Letter to the Editor

A DISCRETE-TIME LAGRANGIAN NETWORK FOR SOLVING CONSTRAINED QUADRATIC PROGRAMS

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A discrete-time recurrent neural network which is called the discrete-time Lagrangian network is proposed in this letter for solving convex quadratic programs. It is developed based on the classical Lagrange optimization method and solves quadratic programs without using any penalty parameter. The condition for the neural network to globally converge to the optimal solution of the quadratic program is given. Simulation results are presented to illustrate its performance.

1. Introduction
Consider the quadratic programming problem in compact form

\[
\begin{aligned}
\text{minimize} & \quad \frac{1}{2} x^T Q x + c^T x \\
\text{subject to} & \quad A x = b
\end{aligned}
\] (1)

where \( x \in \mathbb{R}^n \) is the decision variable, \( c \in \mathbb{R}^n \), \( b \in \mathbb{R}^m \), \( Q \in \mathbb{R}^{n \times n} \) is a positive definite matrix and \( A \in \mathbb{R}^{m \times n} \) with \( m < n \) and full rank. This optimization problem has a unique optimal solution because the objective function of (1) is strictly convex and it is referred to as convex quadratic program.

Since Hopfield and Tank’s seminal work, many neural networks have been developed for solving optimization problems, e.g., Refs. 2–7. In particular, Wang and Li\(^5\) presented a continuous-time recurrent neural network developed from the classical Lagrange optimization method to solve convex quadratic programs like (1). Unlike other penalty parameter based neural networks for solving optimization problems where very large penalty parameter has to be employed to ensure an accurate solution, Refs. 2–4, their model is asymptotically convergent to the optimal solution of the desired quadratic program without using penalty parameter. Wang et al.\(^6\) and Tang and Wang\(^7\) extended this Lagrangian network to solve dynamic problems, i.e., the matrix \( Q \) in (1) is time-variant, and then applied it to real-time kinematic and dynamic control of redundant manipulators, respectively. However, in view of the availability of the digital hardware and the compatibility to the digital computers, the discrete-time version is more desirable in practical implementation. In this letter, we propose the discrete-time counterpart of their network and give the condition for the discrete-time recurrent neural network with global convergence.

This letter is organized as follows. Section 2 first reviews the development of the continuous-time Lagrangian networks and followed by the dynamical equations of the discrete-time Lagrangian network which are obtained through discretization of its continuous-time counterpart. The global convergence
analysis and the condition for ensuring the global convergence of the proposed discrete-time network are given in Sec. 3. Computer simulations are shown and discussed in Sec. 4. Section 5 concludes the letter with final remarks.

2. Network Descriptions

In this section, we first review the continuous-time Lagrangian network introduced in Ref. 5, which is represented by a system of differential equations. By discretization we then propose its discrete-time version, which is expressed by a system of difference equations.

2.1. Continuous-time Lagrangian network

By the classical Lagrange optimization method, the Lagrangian of (1) is defined as

$$L(x, \lambda) = \frac{1}{2} x^T Q x + c^T x + \lambda^T (A x - b)$$  \hspace{1cm} (2)

where $\lambda \in \mathbb{R}^m$ is the Lagrange multiplier vector. The necessary conditions for minimal solution of (1) are found by taking the partial derivatives of $L(x, \lambda)$ with respect to $x$ and $\lambda$, respectively, and setting them to zero

$$\frac{\partial L(x, \lambda)}{\partial x} = Q x + c + A^T \lambda = 0$$ \hspace{1cm} (3)

$$\frac{\partial L(x, \lambda)}{\partial \lambda} = A x - b = 0.$$ \hspace{1cm} (4)

Multiplying (3) by $-1$, we have the Lagrangian optimality conditions

$$-Q x - A^T \lambda - c = 0$$ \hspace{1cm} (5)

$$A x - b = 0.$$ \hspace{1cm} (6)

Hence, the dynamical equation of the continuous-time Lagrangian network in compact matrix form is defined as

$$\frac{dy}{dt} = W y - s$$ \hspace{1cm} (7)

where $y = \begin{bmatrix} x \\ \lambda \end{bmatrix}$, $W = \begin{bmatrix} -Q & -A^T \\ A & N \end{bmatrix}$, $s = \begin{bmatrix} c \\ b \end{bmatrix}$, and $N$ is an $m \times m$ null matrix.

The continuous-time Lagrangian network is proven to be asymptotically stable in Ref. 5.

2.2. Discrete-time Lagrangian network

Taking discretization of the dynamical equation of the continuous-time Lagrangian network (7), we propose the dynamical equation for its discrete-time counterpart as

$$y(k + 1) = y(k) + h(W y(k) - s)$$

$$= (I + hW)y(k) - hs$$ \hspace{1cm} (8)

where $y(k) = \begin{bmatrix} x(k) \\ \lambda(k) \end{bmatrix}$, $h$ is a positive fixed step size and $I$ is an identity matrix.

Figure 1 shows the block diagram for the configuration of the discrete-time Lagrangian network. In this neural network, the $m$-dimensional column vector $b$ is fed into the network as its input while the $n$-dimensional decision variable vector $x$ is generated as its output.

3. Convergence Analysis

Lemma 1

The equilibrium point of the dynamical system (8) is equal to the optimal solution of the quadratic program (1).
The discrete-time Lagrangian network (8) is asymptotically convergent to the optimal solution of the quadratic program (1) if

\[ h < \min_{1 \leq i \leq n+m} \left\{ -\frac{2\text{Re}[\rho_i(W)]}{|\rho_i(W)|^2} \right\} \]  

(11)

where \( \rho_i(W) \) denotes the \( i \)th eigenvalue of the matrix \( W \), \( \text{Re}[\rho_i(W)] \) and \( |\rho_i(W)| \) are the real part and the absolute value of \( \rho_i(W) \), respectively.

\textbf{Proof}

The discrete-time Lagrangian network (8) is described by first-order difference equation. Therefore, it is a linear system. From the linear system theory,\(^8\) system (8) is asymptotically stable if all eigenvalues of the matrix \((I + hW)\) have absolute values less than 1, i.e.,

\[ |\rho_i(I + hW)| < 1, \quad \forall i = 1, 2, \ldots, n + m. \]  

(12)

With the properties of eigenvalue of matrices,\(^9\) the inequality (12) can be written as

\[ |\rho_i(I) + h\rho_i(W)| < 1, \quad \forall i = 1, 2, \ldots, n + m, \]  

(13)

Since for all \( i \), \( \rho_i(I) = 1 \), and let \( \rho_i(W) = \text{Re}[\rho_i(W)] + j\text{Im}[\rho_i(W)] \) where \( j = \sqrt{-1} \), inequality (13) becomes for all \( i = 1, 2, \ldots, n + m, \)

\[ |1 + h\text{Re}[\rho_i(W)] + jh\text{Im}[\rho_i(W)]| < 1, \]  

(14)

or

\[ \{1 + h\text{Re}[\rho_i(W)]\}^2 + \{jh\text{Im}[\rho_i(W)]\}^2 < 1. \]  

(15)

After some algebraic manipulations, we have

\[ h < \frac{-2\text{Re}[\rho_i(W)]}{|\rho_i(W)|^2}, \quad \forall i = 1, 2, \ldots, n + m. \]  

(16)

Since inequality (16) has to be satisfied for all \( i = 1, 2, \ldots, n + m \), which implies that \( h \) has an upper bound which equals the minimum of the right hand side of (16). Therefore, the condition of the positive fixed step size \( h \) which makes the discrete-time Lagrangian network to be asymptotically stable is

\[ h < \min_{1 \leq i \leq n+m} \left\{ -\frac{2\text{Re}[\rho_i(W)]}{|\rho_i(W)|^2} \right\} . \]  

(17)

It should be noted that the right hand side of (17) is always greater than zero though it has a negative sign. It has been proven in Ref. 5 the continuous-time Lagrangian network (7) is asymptotically stable and hence all eigenvalues of the matrix \( W \) have non-positive real part if matrices \( W \) and \( A \) are of full rank. The fixed step size \( h \) obtained by (17) is then always positive.

From Lemma 1 and with the convergence condition (17), the discrete-time Lagrangian network is convergent to the equilibrium point, which is equal to the global solution of the quadratic program (1). The proof is thus complete.

\section*{4. Simulation Results}

Consider the quadratic program (1) with

\[ Q = \begin{bmatrix} 4 & 1 & 3 \\ 1 & 4 & 3 \\ 3 & 3 & 6 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix}, \quad c = [2 \ 2 \ 4]^T \quad \text{and} \quad b = [2 \ 0]^T. \]

The theoretical optimal solution for this quadratic program is \( x = [0 \ 0 \ 1]^T \). Using the discrete-time Lagrangian network (8) to solve this quadratic program, we have

\[ W = \begin{bmatrix} -4 & -1 & -3 & -1 & -1 \\ -1 & -4 & -3 & -1 & 1 \\ -3 & -3 & -6 & -2 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad s = \begin{bmatrix} 2 \\ 2 \\ 4 \\ 2 \\ 0 \end{bmatrix}. \]

Since

\[ \min_{1 \leq i \leq 5} \left\{ -\frac{2\text{Re}[\rho_i(W)]}{|\rho_i(W)|^2} \right\} = 0.219, \]

we have

\[ h < \frac{-2\text{Re}[\rho_i(W)]}{|\rho_i(W)|^2}, \quad \forall i = 1, 2, \ldots, 5. \]  

(18)

This satisfies the condition (17) of the asymptotic stability of the discrete-time Lagrangian network.
by the convergence condition (11), the discrete-time Lagrangian network is globally convergent to the optimal solution of the desired quadratic program if we select $h < 0.219$. Figure 2 shows the time evolution of the decision variable generated by the discrete-time Lagrangian network with different values of the fixed step size $h$.

In subplot (a), where the fixed step size is chosen to be $h = 0.1$. It is noted that the discrete-time Lagrangian network generates the optimal solution $x = [0.017 \ 0.027 \ 0.929]^T$ after 50 iterations. In subplot (b), where the fixed step size is selected to be $h = 0.15$, and the computed optimal solution after 50 iterations is $x = [0.002 \ 0.003 \ 0.993]^T$. However, when the fixed step size is set as $h = 0.2$, as shown in subplot (c), the estimated optimal solution after 50 iterations is $x = [0 \ 0 \ 1]^T$ which is equal to the theoretical optimal solution. Subplot (d) shows the Lagrangian network is divergent when the fixed step size $h = 0.23$ which is greater than its upper bound 0.219. Therefore, we can speed up the rate of convergence by selecting a larger fixed step size below its upper bound.
5. Concluding Remarks

This letter presents a discrete-time Lagrangian network for solving convex quadratic programs. The discrete-time recurrent neural network is obtained through discretization of its continuous-time counterpart which was introduced in Ref. 5. We have given the condition of the fixed step size used in the discrete-time model, which guarantees the proposed discrete-time Lagrangian network to be asymptotically convergent to the optimal solution of the desired quadratic program. This discrete-time model is easier to be implemented in practical applications because of the availability of the digital hardware. Realized in dedicated digital hardware such as field-programmable gate arrays (FPGA), the discrete-time recurrent neural network is suitable for many applications where intensive computation is essential such as robot control and filter design.

References