Global Stability of a Recurrent Neural Network for Solving Pseudomonotone Variational Inequalities

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Abstract—Solving variational inequality problems by using neural networks are of great interest in recent years. To date, most work in this direction focuses on solving monotone variational inequalities. In this paper, we show that an existing recurrent neural network proposed originally for solving monotone variational inequalities can be used to solve pseudomonotone variational inequalities with proper choice of a system parameter. The global convergence, global asymptotic stability and global exponential stability of the neural network are discussed under various conditions (e.g., see [6], [7], and later in [8] in terms of convergence to and stability at the solutions of VI(Ω,F)) or the combination of pseudomonotonicity, a weaker condition than monotonicity, of F and some other suitable conditions. Moreover, our results do not require that the Jacobian of F is symmetric.

II. PRELIMINARIES

Consider a recurrent neural network as follows:
\[ \frac{dx}{dt} = \lambda(-x + P_\Omega(x - \alpha F(x)) + \alpha F(x) - \alpha F[P_\Omega(x - \alpha F(x))]), \]
where \( \lambda > 0 \) and \( \alpha > 0 \) are two scaling factors and \( P_\Omega : R^n \to \Omega \) is a projection operator defined by
\[ P_\Omega(x) = \arg \min_{y \in \Omega} ||x - y||. \]
It is proved in [6] that \( x^* \in \Omega \) is a solution to VI(Ω,F) in (1) if and only if it is an equilibrium point of the neural network in (2). The neural dynamic system (2) was first discussed in [6], and later in [7] in terms of convergence to and stability at the solutions of VI(Ω,F) with different monotonicity assumptions. We will show that the corresponding results are still valid when the neural network is employed to solve pseudomonotone variational inequalities (of course with some additional assumptions).

For the convenience of later discussion, it is necessary to introduce the following definition and lemma.

**Definition 1:** A function \( F : R^n \to R^n \) is said to be pseudomonotone on \( \Omega \) if, for every pair of distinct points \( x, y \in \Omega \),
\[ F(x)^T (y - x) \geq 0 \iff F(y)^T (y - x) \geq 0. \]
\( F \) is said to be strictly pseudomonotone on \( \Omega \) if, for every pair of distinct points \( x, y \in \Omega \),
\[ F(x)^T (y - x) \geq 0 \iff F(y)^T (y - x) > 0, \]
and strongly pseudomonotone on \( \Omega \) if there exists a constant \( \gamma > 0 \) such that for every pair of points \( x, y \in \Omega \),
\[ F(x)^T (y - x) \geq 0 \iff F(y)^T (y - x) \geq \gamma ||x - y||^2. \]
The above definitions of pseudomonotonicity are easily seen as listed in an order from weak to strong. Moreover, they generalize the definitions of various monotonicities that can be found in many literatures (e.g., [7], [9]). Clearly,
monotonicity implies pseudomononicity, strict monotonicity implies strict pseudomononicity and strong monotonicity implies strong pseudomononicity; but not vice versa.

The projection operator $P_{Ω}(\cdot)$ defined by (3) has the following properties.

**Lemma 1** ([1], pp. 9-10): For all $u \in \mathbb{R}^n$ and all $v \in Ω \subset \mathbb{R}^n$, we have

$$\langle P_{Ω}(u) - u \rangle = 0$$

and for all $u, v \in \mathbb{R}^n$ we have

$$\|P_{Ω}(u) - P_{Ω}(v)\| \leq \|u - v\|.$$  \hfill (6)

### III. Convergence and Stability Analysis

In this section, we present some basic properties of the dynamic system (2) and analyze its convergence and stability. For a complete analysis we first give some conditions on the existence of the solution to the ordinary differential equation (2).

**Theorem 1:** Assume that $F(\cdot)x$ is Lipschitz continuous in $Ω$ and locally Lipschitz continuous in $\mathbb{R}^n$, then for each $x_0 \in \mathbb{R}^n$ there exists a unique solution $x(t)$ of (2) with $x(t_0) = x_0$ over $[t_0, T)$, where $T \geq t_0$. If furthermore $F(x)$ satisfies

$$\|F(x)\| \leq k_1 + k_2\|x\|$$

then $T = \infty$, where $k_1, k_2$ are two constants.

**Proof:** Let

$$T(x) = \lambda(−x + P_{Ω}(x - αF(x)) + αF(x) − αF[P_{Ω}(x - αF(x))].$$

(7)

Since $F$ is locally Lipschitz continuous in $\mathbb{R}^n$, there exists a neighborhood $D_0 \subset \mathbb{R}^n$ for each point $x \in \mathbb{R}^n$ such that $\|F(x) - F(y)\| \leq L_0\|x - y\|$, for all $x, y \in D_0$, where $L_0$ denotes the local Lipschitz constant. Consider the same neighborhood $D_0$ for each point $x \in \mathbb{R}^n$. Since $F$ is Lipschitz continuous in $Ω$, one has

$$\|F[P_{Ω}(x - αF(x)) + F[P_{Ω}(y - αF(y))] \leq L\|P_{Ω}(x - αF(x)) - P_{Ω}(y - αF(y))\|.$$ 

where $L$ denotes the Lipschitz constant in $Ω$. Then,

$$\|T(x) - T(y)\| \leq \lambda\|x - y\| + \alpha\|F(x) - F(y)\|

+ \|P_{Ω}(x - αF(x)) - P_{Ω}(y - αF(y))\|

+ \alpha\|F[P_{Ω}(x - αF(x)) - F[P_{Ω}(y - αF(y))]\|

\leq \lambda(1 + \alpha L_0)\|x - y\|

+ (1 + \alpha L_0)\|P_{Ω}(x - αF(x)) - P_{Ω}(y - αF(y))\|

\leq \lambda(1 + \alpha L_0)\|x - y\|

+ (1 + \alpha L_0)\|x - y\| + \alpha(1 + \alpha L_0)\|F(x) - F(y)\|

\leq \lambda(2 + 2\alpha L_0 + \alpha^2 L_0)\|x - y\|.$$ 

In the above inequalities, we have used Lemma 1. Thus, $T(x)$ is locally Lipschitz continuous in $\mathbb{R}^n$. Therefore, there is a unique solution $x(t)$ of (2) over $[t_0, T)$ with the initial point $x_0$. Below we prove the second part of the theorem. In view of the fact that $dx/dt = T(x), \forall t_0 \leq t < T$, that is, $x(t) - x(t_0) = \int_{t_0}^{t} F(x(s))ds$, then

$$\|x(t)\| \leq \|x_0\| + \int_{t_0}^{t} \|F(x(s))\| ds$$

$$\leq \|x_0\| + k_1(t - t_0) + k_2\int_{t_0}^{t} \|x(s)\| ds.$$ 

According to the well known Gronwall Theorem [13], we have

$$\|x(t)\| \leq (\|x_0\| + k_1(t - t_0))\exp[k_2(t_0 - t)], \forall t \in [t_0, T).$$

Hence, $x(t)$ is bounded on $[t_0, T)$ which implies $T = \infty$. \hfill \square

We would like to remark that when $F$ is affine, i.e., $F(x) = Mx + q$ where $M \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$, all conditions in Theorem 1 are satisfied automatically. See the following corollary.

**Corollary 1:** For each $x_0 \in \mathbb{R}^n$, there exists a unique continuous solution $x(t)$ of (2) with $x(t_0) = x_0$ for all $t \geq t_0$.

**Proof:** Let

$$\begin{align*}
T(x) &= λ\{−x + P_{Ω}(x - α(Mx + q)) + \\
&\quad α(Mx + q) − α[M\{P_{Ω}(x - α(Mx + q) + q)\}].
\end{align*}$$

For any $x, y \in \mathbb{R}^n$,

$$\|T(x) - T(y)\| \leq λ\|x - y\| + α\|Mx - My\|

+ \|P_{Ω}(x - α(Mx + q)) - P_{Ω}(y - α(My + q))\|

+ α\|M[P_{Ω}(x - α(Mx + q)) - M[P_{Ω}(y - α(My + q))]\|

\leq λ(1 + α\|M\|)\|x - y\|

+ \|P_{Ω}(x - α(Mx + q)) - P_{Ω}(y - α(My + q))\|

\leq λ(1 + α\|M\|)\|x - y\| + α\|M\|\|x - y\|

\leq λ(1 + α\|M\|)(2 + α\|M\|)\|x - y\|,
$$

where we used Lemma 1. Thus $T(x)$ is Lipschitz continuous in $\mathbb{R}^n$. Furthermore,

$$\|T(x)\| \leq λx + P_{Ω}(x - α(Mx + q))

\leq λ(1 + α\|M\|)\|x - P_{Ω}(x - α(Mx + q))\|

\leq λ(1 + α\|M\|)(\|P_{Ω}(x∗) - x\| + \|P_{Ω}(x - α(Mx + q)) - P_{Ω}(x)\|)

\leq λ(1 + α\|M\|)(\|P_{Ω}(x∗) - x\| + \|x - x∗\|

+ α\|Mx + q\|

\leq λ(1 + α\|M\|)(\|P_{Ω}(x∗) - x\| + \|x - x∗\| + α\|q\|

\leq (2 + α\|M\|)\|x∗\|)

= k_1 + k_2\|x∗\|,$$

where $x∗ \in \mathbb{R}^n$ is a constant and

$$k_1 = λ(1 + α\|M\|)(\|P_{Ω}(x∗)\| + \|x∗\| + α\|q\|)$$

$$k_2 = λ(1 + α\|M\|)(2 + α\|M\|).$$

According to Theorem 1, there exists a unique continuous solution $x(t)$ with $x(t_0) = x_0$ over $[t_0, \infty)$. \hfill \square

The next two theorems are main results of this paper.

**Theorem 2:** Assume that $F(x)$ is Lipschitz continuous in $\mathbb{R}^n$ with constant $L$ and pseudomononic on $Ω$, then the neural network in (2) is globally convergent to the solution
set of (1) provided that \( \alpha < 1/L \). In particular, if (1) has a unique solution, the projection neural network is globally asymptotically stable.

**Proof:** Since \( F(x) \) is Lipschitz continuous in \( \mathbb{R}^n \), by Theorem 1, there is a unique continuous solution \( x(t) \) of (2) over \([t, T)\) for any initial point \( x(t_0) \in \mathbb{R}^n \). Let \([t, T)\) be the maximal interval of existence of the solution, we will show \( t \to T \). According to the Lyapunov theorem, there exists a convergent subsequence \( \{x_k\} \). Hence a unique solution, the projection neural network is globally asymptotically stable. \( \square \)

Consider the following Lyapunov function
\[
V(x) = \frac{1}{2} \|x - x^*\|^2,
\]
where \( x^* \) is a solution to (1). Its time derivative along the neural network system is
\[
\frac{dV}{dt} = (x - x^*)^T T(x) \leq -\lambda(1 - \alpha L) \|x - x^*\|^2 \leq 0,
\]
since \( \alpha < 1/L \). It follows that
\[
\|x(t) - x^*\|^2 \leq \|x_0 - x^*\|^2, \quad \forall t \geq t_0.
\]
Hence \( x(t) \) is bounded and \( T = \infty \). Note \( dV/dt = 0 \) if and only if \( P\alpha(x - \alpha F(x)) = x \) which implies \( x = x^* \) [1].

According to the Lyapunov theorem, there exists a convergent subsequence \( \{x(t_k)\} \) such that
\[
\lim_{k \to \infty} x(t_k) = \hat{x},
\]
where \( \hat{x} \in \Omega^* \) and \( \Omega^* \) is the solution set of (1). Finally, define a Lyapunov function again,
\[
\dot{V}(x) = \frac{1}{2} \|\dot{x} - x^*\|^2.
\]
It is easy to see that \( \dot{V}(x) \) decreases along the trajectory of (2) and satisfies \( \dot{V}(\hat{x}) = 0 \). Therefore, for any \( \varepsilon > 0 \), there exists \( q > 0 \) such that, for all \( t \geq t_q \),
\[
\|x(t) - \hat{x}\| \leq \dot{V}(x(t)) \leq \dot{\hat{V}}(x(t_q)) < \varepsilon,
\]
and thus \( \lim_{t \to \infty} \|x(t) - \hat{x}\| = 0 \). Therefore,
\[
\lim_{t \to \infty} x(t) = \hat{x}.
\]
It is to say, the neural network in (2) is globally convergent to a solution of (1). In particular, if (1) has a unique solution, the projection neural network is globally asymptotically stable. \( \square \)

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\|x(t) - x^*\|^2 \leq \|x_0 - x^*\|^2, \quad \forall t \geq t_0.
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Hence \( x(t) \) is bounded and \( T = \infty \). Note \( dV/dt = 0 \) if and only if \( P\alpha(x - \alpha F(x)) = x \) which implies \( x = x^* \) [1].

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\]
and thus \( \lim_{t \to \infty} \|x(t) - \hat{x}\| = 0 \). Therefore,
\[
\lim_{t \to \infty} x(t) = \hat{x}.
\]

**Theorem 3:** Assume that \( F(x) \) is Lipschitz continuous in \( \mathbb{R}^n \) with constant \( L \) and strongly pseudomonotone on \( \Omega \) with constant \( \gamma \), where \( \gamma > 4L \), then the neural network in (2) is globally exponentially stable. Naturally, one should address himself to the question: does the global exponential stability of the neural network still hold if strong monotonicity of \( F \) is replaced by strong pseudomonotonicity? We have made several attempts in this direction but gain no success. However, we found that the global exponential stability can be ensured by some additional conditions.

**Proof:** Since \( F(x) \) is strongly pseudomonotone, there is only one solution \( x^* \) to \( VI(\Omega) \). Let \( x = P\alpha(x - \alpha F(x)) \) and \( x^* = x^* \) in (1), then
\[
F(x^*)^T (P\alpha(x - \alpha F(x)) - x^*) \geq 0.
\]
By the strong pseudomonotonicity of \( F(x) \), we have
\[
F(P\alpha(x - \alpha F(x)))^T (P\alpha(x - \alpha F(x)) - x^*) \geq \gamma \|P\alpha(x - \alpha F(x)) - x^*\|^2,
\]
Let \( u = x - \alpha F(x) \) and \( v = x^* \) in (6), then
\[
(P\alpha(x - \alpha F(x)) - x + \alpha F(x))^T (x^* - P\alpha(x - \alpha F(x))) \geq 0.
\]
Adding the two resulting inequalities yields
\[
\geq \gamma \|P\alpha(x - \alpha F(x)) - x^*\|^2,
\]
which follows
\[
\|x^* - x + \alpha F(x) - \alpha F(x^*)\|^2 \geq \gamma \|P\alpha(x - \alpha F(x)) - x^*\|^2,
\]
\[
= \gamma \|P\alpha(x - \alpha F(x)) - x^*\|^2.
\]
and

\[
(1 + \alpha \gamma)\|P_{\Omega}(x - \alpha F(x)) - x^*\|^2 \\
\leq [x^* - x + \alpha F(x) - \alpha F(x^*)]^T [x^* - P_{\Omega}(x - \alpha F(x))] \\
+ \alpha \|F(x^*) - F(P_{\Omega}(x - \alpha F(x)))\|^2 [x^* - P_{\Omega}(x - \alpha F(x))] \\
\leq [x^* - x + \alpha F(x) - \alpha F(x^*)]^T [x^* - P_{\Omega}(x - \alpha F(x))] \\
+ \alpha L \|P_{\Omega}(x - \alpha F(x)) - x^*\|^2, 
\]

where the last inequality follows from the fact that \( F \) is Lipschitz continuous. Then,

\[
(1 + \alpha (\gamma - L))\|P_{\Omega}(x - \alpha F(x)) - x^*\|^2 \\
\leq [x^* - x + \alpha F(x) - \alpha F(x^*)] [x^* - P_{\Omega}(x - \alpha F(x))] \\
\leq \|x^* - x + \alpha F(x) - \alpha F(x^*)\| \|x^* - P_{\Omega}(x - \alpha F(x))\|. 
\]

By noting that \((1 + \alpha (\gamma - L)) > 0\), we have

\[
\|P_{\Omega}(x - \alpha F(x)) - x^*\| \\
\leq \frac{1}{1 + \alpha (\gamma - L)} \|x^* - x + \alpha F(x) - \alpha F(x^*)\| \\
\leq \frac{1}{1 + \alpha (\gamma - L)} [\|x^* - x\| + \alpha \|F(x) - F(x^*)\|], 
\]

(8)

Consider the function

\[
V = \frac{1}{2} \|x - x^*\|^2, 
\]

and its time derivative

\[
\frac{dV}{dt} = (x - x^*)^T \frac{dx}{dt} \\
= (x - x^*)^T \lambda \{-x + P_{\Omega}(x - \alpha F(x)) + \alpha F(x) \\
- \alpha F(P_{\Omega}(x - \alpha F(x)))\} \\
= \lambda (x - x^*)^T \{P_{\Omega}(x - \alpha F(x)) - x^* + x^* - x + \alpha F(x) - \alpha F(x^*) + \alpha F(x^*) - \alpha F(P_{\Omega}(x - \alpha F(x)))\} \\
= \lambda (x - x^*)^T \{P_{\Omega}(x - \alpha F(x)) - x^*\} - \|x - x^*\|^2 \\
+ \alpha (x - x^*)^T (F(x) - F(x^*)) \\
+ \alpha (x - x^*)^T (F(x^*) - F(P_{\Omega}(x - \alpha F(x)))) \\
\leq \lambda (1 + \alpha L) \|x - x^*\| \|P_{\Omega}(x - \alpha F(x)) - x^*\| \\
+ (\alpha L - 1) \|x - x^*\|^2. 
\]

Substituting (8) into above yields

\[
\frac{dV}{dt} \leq -\lambda \alpha (\gamma - 4L - \alpha \gamma L) \|x - x^*\|^2. 
\]

Let

\[
k = \frac{\lambda \alpha (\gamma - 4L - \alpha \gamma L)}{1 + \alpha \gamma - \alpha L}. 
\]

Taking into account \( \gamma > 4L \) and \( \alpha < (\gamma - 4L)/\gamma L \), we have \( k > 0 \) and

\[
\|x(t) - x^*\| \leq \|x(t_0) - x^*\| e^{-k(t-t_0)} \quad \forall t \geq t_0. 
\]

Hence the neural network in (2) is globally exponentially stable.

IV. CONCLUDING REMARKS

This paper extends the stability results of a recurrent neural network proposed by Xia and Wang in 1998 [6] and 2000 [7] for solving monotone variational inequalities to pseudomonotone variational inequalities. It is shown that the global convergence, global asymptotic stability and global exponential stability of the neural network can be ensured by Lipschitz condition and various pseudomonotonicity conditions with proper choice of a scaling factor. The convergence and stability results do not require the monotonicity and the symmetry conditions on the nonlinear mapping in the variational inequality that are often crucial to assure convergence and stability properties for many neural networks. In addition, we would like to address that, as pseudomonotone mappings are closely related to pseudoconvex functions [14], another significant application of the neural network discussed in this paper is to solve pseudoconvex optimization problems besides solving pseudomonotone variational inequalities.

REFERENCES