

\[P_{e,1} \approx (1/2)Q \left( \sqrt{\text{SNR}_1} - \sqrt{\text{SNR}_2} \right) \Rightarrow \text{SNR}_1 = 20.22 \text{ dB for } P_{e,1} = 10^{-7}.
\]

Intuitively, user 1 needs to employ a higher SNR when 2-PAM is used compared to 10.8 dB, that was found to be the required minimum value for \( \text{SNR}_1 \) when both users employ optimal, capacity-achieving transmission schemes. Nevertheless, the difference with \( 1 + \text{SNR}_2 \) is smaller than \( 3\Gamma(P_e,2) \) (5.75 dB instead of 14.32 dB). Intuitively this happens because, given the source distribution, Gaussian additive interference is the worst in terms of capacity [10]. The fact that the interference seen by each user that is decoded successively (except for the last one) is non-Gaussian means that, for given \( \text{SNR}_1 \), a higher rate can be transmitted compared to the case of Gaussian interference. In the following section, this difference is quantified for any PAM size. The reason why PAM is chosen instead of QAM is because the analysis for the latter is more complicated owing to the constellation rotation caused by a possible non-zero phase value of the channel. Use of PAM can help focus on the main goal of this paper. The gap analysis for QAM is deferred to a future paper.

---

1When the symbol error probability of both users is small, it can be assumed that \( P_e \approx P_{e,1} = P_{e,2} \).
The first term converges to 1 as the rate of user 2 increases (since both $M_2$ and $\text{SNR}_2$ increase), whereas the second term converges to zero. This means that the SNR of the signal of user 1 needs to exceed $(1 + \text{SNR}_2)$ only by the quantity required by Shannon’s capacity expression, or, equivalently, that the additional gap, $\tilde{\Gamma}$, for user 1 when $\frac{1}{2} \log_2 \left( 1 + \frac{1}{1 + \text{SNR}_1} \right)$ is used to calculate the rate of user 1 is almost zero. Hence, the gap penalty because of PAM needs to be taken into account once, i.e., when determining the SNR of user 2 that is decoded after user 1 and is subject to purely Gaussian noise. Of course, this also means that the power of user 1 needs to be $\tilde{\Gamma}$ times larger than in the case where both users employ Gaussian, capacity-achieving signals, but not $\Gamma^2$ as would be the case if the gap approximation were not used with caution to determine the SNR of the stronger user.

As long as the $\text{SNR}_1$ exceed the minimum required values to guarantee the target $P_e$, successive interference cancellation (SIC) can still be used for detection, as in the capacity-achieving case. The receiver can first decide on the strongest user 1, and then decode the signal of user 2. Therefore, although SIC is not optimal, in general, when the distributions of the transmitted signals are not Gaussian, when the condition $d_{min} \geq \frac{M_2 - 1}{2} \sqrt{\frac{12P_2}{M_2 - 1}}$ is satisfied, SIC is optimal and there is no performance penalty compared to a Maximum-Likelihood (ML) detector that decodes all users jointly. This can greatly simplify the receiver since, instead of joint ML detection of users 1 and 2, a single-user ML (PAM) decoder can be used twice, combined with SIC. It should be stressed here that this does not hold for all multiuser channels.

The value of the additional gap, $\tilde{\Gamma}$, quantified in (5), is verified by obtaining $P_1$ using a computer program. More specifically, for fixed $P_2$, $P_1$ is increased until the target $P_e$ is achieved. As can be seen in Table I, the obtained values agree with the analytical expression. The values also confirm that $d_{min} \geq \frac{M_2 - 1}{2} \sqrt{\frac{12P_2}{M_2 - 1}}$ has to be satisfied in order to attain $P_e$. The reason for this is explained in the following.

### B. A different viewpoint

The problem is now re-examined from the viewpoint of a receiver that decodes the signals of all users jointly and wishes to distinguish among all possible received messages regardless of how they are formed and which transmit symbol combinations they correspond to. If user 1 sends one of $M_1$ messages and user 2 one of $M_2$ messages, since, in a MAC channel, the users transmit independently, the receiver should be able to distinguish among $M_1 M_2$ different messages. In order for the probability of error of both users to be kept below a target value, $P_e$, the receiver should be able to find the combination that was actually transmitted, among the $M_1 M_2$ messages, with probability of error not exceeding $P_e$. If the signal at the receiver is seen as a PAM constellation coming from a single user, the power that is required in order to guarantee $P_e$ equals [1]

$$P_{tot} = \frac{d_{min}^2}{2} \left[ (M_1 M_2)^2 - 1 \right] \sigma^2.$$  

Similar to the derivation of the gap in Section II-A,

$$P_{tot} = \frac{Q^{-1}\left(\frac{M_1 M_2 P_e}{2(M_1 M_2 - 1)}\right)}{3} \left[ (M_1 M_2)^2 - 1 \right] \sigma^2.$$  

As $M_1$ and $M_2$ grow,

$$P_{tot} \rightarrow \frac{Q^{-1}(P_e)^2}{3} \left[ (M_1 M_2)^2 - 1 \right] \sigma^2 = \Gamma_\infty(P_e) \left[ (M_1 M_2)^2 - 1 \right] \sigma^2 \geq \Gamma_\infty(P_e) P_{C-A},$$

where $P_{C-A}$ is the aggregate power that attains a rate of $\log_2(M_1 M_2)$ bits when a capacity-achieving transmission scheme is employed by both users. However, this result is not by itself a proof that if $P_1 = \Gamma_\infty(P_e) P_{1,C-A}$ and $P_2 = \Gamma_\infty(P_e) P_{2,C-A}$ the symbol error probability at the receiver is equal to $P_e$. This was shown in the previous section.
The values of $P_{1,CA}$ (and, therefore, of $P_1$ and $P_2$) depend on the cancellation order at the receiver. A given pair $(M_1, M_2)$ can be achieved in two ways, depending on which user is the strongest (and, therefore, decoded first). The aggregate power $P_{tot}$ is the same for both schemes.

Consider, again, the case of user $i$ employing a constellation of $M_i$ points. One way to construct the constellations so that the aggregate signal at the receiver appear as $M_1M_2$-PAM, is to place the signals of user 2 $d_{min}$ apart (where $d_{min}$ is the minimum distance of the overall constellation) and place the signals of user 1 $M_2d_{min}$ apart. This construction guarantees that $P_2$ is not exceeded and is shown in Fig. 4(a), where $M_1 = 4$ and $M_2 = 2$. The required powers are $P_2 = \frac{d_{min}}{M_2^2} [M_2^2 - 1] \sigma^2$ and $P_1 = \frac{4d_{min}^2}{12} [M_1^2 - 1] \sigma^2$. Hence, $P_{tot} = P_1 + P_2 = \frac{d_{min}^2}{12} (M_1M_2^2 - 1) \sigma^2$. However, the same point of the rate region can be achieved by having user 1 transmit a constellation with $d_{min}$ as shown in Fig. 4(b). In this case, $P_1 = \frac{d_{min}^2}{12} [M_2^2 - 1] \sigma^2$ and $P_2 = \frac{M_2d_{min}^2}{12} [M_2^2 - 1] \sigma^2$. Therefore, again, $P_{tot} = P_1 + P_2 = \frac{d_{min}^2}{12} (M_1M_2^2 - 1) \sigma^2$. This agrees very well with the results on the capacity of the MAC (where Gaussian inputs are employed). A given rate point $(R_1,R_2)$ can be achieved using powers $((2^2R_1 - 1)\sigma^2, (2^2R_2 - 1)\sigma^2)$ when user 2 is decoded first and 1 second, or using $((2^2R_1 - 1)2^{2R_2}\sigma^2, (2^2R_2 - 1)\sigma^2)$ when the interference cancellation order is reversed. From the above, it can be seen that the same happens when PAM is used, the only difference being that both powers should be scaled by the gap.

Using the Maximum Likelihood viewpoint, it becomes clear why the gap should only be taken into account once. The powers of both users need to be boosted by $\Gamma$. However, the difference between the powers of the users only needs to be equal to the value that is predicted by Shannon’s capacity formula. From the ML perspective, this is because the power of the aggregate signal that is affected by purely Gaussian noise is already boosted by $\Gamma$. From the SIC point of view, the user that is decoded first is subject to interference that is not Gaussian. Moreover, when the power is allocated optimally for a given target $P_e$, both ML and SIC receivers can be employed, similar to when Gaussian capacity-achieving signals are used by the transmitters.

It can now be seen why, for small constellations, the value of the additional gap, $\tilde{\Gamma}$, (the ratio of Table I) is not zero. The total power required to transmit a $M_1M_2$ PAM constellation is equal to

$$P_{tot} = ((M_1M_2)^2 - 1)\Gamma(P_e, M_1M_2)\sigma^2.$$  

(6)

Moreover, if user 1 is decoded first, $P_{tot}$ can be written as

$$P_{tot} = P_1 + P_2 = \Gamma(M_1^2 - 1)(\sigma^2 + P_2) + P_2 = \Gamma(M_1^2 - 1)(\sigma^2 + \Gamma(P_e, M_2)(M_2^2 - 1)\sigma^2) + \Gamma(P_e, M_2)(M_2^2 - 1)\sigma^2.$$  

(7)

Combining (6) and (7),

$$\tilde{\Gamma} = \frac{(M_1M_2)^2 - 1}{(M_1^2 - 1)(1 + \Gamma(P_e, M_2)(M_2^2 - 1))}.$$  

(8)

For small values of $M_2$, $\tilde{\Gamma}$ is positive. For $M_1, M_2 \rightarrow \infty$, $\tilde{\Gamma} \rightarrow 1$, in agreement with (5).

<table>
<thead>
<tr>
<th>$b_1=1$</th>
<th>$b_1=2$</th>
<th>$b_1=3$</th>
<th>$b_1=4$</th>
<th>$b_1=5$</th>
<th>$b_1=6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_2=1$</td>
<td>+0.98 dB</td>
<td>+1.05 dB</td>
<td>+1.07 dB</td>
<td>+1.08 dB</td>
<td>+1.09 dB</td>
</tr>
<tr>
<td></td>
<td>(+0.98 dB)</td>
<td>(+1.04 dB)</td>
<td>(+1.07 dB)</td>
<td>(+1.08 dB)</td>
<td>(+1.09 dB)</td>
</tr>
<tr>
<td>$b_2=2$</td>
<td>+0.11 dB</td>
<td>+0.14 dB</td>
<td>+0.15 dB</td>
<td>+0.16 dB</td>
<td>+0.16 dB</td>
</tr>
<tr>
<td></td>
<td>(+0.11 dB)</td>
<td>(+0.14 dB)</td>
<td>(+0.15 dB)</td>
<td>(+0.16 dB)</td>
<td>(+0.16 dB)</td>
</tr>
<tr>
<td>$b_2=3$</td>
<td>-0.04 dB</td>
<td>-0.02 dB</td>
<td>-0.02 dB</td>
<td>-0.02 dB</td>
<td>-0.02 dB</td>
</tr>
<tr>
<td></td>
<td>(-0.04 dB)</td>
<td>(-0.03 dB)</td>
<td>(-0.02 dB)</td>
<td>(-0.02 dB)</td>
<td>(-0.02 dB)</td>
</tr>
<tr>
<td>$b_2=4$</td>
<td>-0.05 dB</td>
<td>-0.04 dB</td>
<td>-0.04 dB</td>
<td>-0.04 dB</td>
<td>-0.02 dB</td>
</tr>
<tr>
<td></td>
<td>(-0.05 dB)</td>
<td>(-0.04 dB)</td>
<td>(-0.04 dB)</td>
<td>(-0.04 dB)</td>
<td>(-0.04 dB)</td>
</tr>
<tr>
<td>$b_2=5$</td>
<td>-0.04 dB</td>
<td>-0.03 dB</td>
<td>-0.03 dB</td>
<td>-0.03 dB</td>
<td>-0.02 dB</td>
</tr>
<tr>
<td></td>
<td>(-0.04 dB)</td>
<td>(-0.03 dB)</td>
<td>(-0.03 dB)</td>
<td>(-0.03 dB)</td>
<td>(-0.03 dB)</td>
</tr>
<tr>
<td>$b_2=6$</td>
<td>-0.02 dB</td>
<td>-0.02 dB</td>
<td>-0.02 dB</td>
<td>-0.02 dB</td>
<td>-0.02 dB</td>
</tr>
<tr>
<td></td>
<td>(-0.02 dB)</td>
<td>(-0.02 dB)</td>
<td>(-0.02 dB)</td>
<td>(-0.02 dB)</td>
<td>(-0.03 dB)</td>
</tr>
</tbody>
</table>

TABLE I
VALUES OF THE RATIO $SNR_{1,9}/SNR_1$ FOR VARYING PAM SIZE USING A COMPUTER PROGRAM. VALUES OBTAINED USING (5) ARE IN PARENTHESES.
Finally, the results in this paper can be extended to the general $K$-user MAC. This is more easily done using the receiver-centric approach, where a PAM signal of size $M_1M_2\cdots M_K$ is seen by the receiver.

IV. SIMULATION RESULTS

The analytical derivations are verified via simulation. $M_1 = 4$ and $M_2 = 8$. In Fig. 5, user 2 is the weakest in terms of power (as in Fig. 4(a)), and its symbols are placed $d_{\text{min}}$ away. When SIC is used at the receiver, user 2 is decoded after user 1. In Fig. 5(a), the SNR of user 2 is kept constant and equal to SNR$_2 = \Gamma(p_e = 10^{-7}, 4)(M_2^2 - 1) \approx 27.7$ dB and SNR$_1$ varies, whereas in Fig. 5(b) the SNR of user 1 is kept constant at SNR$_1 = \Gamma(p_e = 10^{-7}, 8)((M_1M_2)^2 - 1) \approx 39.4$ dB and SNR$_2$ is modified. The difference of SNR$_1$ and SNR$_2$ with the corresponding values when a capacity-achieving code is used are 9.58 dB and 9.7 dB, respectively, i.e., both are $\Gamma$ away from capacity. Moreover, $(M_1^2 - 1)(\text{SNR}_2 + 1) = 39.47$ dB, meaning that the additional gap when calculating SNR$_1$ from SNR$_2$ is almost 0 dB.

Fig. 5. Simulated $P_e$ for weak user 2 and strong user 1. $M_1 = 4$, $M_2 = 8$. Similar observations can be made in Fig. 6 where the weak user is now 1 and is decoded after user 2. The SNR of user 1 is now equal to SNR$_1 = 21.44$ dB. It is smaller compared to the SNR of the weak user 2 in the previous case because its constellation is smaller. Conversely, SNR$_2$ is larger than SNR$_1$ in the previous case and equal to 39.78 dB. However, the sum of the SNRs is the same for both cases of Figs. 5 and 6 and equal to 39.84 dB.

V. CONCLUSION

The purpose of this paper was to study if and how the gap approximation applies to the MAC and if SIC at the receiver can be coupled with suboptimal transmission schemes. It was shown that, for the case of the Gaussian MAC, it is possible to employ the gap approximation, albeit with caution in order to avoid taking the gap into account twice. Moreover, similar to the capacity-achieving case, successive interference cancellation can be used in place of a joint ML decoder leading to a simplification of the receiver. Having established that the gap can be used for the Gaussian MAC, it is now easy to derive rate regions by appropriately inserting the gap in the expressions for the capacity region. Not only does this facilitate the calculation of the rate region, but can also simplify the analysis and the design of resource allocation algorithms of systems operating over multiuser channels.

REFERENCES