Sampling theorems in function spaces for frames associated with linear canonical transform

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A B S T R A C T

The linear canonical transform (LCT) has proven to be a powerful tool in optics and signal processing. Most existing sampling theories of this transform were derived from the LCT band-limited signal viewpoint. However, in the real world, many analog signals encountered in practical engineering applications are non-bandlimited. The purpose of this paper is to derive sampling theorems of the LCT in function spaces for frames without band-limiting constraints. We extend the notion of shift-invariant spaces to the LCT domain and then derive a sampling theorem of the LCT for regular sampling in function spaces with frames. Further, the theorem is modified to the shift sampling in function spaces by using the Zak transform. Sampling and reconstructing signals associated with the LCT are also discussed in the case of Riesz bases. Moreover, some examples and applications of the derived theory are presented. The validity of the theoretical derivations is demonstrated via simulations.

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1. Introduction

The linear canonical transform (LCT) [1] is also known as ABCD transform, generalized Fresnel transform, generalized Huygens integral, affine Fourier transform, and quadratic phase systems. Many signal processing operations, such as the Fourier transform (FT), the fractional Fourier transform (FRFT), the Fresnel transform, and the scaling operations are special cases of this transform. The LCT has proven to be a powerful tool for optical systems, gradient-index medium system analysis, filter design, time–frequency analysis, radar system analysis, pattern recognition, communications, and many others [2–14]. The continuous-time LCT of a signal or function, \( f(t) \in L^2(\mathbb{R}) \), is defined as [2]

\[
F_M(u) = L^M(f(t))(u) = \left\{ \begin{array}{ll}
\int_{-\infty}^{\infty} f(t) K_M(u, t) \, dt, & b \neq 0 \\
\sqrt{\alpha} e^{i\phi}/2w^2 f(u), & b = 0
\end{array} \right.
\]  

(1)

where \( L^M \) denotes the LCT operator, and kernel \( K_M(u, t) \) is given by

\[
K_M(u, t) = A_b e^{i\phi/2b^2} + (\phi / 2bw^2 - i / b) \alpha
\]  

(2)

where \( M = (a, b, c, d), a, b, c, d \in \mathbb{R} \) satisfying \( ad - bc = 1 \), and \( A_b = 1 / \sqrt{2\pi b} \). The \( u \)-axis is regarded as the LCT domain. In general, we only consider the case of \( b \neq 0 \), since the LCT with \( b = 0 \) is just a chirp multiplication operation. Conversely, the inverse LCT is expressed as \( f(t) = \int_{-\infty}^{\infty} F_M(u) K_M^*(u, t) \, du \), where \( * \) in the superscript denotes the complex conjugate.

In digital signal and image processing, digital communications, etc., a continuous signal is usually represented by its discrete samples. A natural question is how to represent a continuous signal in terms of a discrete sequence. For a band-limited signal, Stern [15] found a Shannon-type sampling theorem associated with the LCT, which provides an exact representation by the signal’s uniform samples with sampling...
rate higher than its Nyquist rate. Although Stern’s sampling theory has had an enormous impact [16–22], it has a number of problems: It relies on the use of ideal filters; the bandlimited hypothesis is in contradiction with the idea of a finite duration signal; the bandlimiting operation generates Gibbs oscillations, and finally, the sinc function has a very slow decay rate at infinity such that the computation in the signal domain is very inefficient. Moreover, many applied problems impose different a priori constraints on the type of signals. For these reasons, the sampling and reconstruction problems associated with the LCT have been investigated in function spaces. In [23], Liu et al. established sampling formulae of the LCT for non-bandlimited signals by introducing certain types of non-bandlimited function spaces. Unfortunately, as pointed out by authors of [23], there are no normative rules for determining the parameters of the non-bandlimited function spaces in practical implementations at present.

In our recent paper [24], we proposed a sampling theorem for the LCT in some function space of the form \( \mathcal{V}_M(\phi) = \text{span}_{t \in \mathbb{Z}} \{ \phi(t - n)e^{-(j/2b)(t^2 - n^2)} \}_{n \in \mathbb{Z}} \) without band-limiting constraints, which can provide a suitable and realistic model of sampling and reconstruction for real applications. However, the results were derived for Riesz bases. The contributions of this paper, compared to our previous work, are threefold:

1. We derive some properties of the function space \( \mathcal{V}_M(\phi) \) with frame generators, which extend conventional shift-invariant spaces.
2. We establish a sampling theorem of the LCT for regular sampling in \( \mathcal{V}_M(\phi) \) with frames. Then, the theorem is modified to the shift sampling in \( \mathcal{V}_M(\phi) \) by using the Zak transform.
3. Some necessary conditions for sampling in the LCT domain are established in frame sense and in Riesz basis sense.

The outline of this paper is organized as follows. Section 2 introduces notation and then gives a generalization of the Zak transform in the LCT domain. Section 3 presents some properties of the function spaces related to the LCT and then derives a sampling theorem for the LCT in the function spaces with frames. Further, a shift-sampling theorem for the LCT is established by using the Zak transform. In addition, some relationships and properties about the relevant functions for sampling in the LCT domain are attained in frame sense and in Riesz basis sense. Some examples and applications of the derived results are also discussed. Finally, concluding remarks are given in Section 4.

2. Preliminaries

2.1. Notation

For a measurable function \( f(t) \) on \( \mathbb{R} \), let
\[
\|f(t)\|_\infty = \text{ess sup}\{f(t)\} \quad \text{and} \quad \|f(t)\|_0 = \text{ess inf}\{f(t)\}
\]
be the essential supremum and infimum of \( |f(t)| \), respectively. The characteristic function of a measurable subset \( E \subseteq \mathbb{R} \) is given by
\[
X_E(t) = \begin{cases} 1, & t \in E \\ 0, & \text{otherwise} \end{cases}
\]

2.2. A generalization of the Zak transform associated with the LCT

One of the important tools used in the study of sampling theory is the Zak transform (ZT) [27]. Here, we give a generalization of the ZT associated with the LCT, which will be used in this paper.

The ordinary ZT can be defined in terms of the time-shift operator \( T_t \), as
\[
Z_f(\sigma, \omega) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} \{ f(n)e^{-j\omega n} \} e^{j\sigma n}
\]
where \( T_t \) is given by \( (T_t f)(\cdot) = f(\cdot - t) \). Since the canonical time-shift operator \( T^F_\tau \) [13] generalizes the ordinary time-shift operator \( T_\tau \), it naturally lends itself to defining a linear canonical ZT (LCZT), i.e.,
\[
Z^F_f(\sigma, u) = \sum_{n \in \mathbb{Z}} \{ f(n)e^{-j\omega n} \} e^{j\sigma n} e^{j\omega u n}
\]
where \( (T^F_\tau f)(\cdot) = f(\cdot - \tau)e^{-j(\omega/2b)(\cdot^2 - n^2)} \) [13]. Whenever \( M = (0, 1, -1, 0) \), (6) reduces to the ordinary ZT. If \( \sigma = 0 \), (6) is identical with the discrete-time LCT (DTLCT) defined in (6) of [24].

3. Main results

3.1. Sampling in function spaces for frames associated with the LCT

For any function \( \phi(t) \in L^2(\mathbb{R}) \), let \( (\phi(t - n))_{n \in \mathbb{Z}} \) be a set of functions from \( L^2(\mathbb{R}) \) and \( \mathcal{V}_M(\phi) = \text{span}_{t \in \mathbb{Z}} \{ \phi(t - n) \} \) the closed subspace of \( L^2(\mathbb{R}) \) spanned by \( \{ \phi(t - n) \}_{n \in \mathbb{Z}} \). From [25], the basic principle behind the conventional shift-invariant space \( \mathcal{V}(\phi) \) is that if \( f(t) \in \mathcal{V}(\phi) \), then \( (T_\tau f)(t) \in \mathcal{V}(\phi) \) for all \( \tau \in \mathbb{Z} \). Similarly, the subspace \( \mathcal{V}_M(\phi) = \text{span}_{t \in \mathbb{Z}} \{ \phi(t - n) e^{-j(\omega/2b)(\cdot^2 - n^2)} \}_{n \in \mathbb{Z}} \) of \( L^2(\mathbb{R}) \) constructed in [24] has a following basic property:

\[
f(t) \in \mathcal{V}_M(\phi) \iff (T^F_\tau f)(t) \in \mathcal{V}_M(\phi)
\]
for all \( \tau \in \mathbb{Z} \), where \( T^F_\tau \) is defined in (6). Whenever \( M = (0, 1, -1, 0) \), \( \mathcal{V}_M(\phi) \) and \( T^F_\tau \) reduces to \( \mathcal{V}(\phi) \) and \( T_\tau \), respectively. Clearly, \( \mathcal{V}(\phi) \) is just a special case of \( \mathcal{V}_M(\phi) \), and the relationship between them is given by

\[
f(t) \in \mathcal{V}_M(\phi) \iff f(t)e^{j(\omega/2b)\cdot^2} \in \mathcal{V}(\phi).
\]

Consequently, \( \{ \phi_{n,M}(t) = \phi(t - n)e^{-(j/2b)(t^2 - n^2)} \}_{n \in \mathbb{Z}} \) is a frame or a Riesz basis for \( \mathcal{V}_M(\phi) \) if and only if \( \{ \phi(t - n) \}_{n \in \mathbb{Z}} \) is a frame or a Riesz basis for \( \mathcal{V}(\phi) \). For simplicity, let

\[
G_{\phi,M}(u) = \sum_{k \in \mathbb{Z}} |\phi(u/2 + 2k)|^2
\]

where \( \phi(u/b) \) denotes the FT (with its argument scaled by 1/b) of \( \phi(t) \). It is easy to see that \( G_{\phi,M}(u) = G_{\phi,M}(u + 2\pi b) \), and \( G_{\phi,M}(u) \in L^1(1) \) where \( 1 \triangleq [0,2\pi b) \). Also, let \( E_{\phi,M} \triangleq \). 

Proof. Theorem 1. Let \( \phi(t) \in L^2(\mathbb{R}) \) and \( B \geq A > 0 \). Then, \( \{\phi_n(t)\}_{n \in \mathbb{Z}} \) is

- a frame for \( V_M(\phi) \subset L^2(\mathbb{R}) \) with bounds \( A \) and \( B \) if and only if
  \[ A \leq G_{\phi,M}(u) \leq B \quad \text{a.e. on } \Xi_{\phi,M}; \]
  \( \{ \phi_n(t) \}_{n \in \mathbb{Z}} \)

- a Riesz basis for \( V_M(\phi) \subset L^2(\mathbb{R}) \) with bounds \( A \) and \( B \) if and only if
  \[ A \leq G_{\phi,M}(u) \leq B \quad \text{a.e. on } I. \]

(10) (11)

With the relationship defined in (8), Theorem 1 can be easily derived by using the results of [26], and the derivation is omitted due to the space limitation. Based on the above facts, we now introduce some relationships and properties about the relevant functions for sampling in the LCT domain.

Theorem 2. Suppose that \( \{ \phi_n(t) \}_{n \in \mathbb{Z}} \) is a frame for the subspace \( V_M(\phi) \subset L^2(\mathbb{R}) \), such that the sequence of sampling at integers \( \{ \phi(n) \}_{n \in \mathbb{Z}} \) belongs to \( \ell^2(\mathbb{Z}) \). If there exists a function \( s(t) \in L^2(\mathbb{R}) \) with \( s(t)e^{-\langle \varphi(t), b^2 \rangle^2} \in \mathcal{V}_M(\phi) \) such that

\[ f(t) = \sum_{n \in \mathbb{Z}} f(n)s(t-n)e^{-\langle \varphi(t), b^2 \rangle^2} \]

holds for any \( f(t) \in \mathcal{V}_M(\phi) \) in the \( L^2(\mathbb{R}) \) sense, then

(i) \( \text{supp } \phi(u/b) = \text{supp } S(u/b) \subset E_{\phi,M} \subset \text{supp } \tilde{\phi}(u/b); \)

(ii) \( G_{\phi,M}(u) = 2\pi G_{\phi,M}(u)/\phi(u/b)^2 \)

where \( \phi(u/b) \) and \( S(u/b) \) denote the FT (with its argument scaled by \( 1/b \)) of \( \phi(t) \) and \( s(t) \), respectively, and \( \phi(u/b) \) represents the DFTT (with its argument scaled by \( 1/b \)) of \( \phi(n) \).

Proof. (i) Since \( \phi_n(t) \in \mathcal{V}_M(\phi) \) for any \( n \in \mathbb{Z} \), we have \( \phi_{\phi,M}(t) \in \mathcal{V}_M(\phi) \), i.e., \( \text{supp } \phi(t)e^{-\langle \varphi(t), b^2 \rangle^2} \in \mathcal{V}_M(\phi) \). Then, replacing \( f(t) \) with \( \phi(t)e^{-\langle \varphi(t), b^2 \rangle^2} \) in (12) yields

\[ \phi(t)e^{-\langle \varphi(t), b^2 \rangle^2} = \sum_{n \in \mathbb{Z}} \phi(n)s(t-n)e^{-\langle \varphi(t), b^2 \rangle^2}. \]

Taking the LCT on both sides of (13) gives rise to

\[ \phi\left( \frac{u}{b} \right) = \sqrt{2\pi} \phi\left( \frac{u}{b} \right) S\left( \frac{n}{b} \right) \]

which implies that \( \text{supp } \phi(u/b) \subset \text{supp } \tilde{\phi}(u/b) \) and \( \text{supp } \phi(u/b) \subset \text{supp } S(u/b) \) hold for a.e. \( u \in \mathbb{R} \). Since \( \phi(u/b) \) is \( 2\pi b \) periodic, it follows that

\[ \text{supp } \phi\left( \frac{u}{b} + 2k\pi \right) \subset \text{supp } \phi\left( \frac{u}{b} \right) \]

for all \( k \in \mathbb{Z} \). Meanwhile, \( \bigcup_{k \in \mathbb{Z}} \text{supp } \phi(u/b + 2k\pi) = E_{\phi,M} \) holds. Otherwise, there is a measurable subset \( \delta(\delta) \neq 0 \) such that

\[ \delta = E_{\phi,M} - \bigcup_{k \in \mathbb{Z}} \text{supp } \phi\left( \frac{u}{b} + 2k\pi \right). \]

Then, \( \phi(u/b + 2k\pi) = 0 \) holds for any \( u \in \delta \) and for all \( k \in \mathbb{Z} \).

Hence,

\[ G_{\phi,M}(u) = \sum_{k \in \mathbb{Z}} \left| \phi\left( \frac{u}{b} + 2k\pi \right) \right|^2 = 0 \]

holds for any \( u \in \delta \). However, it is a contradiction \( \delta \subset E_{\phi,M} \). Therefore, \( \bigcup_{k \in \mathbb{Z}} \text{supp } \phi(u/b + 2k\pi) \subset E_{\phi,M} \). On the other hand, it is easy to see that

\[ \bigcup_{k \in \mathbb{Z}} \text{supp } \phi\left( \frac{u}{b} + 2k\pi \right) \supset E_{\phi,M}. \]

Hence,

\[ \text{supp } \phi\left( \frac{u}{b} + 2k\pi \right) \supset \bigcup_{k \in \mathbb{Z}} \text{supp } \phi\left( \frac{u}{b} + 2k\pi \right) = E_{\phi,M}. \]

Since \( \{ \phi_n(t) \}_{n \in \mathbb{Z}} \) is a frame for \( \mathcal{V}_M(\phi) \), there exists a sequence \( d(n) \in \ell^2(\mathbb{Z}) \) such that

\[ s(t)e^{-\langle \varphi(t), b^2 \rangle^2} = \sum_{n \in \mathbb{Z}} d(n)\phi_n(t). \]

Then, we have

\[ s(t)e^{-\langle \varphi(t), b^2 \rangle^2} = \sum_{n \in \mathbb{Z}} d(n)\phi_n(t-n)e^{-\langle \varphi(t-n), b^2 \rangle^2 - n^2}. \]

Taking the LCT on both sides of (21) results in

\[ S\left( \frac{u}{b} \right) = \mathcal{M}(t) \theta^{-\langle \varphi(t), b^2 \rangle^2} D_M(u) \Phi\left( \frac{u}{b} \right) \]

where \( D_M(u) \) is the DTLCT of \( d(n) \). Therefore, \( \text{supp } S(u/b) \subset \text{supp } \phi(u/b) \). Overall, the above description shows that

\[ \text{supp } \phi\left( \frac{u}{b} \right) = \text{supp } S\left( \frac{u}{b} \right) \subset \text{supp } \phi\left( \frac{u}{b} \right) \]

Thus, (i) of Theorem 2 is established.

(ii) Since \( \phi(u/b) \) is \( 2\pi b \) periodic, it follow from (14) that

\[ \phi\left( \frac{u}{b} + 2k\pi \right) = \sqrt{2\pi} \phi\left( \frac{u}{b} + 2k\pi \right) S\left( \frac{u}{b} + 2k\pi \right), \quad k \in \mathbb{Z} \]

such that

\[ \left| \phi\left( \frac{u}{b} + 2k\pi \right) \right|^2 = 2\pi \left| \phi\left( \frac{u}{b} + 2k\pi \right) \right|^2 S\left( \frac{u}{b} + 2k\pi \right)^2. \]

Then, taking a sum for all \( k \) on both sides of (26) yields (ii) of Theorem 2.

Based on the derived results, we have the following sampling theorem for the LCT in the space \( \mathcal{V}_M(\phi) \) with frames.

Theorem 3. Suppose that \( \{ \phi_n(t) \}_{n \in \mathbb{Z}} \) is a frame for the subspace \( \mathcal{V}_M(\phi) \subset L^2(\mathbb{R}) \), such that the sampling sequence at integers \( \{ \phi(n) \}_{n \in \mathbb{Z}} \) belongs to \( \ell^2(\mathbb{Z}) \). Then, for any \( f(t) \in \mathcal{V}_M(\phi) \), there exists a function \( s(t) \in L^2(\mathbb{R}) \) with \( s(t)e^{-\langle \varphi(t), b^2 \rangle^2} \in \mathcal{V}_M(\phi) \) such that

\[ f(t) = \sum_{n \in \mathbb{Z}} f(n)s(t-n)e^{-\langle \varphi(t-n), b^2 \rangle^2 - n^2} \]

holds in the \( L^2(\mathbb{R}) \) sense if and only if

\[ \frac{1}{\sqrt{2\pi}} \phi\left( \frac{u}{b} \right) \mathcal{M}(t) \phi_{\phi,M}(u) \in L^2(I) \]
holds. In this case,
\begin{equation}
S^u_b(u) = \begin{cases} 
\frac{\phi^u_b}{\sqrt{2\pi\phi^u_b}}, & u \in E_{\phi,M} \\
0, & u \notin E_{\phi,M}
\end{cases}
\end{equation}
holds for a.e. \( u \in \mathbb{R} \). Here, \( S(u/b) \) and \( \phi(u/b) \) denote the FTs (with their argument scaled by \( 1/b \)) of \( s(t) \) and \( \phi(t) \), respectively. \( \hat{\phi}(u/b) \) indicates the DTFT (with its argument scaled by \( 1/b \)) of \( \phi(n) \).

**Proof.** Sufficiency: First assume that (28) holds. Hence, \( \hat{\phi}(u/b) \neq 0 \) holds for a.e. \( u \in E_{\phi,M} \). By (9) of [24], there exists a sequence \( \{ c[n]\}_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z}) \) such that
\begin{equation}
\int_{E_{\phi,M}} \left| \frac{\phi^u_b}{\sqrt{2\pi\phi^u_b}} \right|^2 du = \int_{\mathbb{R}} \left| \frac{\phi^u_b}{\sqrt{2\pi\phi^u_b}} \right|^2 X_{E_{\phi,M}}(u) du = \int_I G_{\phi,M}(u)X_{E_{\phi,M}}(u) du \leq \| G_{\phi,M}(u)X_{E_{\phi,M}}(u) \|_{\infty} \int_{\mathbb{R}} \left| \frac{X_{E_{\phi,M}}(u)}{\sqrt{2\pi\phi^u_b}} \right|^2 du
\end{equation}
which implies that \( \left( \phi(u/b)/\sqrt{2\pi\phi(u/b)} \right)X_{E_{\phi,M}}(u) \in L^2(\mathbb{R}) \). Thus, we can obtain
\begin{equation}
S^u_b(u) = \mathcal{F}\{s(t)\} \bigg| \frac{u}{b} \bigg. \Phi^u_b = \int_{E_{\phi,M}} X_{E_{\phi,M}}(u) du
\end{equation}
where \( \mathcal{F} \) denotes the FT operator. Then, inserting (30) into (32) results in
\begin{equation}
S^u_b(u) = \Phi^u_b \sum_{n \in \mathbb{Z}} c[n]e^{i(2\pi/n)u} \Phi(2\pi/n)
\end{equation}
Next, by the relationship between the LCT and the FT [24], we have
\begin{equation}
L^M[s(t)e^{i(2\pi/n)t}](u) = \sqrt{2\pi A_{\phi}\Phi^u_b} X_{E_{\phi,M}}(u/n)
\end{equation}
where \( \tilde{C}_M(u) \) denotes the DTFT of \( c[n] \). Therefore, by applying the semi-discrete canonical convolution theorem [24], we derive
\begin{equation}
s(t)e^{-i(\omega_0/2\pi)t} = \sum_{n \in \mathbb{Z}} c[n]\phi(t-n)e^{-i(\omega_0/2\pi)(t-n)^2} = \sum_{n \in \mathbb{Z}} c[n]\phi_{h,n}(t)
\end{equation}
which implies that \( s(t)e^{-i(\omega_0/2\pi)t} \in \mathcal{V}_M(\phi) \) due to the fact that \( \phi_{h,n}(t) \) is a frame for \( \mathcal{V}_M(\phi) \). Moreover, by assumption, for any \( f(t) \in \mathcal{V}_M(\phi) \), we have
\begin{equation}
f(t) = \sum_{m \in \mathbb{Z}} p[m]\phi(t-m)e^{-i(\omega_0/2\pi)(t-m)^2}
\end{equation}
where \( p[m] \in \ell^2(\mathbb{Z}) \). Using the semi-discrete canonical convolution theorem [24] and (36) gives rise to
\begin{equation}
F_M(u) = \sqrt{2\pi}\hat{p}_M(u)\Phi^u_b
\end{equation}
where \( F_M(u) \) and \( \hat{p}_M(u) \) denote the LCT of \( f(t) \) and the DTFT of \( p[m] \), respectively. Then, by (37) and (32), it follows that
\begin{equation}
F_M(u) = 2\pi\hat{p}_M(u)\Phi^u_b S^u_b(u)
\end{equation}
Next, by (36), we let
\begin{equation}
f[n] = \sum_{n \in \mathbb{Z}} p[m]\phi(n-m)e^{-i(\omega_0/2\pi)(n-m)^2}
\end{equation}
Obviously, \( \{ f[n]\}_{n \in \mathbb{Z}} \) is well defined since \( p[n], \phi[n] \in \ell^2(\mathbb{Z}) \). Then, applying (39) and the fully discrete canonical convolution theorem [24] results in
\begin{equation}
\tilde{F}_M(u) = \sqrt{2\pi}\hat{p}_M(u)\Phi^u_b S^u_b(u)
\end{equation}
where \( \tilde{F}_M(u) \) denotes the DTLCT of \( f[n] \). Inserting (40) into (38) yields
\begin{equation}
F_M(u) = \sqrt{2\pi}\tilde{F}_M(u)
\end{equation}
Then, from (41) and the semi-discrete canonical convolution theorem [24], (27) can be established.

**Necessity:** On the contrary, assume that there exists a function \( s(t) \in L^2(\mathbb{R}) \) with \( s(t)e^{-i(\omega_0/2\pi)t} \in \mathcal{V}_M(\phi) \) such that (27) holds in the \( L^2(\mathbb{R}) \) sense. Since \( \phi_M(t) \in \mathcal{V}_M(\phi) \) holds for any \( n \in \mathbb{Z} \), it follows that \( \phi_{h,n}(t) \in \mathcal{V}_M(\phi) \), i.e., \( \phi(t)e^{-i(\omega_0/2\pi)t} \in \mathcal{V}_M(\phi) \). Then, replacing \( f(t) \) with \( \phi(t)e^{-i(2\pi/n)t} \) in (27) yields
\begin{equation}
\phi(t)e^{-i(2\pi/n)t} = \sum_{n \in \mathbb{Z}} c[n]e^{i(2\pi/n)t} s(t-n)e^{-i(\omega_0/2\pi)(t-n)^2}.
\end{equation}
Taking the LCT of both sides of (42) yields
\begin{equation}
\Phi^u_b = \sqrt{2\pi}\tilde{C}_M(u) S^u_b(u)
\end{equation}
Then, by Theorem 2, we derive
\begin{equation}
\sup \delta \Phi^u_b \geq \bigcup_{k \in \mathbb{Z}} \sup \delta \Phi^u_b + 2\pi\mathcal{E}_{\phi,M} = E_{\phi,M} \sup \delta \Phi^u_b.
\end{equation}
Hence, (43) can be rewritten as
\begin{equation}
S^u_b(u) = \begin{cases}
\frac{\phi^u_b}{\sqrt{2\pi\phi^u_b}}, & u \in E_{\phi,M} \\
0, & u \notin E_{\phi,M}
\end{cases}
\end{equation}
Since \( S(u/b) \in L^2(\mathbb{R}) \), it follows from (45) that
\begin{equation}
\infty > \int_{\mathbb{R}} \left| S^u_b(u) \right|^2 du = \int_{\mathbb{R}} \left| \frac{\phi^u_b}{\sqrt{2\pi\phi^u_b}} \right|^2 X_{E_{\phi,M}}(u) du \\
\geq \| G_{\phi,M}(u) \|_2 \int_{\mathbb{R}} \left| X_{E_{\phi,M}}(u) \right|^2 du
\end{equation}
which implies that (28) holds. This proves Theorem 3.
Note that Theorem 3 extends the result of Riesz bases in [24] to frames. Moreover, if \( \phi(t) \in L^2(\mathbb{R}) \) is such that the sampling sequence \( \{\phi[n]\}_{n \in \mathbb{Z}} \) makes sense, and \( \{\phi[n+\sigma]\}_{n \in \mathbb{Z}} \) belongs to \( \ell^2(\mathbb{Z}) \) for some \( \sigma \in [0,1) \). By (5), the ZT (with its argument scaled by \( 1/b \)) of \( \phi(t) \) is given by

\[
Z_{\phi}(\sigma, \frac{u}{b}) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} \phi[n+\sigma]e^{-in\frac{u}{b}}. \tag{47}
\]

A generating function \( \Phi(t) \) of \( \mathcal{V}_M(\phi) \) may not satisfy \( (1/\sqrt{2\pi})(\phi(u/b))_{\infty} \in L^2(\mathbb{R}) \) but may satisfy \( (1/\sqrt{2\pi})Z_{\phi}(\sigma, u/b)_{\infty} \) for some \( \sigma \in [0,1) \). Then, Theorem 3 can be extended to the case of sampling \( \{\phi[n+\sigma]\}_{n \in \mathbb{Z}} \) for some \( \sigma \in [0,1) \) as follows.

**Theorem 4.** Assume that \( \{\phi_n,M(t)\}_{n \in \mathbb{Z}} \) is a frame for the subspace \( \mathcal{V}_M(\phi) \) of \( L^2(\mathbb{R}) \), such that the sampling sequence at integers \( \{\phi[n+\sigma]\}_{n \in \mathbb{Z}} \) belongs to \( \ell^2(\mathbb{Z}) \) for some \( \sigma \in [0,1) \). Then, for any \( f(t) \in \mathcal{V}_M(\phi) \), there exists a function \( s_u(t) \in L^2(\mathbb{R}) \) such that \( \Phi(u/b)_{\infty} \in \mathcal{V}_M(\phi) \) such that

\[
f(t) = \sum_{n \in \mathbb{Z}} (\gamma^M_{-f}(n))s_u(t-n)e^{-in\frac{u}{b}(x^2-x^2)} \tag{48}
\]

holds in the \( L^2(\mathbb{R}) \) sense if and only if

\[
\frac{1}{\sqrt{2\pi}}Z_{\phi}(\sigma, \frac{u}{b})_{\infty} \in L^2(\mathbb{R}) \tag{49}
\]

holds. In this case,

\[
S_u(u/b) = \begin{cases} \Phi(u/b)_{\infty}, & u \in E_{\phi,M} \\ 0, & u \notin E_{\phi,M} \end{cases} \tag{50}
\]

holds for a.e. \( u \in \mathbb{R} \), where \( s_u(t) \) and \( \Phi(u/b) \) denote the FTs (with their argument scaled by \( 1/b \)) of \( s_u(t) \) and \( \phi(t) \), respectively.

By applying the ZT and its generalized form defined in (6), the proof of Theorem 4 is similar to that of Theorem 3 and is omitted due to space limitation. Meanwhile, by Theorem 4, we have the following theorem of the LCT for Riesz bases, which is an improvement of the result in [24].

**Theorem 5.** Suppose that \( \{\phi_n,M(t)\}_{n \in \mathbb{Z}} \) is a Riesz basis for the subspace \( \mathcal{V}_M(\phi) \) of \( L^2(\mathbb{R}) \), such that the sampling sequence at integers \( \{\phi[n+\sigma]\}_{n \in \mathbb{Z}} \) belongs to \( \ell^2(\mathbb{Z}) \) for some \( \sigma \in [0,1) \). Then, for any \( f(t) \in \mathcal{V}_M(\phi) \), there exists a function \( s_u(t) \in L^2(\mathbb{R}) \) such that \( \Phi(u/b)_{\infty} \in \mathcal{V}_M(\phi) \) such that

\[
f(t) = \sum_{n \in \mathbb{Z}} (\gamma^M_{-f}(n))s_u(t-n)e^{-in\frac{u}{b}(x^2-x^2)} \tag{51}
\]

holds in the \( L^2(\mathbb{R}) \) sense if and only if

\[
\frac{1}{\sqrt{2\pi}}Z_{\phi}(\sigma, \frac{u}{b})_{\infty} \in L^2(\mathbb{R}) \tag{52}
\]

holds. In this case, \( s_u(t) = \Phi(u/b)/(\sqrt{2\pi})Z_{\phi}(\sigma, u/b) \) holds for a.e. \( u \in \mathbb{R} \).

Finally, we give some necessary conditions for sampling in the case of Riesz bases, which are not addressed in [24].

**Theorem 6.** Suppose that \( \{\phi_n,M(t)\}_{n \in \mathbb{Z}} \) is a Riesz basis for the subspace \( \mathcal{V}_M(\phi) \) of \( L^2(\mathbb{R}) \), such that the sampling sequence at the integers \( \{\phi[n]\}_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z}) \). For any \( f(t) \in \mathcal{V}_M(\phi) \), if there exists a function \( s(t) \in L^2(\mathbb{R}) \) with \( ste^{-(\gamma/2b)/t^2} \in \mathcal{V}_M(\phi) \) such that

\[
f(t) = \sum_{n \in \mathbb{Z}} f(n)s(t-n)e^{-(\gamma/2b)/t^2} \tag{53}
\]

holds in the \( L^2(\mathbb{R}) \) sense, then satisfied are the following relations:

(i) \( \hat{\phi}(u/b) \neq 0 \) a.e. \( u \in \mathbb{R} \), and \( E_{\phi,M} = E_s,M = \text{supp} \hat{\phi}(u/b) = \mathbb{R} \).

(ii) \( \sqrt{2\pi}\tilde{s}(u/b) = 1 \) a.e. \( u \in \mathbb{R} \).

(iii) \( \sqrt{2\pi}|\sum_{n \in \mathbb{Z}} s(u/b + 2ka) = 1 \) a.e. \( u \in \mathbb{R} \).

where \( \hat{\phi}(u/b) \) and \( \tilde{s}(u/b) \) indicate the DTFTs (with their argument scaled by \( 1/b \)) of \( \phi[n] \) and \( s[n] \), respectively, and \( S(u/b) \) denotes the FT (with its argument scaled by \( 1/b \)) of \( s(t) \).

**Proof.** (i) By Theorem 2, we have \( \text{supp} \hat{\phi}(u/b) \subseteq E_{\phi,M} \). Thus, \( \text{supp} \phi(u/b + 2ka) \subseteq \text{supp} \hat{\phi}(u/b) \) holds for all \( k \in \mathbb{Z} \) and \( u \in \mathbb{R} \). Moreover, \( \bigcup_{k \in \mathbb{Z}} \text{supp} \phi(u/b + 2ka) = \mathbb{R} \) holds. Otherwise, there is a measurable subset \( \delta(\delta \neq 0) \) such that

\[
\delta = \mathbb{R} \setminus \bigcup_{k \in \mathbb{Z}} \text{supp} \phi(u/b + 2ka) \tag{54}
\]

Then, \( \phi(u/b + 2ka) = 0 \) holds for any \( u \in \delta \) and for all \( k \in \mathbb{Z} \). Hence,

\[
G_{\phi,M}(u) = \sum_{k \in \mathbb{Z}} |\phi(u/b + 2ka)|^2 = 0 \tag{55}
\]

holds for any \( u \in \delta \). By Theorem 1, \( G_{\phi,M}(u) \neq 0 \) for a.e. \( u \in \mathbb{R} \) holds. However, it is a contradiction with \( \delta \subseteq \mathbb{R} \). Thus,

\[
\hat{\phi}(u/b) \neq 0 \text{ a.e. } u \in \mathbb{R} \tag{56}
\]

and

\[
E_{\phi,M} = E_s,M = \text{supp} \hat{\phi}(u/b) = \mathbb{R} \tag{57}
\]

(ii) Using (53) with \( f(t) = s(t)e^{-(\gamma/2b)/t^2} \) yields

\[
s(t)e^{-(\gamma/2b)/t^2} = \sum_{n \in \mathbb{Z}} s(n)\tilde{s}(t-n)e^{-(\gamma/2b)/t^2} \tag{58}
\]

Then, taking the LCT of both sides of (58) gives

\[
S(u/b) = \sqrt{2\pi}\tilde{s}(u/b)s(u/b) \tag{59}
\]

Next, by (ii) of Theorem 2, it follows that

\[
G_{s,M}(u) = 2\pi|\tilde{s}(u/b)|^2G_{s,M}(u) \tag{60}
\]

Then, from Theorem 2 and Proof (i) of Theorem 6, we have

\[
G_{s,M}(u) \neq 0 \text{ a.e. } u \in \mathbb{R} \tag{61}
\]

Hence, using (60) and (61) yields \( \sqrt{2\pi}\tilde{s}(u/b) = 1 \) a.e. \( u \in \mathbb{R} \).

(iii) From Poisson’s summation formula of the FT [2], we have \( \tilde{s}(u/b) = \sum_{k \in \mathbb{Z}} s(u/b + 2ka) \). Then, according to the proof of (ii) in Theorem 6, we can derive this result. This completes the proof of Theorem 6. □

Now, we give two examples that satisfy our conditions but not those in previous papers.
Example 1. Let \( \phi(t) \) be a function in \( L^2(\mathbb{R}) \) with its scaled FT \( \Phi(u/b) \) satisfying \( \Phi(u/b) = \chi_{[0,4\pi b]}(u) \), \( 0 < \sigma < \frac{1}{b} \). By (9), we have \( G_{\phi,M}(u) = \chi_{[0,4\pi b]}(u) \) on \( I \). Moreover, applying Poisson’s summation formula of the FT [2] yields \( \Phi(u/b) = \sum_{k \in \mathbb{Z}} \Phi(u/b + 2k\sigma) \) so that \( \Phi(u/b) \chi_{[0,4\pi b]}(u) \). By Theorem 1, it is easy to see that \( \{\phi_{n,M}(t)\}_{n \in \mathbb{Z}} \) is a frame but not a Riesz basis for \( \mathcal{V}_M(\phi) \). Hence, the sampling theory in [24] cannot be applied to deal with the function \( \phi(t) \). However, \( (1/\sqrt{2\pi} \Phi(u/b) \chi_{[0,4\pi b]}(u) \) in \( L^2(I) \) implies that the proposed sampling theorem in Theorem 3 is available. The \( S(u/b) \) is given by

\[
S(u/b) = \frac{\Phi(u/b)}{\sqrt{2\pi}} \chi_{E_M}(u) = \frac{1}{\sqrt{2\pi}} \chi_{[0,4\pi b]}(u). \tag{62}
\]

Taking the inverse FT of both sides of (62) results in the interpolation function \( s(t) \) defined in (27).

Example 2. We choose \( \phi(t) = \beta^2(t) \), the 2th-order B-spline function [28], \( \beta^2(t) \) has support on \( [0,3) \), i.e.,

\[
\beta^2(t) = \frac{t^2}{2} \chi_{[0,1)}(t) + \frac{6t - 2t^2 - 3}{2} \chi_{[1,2)}(t) + \frac{(3 - t)^2}{2} \chi_{[2,3)}(t) \tag{63}
\]

Following the results of [28], we have

\[
\Phi(u/b) = \frac{1}{\sqrt{2\pi}} \left( 1 - e^{-ju/b} \right)^3 \nonumber
\]

so that

\[
G_{\phi,M}(u) = \sum_{k \in \mathbb{Z}} \left| \Phi(u/b + 2k\sigma) \right|^2 = \frac{1}{6\sigma} + \frac{1}{3\pi} \cos^2 \left( \frac{u}{2b} \right). \tag{64}
\]

By Theorem 1, it is shown that \( \{\phi_{n,M}(t)\}_{n \in \mathbb{Z}} \) is a Riesz basis for \( \mathcal{V}_M(\phi) \). Clearly, \( \beta^2(t) \) has its values on the integers \( 0, 1, 2, 3 \). Then, we derive

\[
\beta^2(u/b) = \frac{1}{2\sqrt{2\pi}} \left( e^{-ju/b} + e^{-ju/b} \right) \nonumber
\]

which equals zero at \( u = \pm b \in I \). Therefore, \( 1/\sqrt{2\pi} \Phi(u/b) \chi_{[0,4\pi b]}(u) \) is a Riesz basis for \( \mathcal{V}_M(\phi) \), the sampling theory established in [24] cannot be applied to deal with \( \phi(t) \). Note that

\[
Z_{\phi} \left( \frac{1}{2}, \frac{u}{b} \right) = \frac{1}{8\sqrt{2\pi}} \left( 1 + 6e^{-ju/b} + e^{-ju/b} \right), \tag{65}
\]

and it is not difficult to show that \( (1/\sqrt{2\pi} Z_{\phi} \chi_{[0,4\pi b]}(u) \) in \( L^2(I) \). Hence, the proposed sampling theorem in Theorem 5 can be applied, and the interpolation function \( s_n(t) \) deﬁned in (48) with \( \sigma = 1/2 \) can be derived by taking the inverse FT of

\[
S_{1/2}(u/b) = \frac{8 \left( 1 - e^{-ju/b} \right)^3}{\sqrt{2\pi} \left( 1 + 6e^{-ju/b} + e^{-ju/b} \right)}. \tag{66}
\]

3.2. Applications

In many problems of practical interest one is interested in the samples of a chirp signal \( f(t) \) that is ubiquitous in radar, sonar, and communications systems. It was shown [29] that the Nyquist rate for sampling a chirp signal \( f(t) \) in the LCT domain is lower than the one used in the Fourier domain. The signal \( f(t) \) can be derived as follows [29]:

\[
f(t) = \sum_{n \in \mathbb{Z}} f[n] \sin(t - n) e^{-(j/2b)\pi t^2 - n^2} \tag{67}
\]

In practice one only has finite number of samples of the function of interest, and therefore, this methodology is rarely used in real applications because of the slow decay rate of the sinc function. Now, we focus on the same problem based on the derived theory. We can rewrite \( f(t) \) as an expansion in terms of a general interpolation function \( s(t) \) or \( s_n(t) \). Our objective is then to choose a generator \( \phi(t) \) of \( \mathcal{V}_M(\phi) \), which has faster time decay rate than the sinc function. For instance, we choose \( \phi(t) = \beta^2(t) \). In this case, (67) can be viewed as a special case of (51) with \( \phi(t) \) chosen as sinc(t) and \( \sigma = 0 \) in Theorem 5. For purpose of illustration, we observe a signal given by

\[
f(t) = \sum_{k = 0} f[n] \sin(2\pi ft) e^{-jk/2\pi t^2} \tag{68}
\]

where \( k = 1, \rho_0 = 2, \rho_1 = 5, \rho_2 = 7, f_0 = 0.1, f_1 = 0.25 \), and \( f_2 = 0.3 \). It is band-limited in the LCT domain with \( M = (1, 1.5, 1.5) \). The maximum LCT-frequency value of \( f \)
In this case, the space $V_M$ should satisfy $\Delta_M \leq \frac{\pi}{2} \times \frac{1}{2 \times 2bf_2}$. In our example, we choose $\Delta_M = \frac{\pi}{2}$. The original signal $f(t)$ and its corresponding samples are shown in Fig. 1.

Now, we try to recover $f(t)$, $t \in [-4, 4]$ using (51) (with $\phi$ chosen as a B-spline of order 2 and $\sigma = \frac{1}{2}$) against the case of sinc interpolation (or $\phi = \text{sinc}$, $\sigma = 0$) under the condition that the number of sampling points is constrained to 19. The original and recovered signals are plotted in Fig. 2. In our example, we choose a B-spline of order 2 and $\Delta = \frac{\pi}{2}$.

Fig. 2. The original and recovered signals: (a) real parts and (b) imaginary parts.

The immediate application of the derived results can also be found in sampling for the Fresnel transform [35,36]. It was shown [15] that the Fresnel transform is just the special case of the LCT when $M = (1, 2z/2\pi, 0, 1)$. In this case, the space $V_M(\phi)$ reduces to the form $V_{(1, 2z/2\pi, 0, 1)}(\phi) = \text{span}_{\lambda \in \mathbb{Z}} \{ \phi(t-\lambda)e^{-j\pi(\lambda^2-z^2)} \}$. Consequently, the derived sampling theorems can also be extended to the Fresnel transform by using the proposed method.

Some of the ideas presented in this paper may be useful for resolving some problems in the LCT domains, such as regular sampling, irregular sampling, generalized sampling, oversampling, and oblique projection [30–34].

4. Conclusion

In this paper, we first extended the notion of shift-invariant spaces to the LCT domain. Then, we derived a sampling theorem of the LCT for regular sampling in function spaces with frames, and further, the theorem was modified to the shift sampling in function spaces by using the Zak transform. We also discussed sampling and reconstructing signals associated with the LCT in the case of Riesz bases. Moreover, some examples and applications of the derived theory were presented. The validity of the theoretical derivations was demonstrated via simulations.

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References


