Some two-sided inequalities for multiple Gamma functions and related results

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Abstract
There is an abundant literature on inequalities for the (Euler’s) Gamma function \( \Gamma \) and its various related functions. Yet, only very recently, several authors began to study inequalities for the (Barnes’) double Gamma function \( \Gamma_2 \). Here, in this paper, we aim at presenting several two-sided inequalities for the multiple Gamma functions \( \Gamma_n \) \( (n = 2, 3, 4, 5) \). In our investigation of these two-sided inequalities for the multiple Gamma functions \( \Gamma_n \) \( (n = 2, 3, 4, 5) \), we employ and extend a method based upon Taylor’s formula and express \( \log \Gamma_n(1 + x) \) as series involving the Zeta functions. We also give a more convenient explicit form of the multiple Gamma functions \( \Gamma_n \) \( (n = 2, 3, 4, 5) \) \( (n \in \mathbb{N}) \), \( \mathbb{N} \) being the set of positive integers. The main two-sided inequalities for the multiple Gamma functions \( \Gamma_n \) \( (n = 2, 3, 4, 5) \) (which we have presented in this paper) are presumably new and their derivations provide a fruitful insight into the corresponding problem for the multiple Gamma functions \( \Gamma_n \) when \( n \geq 6 \).

1. Introduction, definitions and preliminaries

The multiple Gamma functions \( \Gamma_n \) were defined and studied systematically by Barnes [9–12] and by others (cf., e.g., [4,38–40]) in about 1900 (see also [37, p. 649, Entry 6.441(4); p. 887, Entry 8.333] and [63, p. 264]). About two decades ago, these functions were revived in the study of the determinants of the Laplacians on the \( n \)-dimensional unit sphere \( S^n \) (cf. [20,21,25,26,43,59,61]) and have since been investigated in various other ways (cf. [57, p. 24, Section 1.3] and [58, p. 38, Section 1.4]); see also [2,3,15–17,22,23,27,35,41,44–49,52,54].

There is a remarkably abundant literature on inequalities for the classical Gamma function \( \Gamma \) and its such related functions as, for example, the Psi (or Digamma) function defined by

\[
\psi(z) := \frac{d}{dz} \log \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)} \quad \text{or} \quad \log \Gamma(z) = \int_1^z \psi(t)dt,
\]

(see, e.g., [5,6,50,51]; see also the references cited in these earlier works). Among several equivalent useful expressions for the Gamma function, its Weierstrass canonical product form is recalled here as follows:

\[
\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{k=1}^{\infty} \left( 1 + \frac{z}{k} \right)^{-1} \exp \left( \frac{z}{k} \right) \quad (z \in \mathbb{C} \setminus \mathbb{Z}_0),
\]

(1.1)
where \( \mathbb{C} \) is the set of complex numbers, \( \mathbb{Z}_0 := \{0, -1, -2, \cdots\} \) and \( \gamma \) denotes the Euler–Mascheroni constant defined by (see also a recent work [29])

\[
\gamma := \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \log n \right) \approx 0.57721 56649 01532 86060 65120 90082 40243 1042 \ldots.
\] (1.2)

On the other hand, there are only a few recent papers on inequalities for the double Gamma function \( \Gamma_2 := 1/G \) (see, for example, the recent works by Batir [13], Batir and Cancan [14], Chen [18], and Chen and Srivastava [19]). Very recently, Koumandos and Pedersen [42] and Choi and Srivastava [31] presented asymptotic formulas for Barnes’ triple Gamma function.

Barnes [9] gave several explicit Weierstrass canonical product forms of the double Gamma function \( \Gamma_2 := 1/G \), one of which is recalled here as follows:

\[
(\Gamma_2(z + 1))^{-1} = G(z + 1) = (2\pi)^{z/2} \exp \left( -\frac{1}{2} z^2 - \frac{1}{2} (z + 1)^2 \right) \prod_{k=1}^{\infty} \left( 1 + \frac{z^2}{k^2} \right). \quad \text{(1.3)}
\]

where \( \gamma \) denotes the Euler–Mascheroni constant given by Eq. (1.2). Analogous to the familiar relations:

\[
\Gamma(1) = 1 \quad \text{and} \quad \Gamma(z + 1) = z \Gamma(z), \quad \text{(1.4)}
\]

the double Gamma function \( \Gamma_2 := 1/G \) satisfies the following fundamental relations:

\[
G(1) = 1 \quad \text{and} \quad G(z + 1) = \Gamma(z) G(z). \quad \text{(1.5)}
\]

Like the Euler–Mascheroni constant \( \gamma \) in Eq. (1.2), there is a set of constants which are naturally involved in the theory of multiple Gamma functions (see [1, 15, 36] and [57, p. 128]). Some of these constants are introduced here for our latter use. First of all, \( A \) denotes the Glaisher–Kinkelin constant defined by (see [9])

\[
\log A = \lim_{n \to \infty} \left[ \sum_{k=1}^{n} k \log k - \left( \frac{n^2}{2} + \frac{n}{2} + \frac{1}{12} \right) \log n + \frac{n^2}{4} \right] \approx 1.282427130 \ldots. \quad \text{(1.6)}
\]

The constants \( B \) and \( C \) are analogous to the Glaisher–Kinkelin constant \( A \) and are defined by (see [26])

\[
\log B = \lim_{n \to \infty} \left[ \sum_{k=1}^{n} k^2 \log k - \left( \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \right) \log n + \frac{n^3}{9} - \frac{n^2}{12} \right], \quad \text{(1.7)}
\]

and

\[
\log C = \lim_{n \to \infty} \left[ \sum_{k=1}^{n} k^3 \log k - \left( \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} - \frac{1}{120} \right) \log n + \frac{n^4}{16} - \frac{n^3}{12} \right], \quad \text{(1.8)}
\]

respectively. The approximate numerical values of the constant \( B \) and \( C \) are given by

\[
B \approx 1.03091675 \ldots \quad \text{and} \quad C \approx 0.97955746 \ldots.
\]

The constants \( A, B, \) and \( C \) are also known to be expressible as follows:

\[
\log A = \frac{1}{12} - \zeta'(-1), \quad \log B = -\zeta'(-2), \quad \text{and} \quad \log C = -\frac{11}{720} - \zeta'(-3). \quad \text{(1.9)}
\]

in terms of special values of the derivative of the Riemann zeta function \( \zeta(s) \) defined by

\[
\zeta(s) := \begin{cases} 
\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1 - 2^{-s}} \sum_{n=1}^{\infty} \frac{1}{n^s} & (\Re(s) > 1), \\
\frac{1}{1 - 2^{-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} & (\Re(s) > 0; \ s \neq 1).
\end{cases} \quad \text{(1.10)}
\]

The Riemann Zeta function \( \zeta(s) \) is a special case of the Hurwitz (or generalized) Zeta function \( \zeta(s,a) \) defined by

\[
\zeta(s,a) := \sum_{k=0}^{\infty} \frac{1}{(k+a)^s}, \quad (\Re(s) > 1; a \in \mathbb{C} \setminus \mathbb{Z}_0). \quad \text{(1.11)}
\]

each of which can be continued meromorphically to the whole complex \( s \)-plane except for a simple pole at \( s = 1 \) with its residue 1 (see, for details, [57, pp. 88–103]).

**Remark 1.** Adamchik [11] (see also [15] and [57, p. 128]) presented a set of mathematical constants which include the above-defined \( A, B, \) and \( C \) as special cases.

The Polygamma functions \( \psi^{(n)}(z) \ (n \in \mathbb{N}) \) are defined by

\[
\psi^{(n)}(z) := \sum_{k=0}^{\infty} \frac{1}{(k+a)^s}, \quad \psi(x) := \frac{d}{dx} \log \Gamma(x),
\]
\[
\psi^{(n)}(z) := \frac{d^{n+1}}{dz^{n+1}} \{ \log \Gamma(z) \} = \frac{d^n}{dz} \{ \psi(z) \} \quad (n \in \mathbb{N}_0; \ z \in \mathbb{C} \setminus \mathbb{Z}_0),
\]
where \( \mathbb{N} \) is the set of positive integers and \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). In terms of the Hurwitz (or generalized) Zeta function \( \zeta(s, a) \), we can write
\[
\psi^{(n)}(z) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(k+z)^{n+1}} = (-1)^{n+1} n! \zeta(n+1, z) \quad (n \in \mathbb{N}; \ z \in \mathbb{C} \setminus \mathbb{Z}_0),
\]
which has turned out to be useful in certain applications (see, e.g., [28]).

Here, in this paper, we aim at presenting some two-sided inequalities for the multiple Gamma functions \( \Gamma_n \) for (especially) \( n = 2, 3, 4, 5 \). We employ and extend the method initiated by Batir and Cancan [14] and express \( \log \Gamma_n(1 + x) \) as series involving the Zeta functions (see, e.g., [7, 21, 24–26, 33, 53, 55–57]). We also derive a more convenient and potentially useful explicit form of the multiple Gamma functions \( \Gamma_n \) (\( n \in \mathbb{N} \)). The main two-sided inequalities for the multiple Gamma functions \( \Gamma_n (n = 2, 3, 4, 5) \) (which we have presented in this paper) are presumably new and their derivations provide a fruitful insight into the corresponding problem for the multiple Gamma functions \( \Gamma_n \) when \( n \geq 6 \).

2. Multiple gamma functions

There are two known ways to define the \( n \)-ple Gamma functions \( \Gamma_n \). First of all, Barnes [12] (see also Vardi [59]) defined \( \Gamma_n \) by using the \( n \)-ple Hurwitz Zeta functions (see, e.g., [22], [57, Chapter 2]). Secondly, a recurrence relation of the Weierstrass canonical product forms of the \( n \)-ple Gamma functions \( \Gamma_n \) was given by Vignéras [60] who used the theorem of Dufresnoy and Pisot [34] which provides the existence, uniqueness, and expansion of the series of Weierstrass satisfying a certain functional equation.

By making use of the aforementioned Dufresnoy–Pisot theorem and starting with
\[
f_1(x) = -\gamma x + \sum_{n=1}^{\infty} \left[ \frac{x}{n} - \log \left( 1 + \frac{x}{n} \right) \right],
\]
Vignéras [60] obtained a recurrence relation of \( \Gamma_n (n \in \mathbb{N}) \) which is stated here as Theorem 1 below.

**Theorem 1.** The \( n \)-ple Gamma functions \( \Gamma_n \) are defined by
\[
\Gamma_n(z) = [G_n(z)]^{(-1)^{n-1}} \quad (n \in \mathbb{N}),
\]
where
\[
G_n(z + 1) = \exp \{ f_n(z) \}
\]
and the functions \( f_n(z) \) are given by
\[
f_n(z) = -z A_n(1) + \sum_{k=1}^{n-1} \frac{p_k(z)}{k!} [f_{n-k-1}(0) - A_{n-k}(1)] + A_n(z),
\]
with
\[
A_n(z) = \sum_{m \in \mathbb{N}_0^{n-1} \times \mathbb{N}} \left[ \frac{1}{n} \left( \frac{z}{L(m)} \right)^n - \frac{1}{n-1} \left( \frac{z}{L(m)} \right)^{n-1} + \cdots + (-1)^{n-1} \frac{z}{L(m)} + (-1)^n \log \left( 1 + \frac{z}{L(m)} \right) \right],
\]
where
\[
L(m) = m_1 + m_2 + m_3 + \cdots + m_n,
\]
if
\[
m = (m_1, m_2, m_3, \ldots, m_n) \in \mathbb{N}_0^{n-1} \times \mathbb{N}
\]
and the polynomials \( p_n(z) \) given by
\[
p_n(z) := \begin{cases} 
1^n + 2^n + 3^n + \cdots + (N - 1)^n & (z = N; \ N \in \mathbb{N} \setminus \{1\}), \\
\frac{B_{n+1}(z) - B_{n+1}}{n+1} & (z \in \mathbb{C}),
\end{cases}
\]
satisfy the following relations:
\[
p_n(z) = \frac{B_{n+1}(z)}{n+1} = B_n(z) \quad \text{and} \quad p_n(0) = 0,
\]
\(B_n(z)\) being the Bernoulli polynomial of degree \(n\) in \(z\).

By analogy with the Bohr–Mollerup theorem (see [8, p. 14]; see also [57, p. 13]), which guarantees the uniqueness of the Gamma function \(\Gamma\), one can give, for the double Gamma function and (more generally) for the multiple Gamma functions of order \(n\) \((n \in \mathbb{N})\), a definition of Artin [8] by means of the following theorem (see Vignéras [60, p. 239]).

**Theorem 2.** For all \(n \in \mathbb{N}\), there exists a unique meromorphic function \(G_n(z)\) satisfying each of the following properties:

1. \(G_n(z + 1) = G_{n-1}(z)G_n(z)\) \((z \in \mathbb{C})\);
2. \(G_n(1) = 1\);
3. For \(x \geq 1\), \(G_n(x)\) are infinitely differentiable and
   \[
   \frac{d^{n+1}}{dx^{n+1}} \{\log G_n(x)\} \geq 0;
   \]
4. \(G_n(x) = x\).

It is not difficult to verify (see, e.g., [57, pp. 40–41]) that \(\Gamma_n(z)^{-1}\) is an entire function with zeros at \(z = -k\) \((k \in \mathbb{N}_0)\) with multiplicities
   \[
   \binom{n + k - 1}{n - 1} \quad (n \in \mathbb{N}; k \in \mathbb{N}_0).
   \]

In our earlier investigations, we gave explicit forms of the multiple Gamma functions \(\Gamma_n\) \((n = 3, 4, 5)\) (see, e.g., [21,33]). Now, by observing Eq. (2.7), we can present the following explicit form of the multiple Gamma functions \(\Gamma_n\) \((n \in \mathbb{N})\) for a potential and easier future use.

**Theorem 3.** The \(n\)-ple Gamma functions \(\Gamma_n\) in Theorem 1 can be written in a more explicit form as follows:

\[
\Gamma_n(1 + z) = \exp Q_n(z) \prod_{k=1}^n \left( 1 + \frac{z}{k} \right)^{\binom{n + k - 2}{n - 1}} \exp \left[ \binom{n + k - 2}{n - 1} \left( \sum_{j=1}^n \frac{(-1)^{n-1} z^j}{j^{n-k}} \right) \right],
\]

where \(Q_n(z)\) is a polynomial in \(z\) of degree \(n\) given by

\[
Q_n(z) := (-1)^{n-1} \left[ -z A_n(1) + \sum_{k=1}^{n-1} \frac{p_k(z)}{k!} \left( f_n^{(k)}(0) - A_n^{(k)}(1) \right) \right].
\]

\[
f_n(z) := -z A_n(1) + \sum_{k=1}^{n-1} \frac{p_k(z)}{k!} \left( f_n^{(k)}(0) - A_n^{(k)}(1) \right) + A_n(z),
\]

\[
A_n(z) := \sum_{k=1}^{n-1} (-1)^{n-k} \binom{n + k - 2}{n - 1} \left[ -\log \left( 1 + \frac{z}{k} \right) + \sum_{j=1}^n \frac{(-1)^{n-1} z^j}{j^{n-k}} \right]
\]

and

\[
p_n(z) = \frac{1}{n + 1} \sum_{k=1}^{n+1} \binom{n + 1}{k} B_{n+1-k} z^k \quad (n \in \mathbb{N}).
\]

**Remark 2.** In order to get explicit forms of the multiple Gamma functions \(\Gamma_n(1 + z)\) in Theorem 3, it is indispensable to compute \(A_n(1)\) explicitly. In fact, by using the Taylor–Maclaurin expansion of \(\log(1 + t)\) in Eq. (2.11) and certain series involving Zeta functions, Choi et al. [21] found that

\[
A_n(z) = \frac{1}{(n-1)!} \sum_{j=0}^{n-1} s(n-1, j) \left[ \sum_{k=0}^{j} \binom{j}{k} (1 + z)^{-k} \sum_{i=0}^{j} \binom{j}{i} (z^{-1})^{i-1} \sum_{l=0}^{j-i} \frac{(-1)^{j-i} \zeta(-l)}{j-l} \sum_{m=0}^{i} \frac{(-1)^{j-i} \zeta(m-l)}{j-i-m} \right],
\]

where \(s(n, k)\) denotes the Stirling numbers of the first kind (see [57, pp. 56–57]) and \(H_n\) denotes the harmonic numbers given by

\[
H_n = \sum_{i=1}^n \frac{1}{i}.
\]
Now, by applying Eq. (2.13) in Theorem 3, we can give explicit forms of the multiple Gamma functions $\Gamma_n$ ($n \in \mathbb{N}$) whose cases ($n = 3, 4, 5$) are recalled here as the following corollary (see [21]).

**Corollary 1.** Each of the following expressions holds true:

\[
\Gamma_n(1+z) = \exp \left( c_1 z + c_2 z^2 + c_3 z^3 \right) \cdot \prod_{k=1}^{n-1} \left( 1 + \frac{z}{k} \right) ^{-\left( k + 1 \right)} \left( k + 1 \right) \left( \frac{z}{k} - \frac{z^2}{2k^2} + \frac{z^3}{3k^3} \right),
\]

where

\[
c_1 = \frac{3}{8} - \frac{1}{4} \log(2\pi) - \log A, \quad c_2 = \frac{1}{8} + \frac{1}{4} \log(2\pi) + \frac{\gamma}{4}, \quad c_3 = -\frac{1}{4} - \frac{\pi^2}{36} - \frac{\gamma}{6};
\]

\[
\Gamma_n(1+z) = \exp \left( d_1 z + d_2 z^2 + d_3 z^3 + d_4 z^4 \right) \cdot \prod_{k=1}^{n-1} \left( 1 + \frac{z}{k} \right) ^{-\left( k + 1 \right)} \left( k + 1 \right) \left( \frac{z}{k} - \frac{z^2}{2k^2} + \frac{z^3}{3k^3} - \frac{z^4}{4k^4} \right),
\]

where

\[
d_1 = \frac{7}{24} - \frac{1}{2} \log A - \frac{1}{2} \log B - \frac{1}{6} \log(2\pi), \quad d_2 = -\frac{1}{144} + \frac{\gamma}{6} + \frac{1}{4} \log(2\pi) + \frac{1}{2} \log A.
\]

\[
d_3 = -\frac{2}{9} - \frac{\gamma}{6} - \frac{1}{12} \log(2\pi) - \frac{\pi^2}{54}, \quad d_4 = \frac{11}{144} + \frac{\gamma}{24} + \frac{\pi^2}{48} + \frac{\zeta(3)}{12};
\]

\[
\Gamma_n(1+z) = \exp \left( e_1 z + e_2 z^2 + e_3 z^3 + e_4 z^4 + e_5 z^5 \right) \cdot \prod_{k=1}^{n-1} \left( 1 + \frac{z}{k} \right) ^{-\left( k + 1 \right)} \left( k + 1 \right) \left( \frac{z}{k} - \frac{z^2}{2k^2} + \frac{z^3}{3k^3} - \frac{z^4}{4k^4} + \frac{z^5}{5k^5} \right),
\]

where

\[
e_1 = \frac{409}{1728} - \frac{1}{8} \log(2\pi) - \frac{11}{12} \log A - \frac{1}{6} \log B - \frac{3}{16} \frac{\zeta(3)}{\pi^2} + \frac{1}{20} \zeta(4) - \frac{1}{20} \zeta(5),
\]

\[
e_2 = -\frac{1}{16} + \frac{\gamma}{8} + \frac{11}{48} \log(2\pi) + \frac{3}{4} \log A + \frac{\zeta(3)}{16 \pi^2},
\]

\[
e_3 = \frac{149}{864} - \frac{11}{72} \log(2\pi) - \frac{1}{6} \log A - \frac{1}{12} \zeta(2),
\]

\[
e_4 = \frac{7}{64} + \frac{1}{16} \gamma + \frac{1}{48} \log(2\pi) + \frac{11}{96} \zeta(2) + \frac{1}{16} \zeta(3),
\]

\[
e_5 = \frac{5}{288} - \frac{1}{120} \gamma - \frac{1}{20} \zeta(2) - \frac{11}{120} \zeta(3) - \frac{1}{20} \zeta(4).
\]

**3. Use of the method based upon Taylor's formula**

By applying Taylor's formula (see, e.g., [62, p. 209, Theorem 7.44]) to the function $h(t)$ given by

\[h(t) = t \log t \quad (t \in [k, k + x]),\]

up to its third derivative, Batir and Cancan [14] found that

\[
\log \left( 1 + \frac{x}{k} \right) = \frac{x}{k} - \frac{x^2}{2k^2} + \frac{x^3}{3(k + \mu(k))^3} \quad (0 < \mu(k) < x),
\]

where $\mu(k)$ can be easily expressed as follows:
\[
\mu(k) = \left[ \frac{3}{x^3} \log \left(1 + \frac{x}{k} \right) - \frac{3}{kx^2} - \frac{3}{2k^2x} \right]^{\frac{1}{2}} - k.
\] (3.2)

Batir and Cancan [14] proved that \( \mu(k) \) is strictly increasing for all \( k \geq 1 \) and \( x > 0 \) and that
\[
\mu(\infty) = \lim_{k \to \infty} \mu(k) = \frac{x}{4}.
\]

By using these observations, Batir and Cancan [14] derived a two-sided inequality for the double Gamma function \( \Gamma = 1/\Gamma_2 \).

More generally, by applying Taylor's formula to the same function as above to get the remainder on \([k, k + x]\) of the \( n \)th derivative \((n \in \mathbb{N})\), we obtain
\[
\log \left(1 + \frac{x}{k} \right) = \sum_{j=1}^{n-1} \frac{(-1)^{j-1}}{j} \left(\frac{x}{k} \right)^j + \frac{(-1)^{n-1}}{n} \left( \frac{x}{k + \mu_n(k;x)} \right)^n
\]
\[
\left( k \geq 1; \; x > 0; \; n \in \mathbb{N}; \; 0 < \mu_n(k;x) < x \right),
\]
which shows that \( \mu_n(k;x) \) can be written in the following explicit form:
\[
\mu_n(k;x) = (g_n(k;x))^{\frac{1}{n}} - k \quad (k \geq 1; \; x > 0; \; n \in \mathbb{N}),
\]
where, for convenience,
\[
g_n(k;x) := \frac{(-1)^{n-1} n}{x^n} \left[ \log \left(1 + \frac{x}{k} \right) - \sum_{j=1}^{n-1} \frac{(-1)^j}{j} \left( \frac{x}{k} \right)^j \right].
\]

We now provide some properties of \( \mu_n(k;x) \) as asserted by the following lemmas.

**Lemma 1.** The following formula holds true:
\[
\lim_{k \to \infty} \mu_n(k;x) = \frac{x}{n+1} \quad (x > 0; \; n \in \mathbb{N}).
\]

**Proof.** In view of the Taylor–Maclaurin series expansion of \( \log(1 + t) \), we find, for any bounded real number \( x \), that
\[
\log \left(1 + \frac{x}{k} \right) - \sum_{j=1}^{n-1} \frac{(-1)^{j-1}}{j} \left(\frac{x}{k} \right)^j = \sum_{j=n}^{\infty} \frac{(-1)^{j-1}}{j} \left(\frac{x}{k} \right)^j = \frac{(-1)^{n-1}}{n} \frac{x^n}{k^n} + \frac{(-1)^n x^{n+1}}{n+1 k^{n+1}} + O \left( \frac{1}{k^{n+2}} \right) \quad (k \to \infty),
\]
so that
\[
\mu_n(k;x) = \left[ \frac{1}{k^n} - \frac{nx}{n+1} \frac{1}{k^{n+1}} + O \left( \frac{1}{k^{n+2}} \right) \right]^{\frac{1}{n}} - k = k \left[ 1 - \frac{nx}{n+1} \frac{1}{k} + O \left( \frac{1}{k^2} \right) \right]^{\frac{1}{n}} - k \quad (k \to \infty).
\]

Using the generalized binomial theorem, we get
\[
\mu_n(k;x) = k \sum_{j=0}^{\infty} \left( \frac{\frac{1}{n}}{j} \right) \left[ \frac{nx}{n+1} \frac{1}{k} + O \left( \frac{1}{k^2} \right) \right]^j - k,
\]
where the generalized binomial coefficient \( \binom{\alpha}{j} \) is defined, for any \( \alpha \in \mathbb{C} \), by
\[
\binom{\alpha}{j} = \begin{cases} 
1 & (j = 0)
\frac{\alpha(\alpha-1) \cdots (\alpha-j+1)}{j!} & (j \in \mathbb{N}).
\end{cases}
\]

We, therefore, find that
\[
\mu_n(k;x) = k \left[ 1 + \frac{x}{n+1} \frac{1}{k} + O \left( \frac{1}{k^2} \right) \right] - k = \frac{x}{n+1} + O \left( \frac{1}{k} \right) \quad (k \to \infty).
\]

Finally, by taking the limit of the last identity as \( k \to \infty \), we complete our proof of **Lemma 1.** □

Our demonstration of **Lemma 3** is based upon Bernoulli’s inequality which, for the sake of ready reference, we recall here as **Lemma 2** below.

**Lemma 2** (Bernoulli’s Inequality). Let \( \alpha \) be a positive real number and \( \delta \geq -1 \). If \( 0 < \alpha \leq 1 \), then \( (1 + \delta)^\alpha \leq 1 + \alpha \delta \). Furthermore, if \( \alpha \geq 1 \), then \( (1 + \delta)^\alpha \geq 1 + \alpha \delta \). The equality holds true only when \( \alpha = 1 \) if \( \delta > 0 \).
**Lemma 3.** The function $\mu_n(k; x)$ is strictly increasing on $k \in [1, \infty)$ and $x > 0$ for all $n \in \mathbb{N}$.

**Proof.** In view of Eq. (3.5), by considering the sum formula of a finite geometric series, we get

$$\frac{\partial}{\partial k} \{g_n(k; x)\} = - \frac{n}{k^2 (k + x)} \quad (n \in \mathbb{N}),$$

which readily yields

$$\frac{\partial}{\partial k} \{\mu_n(k; x)\} = \frac{1}{k^n (k + x)} \{g_n(k; x)\} \frac{n^\alpha}{x^\alpha} - n \quad (n \in \mathbb{N}).$$

It is thus observed that the function $\mu_n(k; x)$ is strictly increasing on $k \in [1, \infty)$ and $x > 0$ if and only if

$$\frac{\partial}{\partial k} \{\mu_n(k; x)\} > 0 \quad (k \in [1, \infty); \ x > 0),$$

that is, if and only if

$$\left\{ (-1)^n \left( \frac{k}{x} \right)^n \left[ \log \left( 1 + \frac{x}{k} \right) - \sum_{j=1}^{n-1} \frac{(-1)^j}{j} \left( \frac{x}{k} \right)^j \right] \right\}^{n-1} < \frac{1}{(1 + \frac{k}{x})^n},$$

(3.7)

$$\left( k \in [1, \infty) \quad \text{and} \quad x > 0 \right).$$

Here, by setting $\frac{1}{x} = t$ in Eq. (3.7), it is seen that the function $\mu_n(k; x)$ is strictly increasing on $k \in [1, \infty)$ and $x > 0$ if and only if

$$H_n(t) := |h_n(t)|^{n-1} - \frac{t^n (n-1)}{n!} (1 + t)^{-n} < 0 \quad (t > 0; \ n \in \mathbb{N}),$$

(3.8)

where, for convenience,

$$h_n(t) := (-1)^{n-1} \left( \log(1 + t) - \sum_{j=1}^{n-1} \frac{(-1)^j}{j} t^j \right) \quad (t > 0).$$

We note that

$$h_n'(t) = \frac{t^{n-1}}{1 + t} \quad (t > 0; \ n \in \mathbb{N}).$$

It is observed that, for the inequality in Eq. (3.8), it is equivalent to prove that

$$G_n(t) := h_n(t) - \frac{t^n}{n (1 + t)^{n+1}} < 0 \quad (t > 0; \ n \in \mathbb{N}).$$

(3.9)

We now want to prove the following inequality:

$$G_n(t) = \frac{t^{n-1}}{1 + t} + \frac{t^n}{(n + 1) (1 + t)^{1 + \frac{1}{n}}} - \frac{t^{n-1}}{(1 + t)^{\frac{1}{n}}} < 0 \quad (t > 0; \ n \in \mathbb{N}),$$

if and only if

$$1 + \frac{t}{(n + 1) (1 + t)^{\frac{1}{n}}} - (1 + t)^{\frac{1}{n}} < 0 \quad (t > 0; \ n \in \mathbb{N}),$$

that is, if and only if

$$1 + \frac{t}{(n + 1) (1 + t)^{\frac{1}{n}}} < 1 + t \quad (t > 0; \ n \in \mathbb{N}),$$

that is, if and only if

$$(1 + t)^{\frac{1}{n}} < 1 + \frac{n}{n + 1} t \quad (t > 0; \ n \in \mathbb{N}),$$

the last inequality of which holds true by means of the first part of the Bernoulli inequality in Lemma 2. Thus $G_n(t)$ is strictly decreasing on $t \in (0, \infty)$ for all $n \in \mathbb{N}$, and so we have

$$G_n(t) < G_n(0) = 0 \quad (t > 0; \ n \in \mathbb{N}).$$

This completes the proof of the inequality in Eq. (3.9).  \qed
From Lemmas 1 and 3, it is easy to deduce Lemma 4 below.

**Lemma 4.** The following two-sided inequality holds true:

\[
\mu_n(1; x) \leq \mu_n(k; x) < \frac{x}{n+1} \quad (k \in [1, \infty); \ x > 0; \ n \in \mathbb{N}).
\]  

(3.10)

*For our convenience and later use, we define the following two functions:*

\[
\alpha_n(x) := 1 + \frac{x}{n+1} \quad \text{and} \quad \beta_n(x) := 1 + \mu_n(1; x) \quad (x > 0; \ n \in \mathbb{N}).
\]  

(3.11)

We are now ready to present our proposed two-sided inequalities for the multiple Gamma functions \( \Gamma_n(1 + x) \) \((n = 2, \ldots, 5; \ x > 0)\).

**Theorem 4.** Each of the following two-sided inequalities holds true:

\[
(2\pi)^{-\frac{1}{2}} \exp[p_2(x)] \exp\left(-\frac{x^3}{3} q_{32}(x)\right) \leq \Gamma_2(1 + x) \leq (2\pi)^{-\frac{1}{2}} \exp[p_2(x)] \exp\left(-\frac{x^3}{3} q_{32}(x)\right).
\]  

where

\[
p_2(x) := \frac{x}{2} + \frac{1}{2} (\gamma + 1)x^2,
\]

\[
q_{32}(x) := \zeta\left(2, \alpha_3(x)\right) + [1 - \alpha_3(x)] \zeta\left(3, \alpha_3(x)\right),
\]

\[
q_{33}(x) := \zeta\left(2, \beta_3(x)\right) + [1 - \beta_3(x)] \zeta\left(3, \beta_3(x)\right);
\]

\[
\exp[p_3(x)] \exp\left(\frac{x^3}{8} q_{33}(x)\right) \leq \Gamma_3(1 + x) \leq \exp[p_3(x)] \exp\left(\frac{x^3}{8} q_{33}(x)\right),
\]  

where

\[
p_3(x) := c_1 x + c_2 x^2 + c_3 x^3,
\]

\[
q_{33}(x) := \zeta\left(2, \alpha_3(x)\right) + [1 - 2\alpha_3(x)] \zeta\left(3, \alpha_3(x)\right) + [\alpha_2(x) - \alpha_3(x)] \zeta\left(4, \alpha_3(x)\right),
\]

\[
q_{33}(x) := \zeta\left(2, \beta_3(x)\right) + [1 - 2\beta_3(x)] \zeta\left(3, \beta_3(x)\right) + [\beta_2(x) - \beta_3(x)] \zeta\left(4, \beta_3(x)\right);
\]

\[
\exp[p_4(x)] \exp\left(-\frac{x^3}{30} q_{33}(x)\right) \leq \Gamma_4(1 + x) \leq \exp[p_4(x)] \exp\left(-\frac{x^3}{30} q_{33}(x)\right),
\]  

where

\[
p_4(x) := d_1 x + d_2 x^2 + d_3 x^3 + d_4 x^4,
\]

\[
q_{34}(x) := \zeta\left(2, \alpha_3(x)\right) + [6 - 3\alpha_3(x)] \zeta\left(3, \alpha_3(x)\right) + [11 - 12\alpha_3(x) + 3\alpha_2^2(x)] \zeta\left(4, \alpha_3(x)\right)
\]

\[+ [1 - \alpha_3(x)][2 - \alpha_3(x)][3 - \alpha_3(x)] \zeta\left(5, \alpha_3(x)\right),
\]

\[
q_{34}(x) := \zeta\left(2, \beta_3(x)\right) + [6 - 3\beta_3(x)] \zeta\left(3, \beta_3(x)\right) + [11 - 12\beta_3(x) + 3\beta_2^2(x)] \zeta\left(4, \beta_3(x)\right)
\]

\[+ [1 - \beta_3(x)][2 - \beta_3(x)][3 - \beta_3(x)] \zeta\left(5, \beta_3(x)\right);
\]

\[
\exp[p_5(x)] \exp\left(\frac{x^6}{144} q_{33}(x)\right) \leq \Gamma_5(1 + x) \leq \exp[p_5(x)] \exp\left(\frac{x^6}{144} q_{33}(x)\right),
\]  

where

\[
p_5(x) := e_1 x + e_2 x^2 + e_3 x^3 + e_4 x^4 + e_5 x^5,
\]
\[ q_{a_6}(x) = \zeta(2, a_6(x)) + [10 - 4a_6(x)] \zeta(3, a_6(x)) + [35 - 30a_6(x) + 6a_6^2(x)] \zeta(4, a_6(x)) \]
\[ + [50 - 70a_6(x) + 30a_6^2(x) - 4a_6^3(x)] \zeta(5, a_6(x)) + [1 - a_6(x)][2 - a_6(x)][3 - a_6(x)][4 - a_6(x)] \zeta(5, a_6(x)). \]

\[ q_{b_6}(x) = \zeta(2, b_6(x)) + [10 - 4b_6(x)] \zeta(3, b_6(x)) + [35 - 30b_6(x) + 6b_6^2(x)] \zeta(4, b_6(x)) \]
\[ + [50 - 70b_6(x) + 30b_6^2(x) - 4b_6^3(x)] \zeta(5, b_6(x)) + [1 - b_6(x)][2 - b_6(x)][3 - b_6(x)][4 - b_6(x)] \zeta(5, b_6(x)). \]

**Proof.** Upon taking logarithms on each of the explicit Weierstrass canonical product forms Eqs. (1.3), (2.14), (2.15) and (2.16) of the multiple Gamma functions and applying the result in Lemma 4, if we make a little simplification, we are led to the inequalities asserted by Theorem 4. □

**Remark 3.** In view of the relation Eq. (1.13), the Hurwitz (or generalized) Zeta function \( \zeta(s, a) \) in Theorem 4 can be replaced by the Polygamma functions \( \psi^{(n)}(z) \) as it was done in [14].

### 4. Use of series involving the zeta functions

Ever since the Goldbach theorem of 1729 (see [53]; see also [57, p. 142]), closed-form evaluations of series involving the Zeta functions have not only attracted many important investigations (see, e.g., [7,21,24–26,33,53,55,57]), but have also found their applications in various ways (see, e.g., [21,25,30]). Our very recent paper [32] deals extensively with the familiar family of the Goldbach–Euler series; in fact, it was motivated essentially by the various developments emerging from the aforementioned Goldbach theorem of 1729.

In light of Eq. (1.10), if we take logarithms of both sides in Eqs. (1.3), (2.14), (2.15) and (2.16) and apply the Taylor-Maclaurin expansion of \( \log(1 + t) \) to each of the resulting identities, we find the series representations given by Theorem 5 below.

**Theorem 5.** Each of the following series representations holds true:

\[
\log \Gamma_2(1 + z) = c_{2.1} z + c_{2.2} z^2 + \sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k+1} \zeta(k) z^{k+1};
\]

\[
\log \Gamma_3(1 + z) = c_{3.1} z + c_{3.2} z^2 + c_{3.3} z^3 + \frac{1}{2} \sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k+1} \zeta(k) z^{k+1} + \frac{1}{2} \sum_{k=2}^{\infty} \frac{(-1)^k}{k+2} \zeta(k) z^{k+2};
\]

\[
\log \Gamma_4(1 + z) = c_{4.1} z + c_{4.2} z^2 + c_{4.3} z^3 + c_{4.4} z^4 + \frac{1}{3} \sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k+1} \zeta(k) z^{k+1} + \frac{1}{2} \sum_{k=2}^{\infty} \frac{(-1)^k}{k+2} \zeta(k) z^{k+2} + \frac{1}{6} \sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k+3} \zeta(k) z^{k+3};
\]

\[
\log \Gamma_5(1 + z) = c_{5.1} z + c_{5.2} z^2 + c_{5.3} z^3 + c_{5.4} z^4 + c_{5.5} z^5 + \frac{1}{4} \sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k+1} \zeta(k) z^{k+1} + \frac{11}{24} \sum_{k=2}^{\infty} \frac{(-1)^k}{k+2} \zeta(k) z^{k+2} + \frac{1}{4} \sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k+3} \zeta(k) z^{k+3} + \frac{1}{24} \sum_{k=2}^{\infty} \frac{(-1)^k}{k+4} \zeta(k) z^{k+4},
\]

where

\[ c_{2.1} = \frac{1}{2} \log(2\pi); \quad c_{2.2} = \frac{1}{2} \gamma; \quad c_{3.1} = \frac{3}{8} \frac{1}{4} \log(2\pi) - \log A; \]

\[ c_{3.2} = \frac{1}{8} \log(2\pi) + \frac{\gamma}{4}; \quad c_{3.3} = -\frac{1}{4} - \frac{\gamma}{6}; \]

\[ c_{4.1} = \frac{7}{24} - \frac{1}{6} \log(2\pi) \log A - \frac{1}{2} \log B; \quad c_{4.2} = -\frac{1}{144} + \frac{\gamma}{6} + \frac{1}{4} \log(2\pi) + \frac{1}{2} \log A; \]

\[ c_{5.1} = \frac{1}{8} \log(2\pi) + \frac{\gamma}{4}; \quad c_{5.2} = -\frac{1}{8} \log A; \quad c_{5.3} = -\frac{1}{144} + \frac{\gamma}{6} + \frac{1}{4} \log(2\pi) + \frac{1}{2} \log A; \]
\[ c_{43} = -\frac{2}{9} - \frac{1}{12} \log(2\pi) - \frac{\gamma}{6}; \quad c_{44} = \frac{11}{144} + \frac{\gamma}{24}; \]

\[ c_{51} = e; \quad c_{52} = e; \quad c_{53} = -\frac{149}{864} - \frac{11}{72} \gamma - \frac{1}{8} \log(2\pi) - \frac{1}{6} \log A; \]

\[ c_{54} = \frac{7}{64} + \frac{\gamma}{16} + \frac{1}{48} \log(2\pi); \quad c_{55} = -\frac{5}{288} + \frac{\gamma}{120}. \]

We next recall here the well-known property of an alternating series of real numbers.

**Lemma 5.** Let \( \{a_k\}_{k=1}^\infty \) be a decreasing sequence of positive real numbers with \( a_k \downarrow 0 \) as \( k \to \infty \). Then the following inequalities hold true:

\[ \sum_{k=1}^{2n} (-1)^{k+1} a_k < \sum_{k=1}^{\infty} (-1)^{k+1} a_k < \sum_{k=1}^{2n-1} (-1)^{k+1} a_k \quad (n \in \mathbb{N}). \]  

By applying Lemma 5 to each of the formulas in Theorem 5, we obtain certain interesting inequalities for the multiple Gamma functions \( \Gamma_n(1+x) \) (\( n = 2, 3, 4, 5 \)).

**Theorem 6.** Each of the following inequalities holds true:

\[ \exp \left( c_{5,1} x + c_{2,2} x^2 + \sum_{k=2}^{2n} \frac{(-1)^{k+1}}{k+1} \zeta(k) x^{k+1} \right) < \Gamma_2(1+x) < \exp \left( c_{3,1} x + c_{2,2} x^2 + \sum_{k=2}^{2n-1} \frac{(-1)^{k+1}}{k+1} \zeta(k) x^{k+1} \right) (0 < x \leq 1; \; n \in \mathbb{N}); \]

\[ \exp \left( c_{5,1} x + c_{3,2} x^2 + c_{3,3} x^3 \right) \cdot \exp \left( \frac{1}{2} \sum_{k=2}^{2n} \frac{(-1)^{k+1}}{k+1} \zeta(k) x^{k+1} + \frac{1}{2} \sum_{k=2}^{2n-1} \frac{(-1)^{k}}{k+2} \zeta(k) x^{k+2} \right) < \Gamma_3(1+x) < \exp \left( c_{3,1} x + c_{3,2} x^2 + c_{3,3} x^3 \right) \cdot \exp \left( \frac{1}{2} \sum_{k=2}^{2n-1} \frac{(-1)^{k+1}}{k+1} \zeta(k) x^{k+1} + \frac{1}{2} \sum_{k=2}^{2n-1} \frac{(-1)^{k}}{k+2} \zeta(k) x^{k+2} \right) (0 < x \leq 1; \; n \in \mathbb{N}); \]

\[ \exp \left( c_{4,1} x + c_{4,2} x^2 + c_{4,3} x^3 + c_{4,4} x^4 + \frac{1}{3} \sum_{k=2}^{2n} \frac{(-1)^{k+1}}{k+1} \zeta(k) x^{k+1} \right) \cdot \exp \left( \frac{1}{2} \sum_{k=2}^{2n} \frac{(-1)^{k}}{k+2} \zeta(k) x^{k+2} + \frac{1}{6} \sum_{k=2}^{2n} \frac{(-1)^{k+1}}{k+3} \zeta(k) x^{k+3} \right) < \Gamma_4(1+x) < \exp \left( c_{4,1} x + c_{4,2} x^2 + c_{4,3} x^3 + c_{4,4} x^4 + \frac{1}{3} \sum_{k=2}^{2n} \frac{(-1)^{k+1}}{k+1} \zeta(k) x^{k+1} + \frac{1}{6} \sum_{k=2}^{2n-1} \frac{(-1)^{k+1}}{k+3} \zeta(k) x^{k+3} \right) (0 < x \leq 1; \; n \in \mathbb{N}); \]

\[ \exp \left( c_{5,1} x + c_{5,2} x^2 + c_{5,3} x^3 + c_{5,4} x^4 + c_{5,5} x^5 \right) \cdot \exp \left( \frac{1}{4} \sum_{k=2}^{2n} \frac{(-1)^{k+1}}{k+1} \zeta(k) x^{k+1} + \frac{11}{24} \sum_{k=2}^{2n} \frac{(-1)^{k}}{k+2} \zeta(k) x^{k+2} \right) \cdot \exp \left( \frac{1}{4} \sum_{k=2}^{2n} \frac{(-1)^{k+1}}{k+3} \zeta(k) x^{k+3} + \frac{11}{24} \sum_{k=2}^{2n} \frac{(-1)^{k}}{k+4} \zeta(k) x^{k+4} \right) < \Gamma_5(1+x) < \exp \left( c_{5,1} x + c_{5,2} x^2 + c_{5,3} x^3 + c_{5,4} x^4 + c_{5,5} x^5 \right) \cdot \exp \left( \frac{1}{4} \sum_{k=2}^{2n} \frac{(-1)^{k+1}}{k+1} \zeta(k) x^{k+1} + \frac{11}{24} \sum_{k=2}^{2n} \frac{(-1)^{k}}{k+2} \zeta(k) x^{k+2} \right) \cdot \exp \left( \frac{1}{4} \sum_{k=2}^{2n} \frac{(-1)^{k+1}}{k+3} \zeta(k) x^{k+3} + \frac{11}{24} \sum_{k=2}^{2n} \frac{(-1)^{k}}{k+4} \zeta(k) x^{k+4} \right) (0 < x \leq 1; \; n \in \mathbb{N}), \]

where the c's are as in Theorem 4.
Remark 4. The range $0 < x \leq 1$ in Theorem 5 may be extended to $x \in (1, \infty)$ by using the fundamental functional relation (a) in Theorem 2.

Upon setting $n = 1$ in each of the inequalities in Theorem 5, we are led to the following result.

Corollary 2. Each of the following inequalities holds true:

$$\exp\left(c_{21}x + c_{22}x^2 - \frac{\pi^2}{18}x^3\right) < \Gamma_2(1 + x) < \exp\left(c_{21}x + c_{22}x^2\right) \quad (0 < x \leq 1);$$

$$\exp\left[c_{31}x + c_{32}x^2 - \left(\frac{1}{4} + \frac{\pi^2}{36} + \frac{\gamma}{6}\right)x^3\right] < \Gamma_3(1 + x) < \exp\left(c_{31}x + c_{32}x^2 + c_{33}x^3 + \frac{\pi^2}{48}x^4\right) \quad (0 < x \leq 1);$$

$$\exp\left[c_{41}x + c_{42}x^2 - \left(\frac{2}{9} + \frac{\pi^2}{54} + \frac{1}{12}\log(2\pi) + \frac{\gamma}{6}\right)x^3 + c_{44}x^4 - \frac{\pi^2}{180}x^5\right] < \Gamma_4(1 + x) < \exp\left[c_{41}x + c_{42}x^2 + c_{43}x^3 + \left(\frac{11}{144} + \frac{\pi^2}{48} + \frac{\gamma}{24}\right)x^4\right] \quad (0 < x \leq 1);$$

$$\exp\left[c_{51}x + c_{52}x^2 + c_{53}x^3 + c_{54}x^4 + c_{55}x^5 - \left(\frac{7}{2}\right)x^6\right] < \Gamma_5(1 + x) < \exp\left[c_{51}x + c_{52}x^2 + c_{53}x^3 + \left(\frac{5}{576} + \frac{\pi^2}{864}\right)x^4\right] \quad (0 < x \leq 1).$$

5. Concluding remarks and observations

In our present investigation, we have employed and extended a method based upon Taylor’s formula, which was initiated recently by Batir and Cancan [14], in order to prove several two-sided inequalities for the multiple Gamma functions $\Gamma_n (n = 2, 3, 4, 5)$. We have expressed $\log \Gamma_n (1 + x)$ as series involving the Zeta functions. We have also given a more convenient and potentially useful explicit form of the multiple Gamma functions $\Gamma_n (n \in \mathbb{N})$.

By using Theorem 3 in conjunction with Eq. (2.13), it is not difficult to find explicit Weierstrass canonical product forms of the multiple Gamma functions $\Gamma_n (n \geq 6)$, corresponding to those which we have already derived in Section 3 and Section 4, can also be derived by using Lemma 4 and the method developed in Section 4. We choose to leave these further extensions as an exercise for the interested reader.

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References


[36] J.W.L. Glaisher, On the product $\prod_{n=1}^{\infty} (1 + \frac{x}{n})$, Messenger Math. 7 (1877) 43–47.


[38] O. Hölder, Über eine Transcendente Funktion, Göttingen, Dieterichsche Verlags-Buchhandlung 1886 (1886) 514–522.


