Extrapolation of Bandlimited Signals in Linear Canonical Transform Domain

Jun Shi, Xuejun Sha, Qinyu Zhang, and Naitong Zhang

Abstract—The linear canonical transform (LCT) has been shown to be a powerful analyzing tool in signal processing. Many results of this transform are already known, including bandlimited extrapolation. The existing algorithm for solving the problem of LCT bandlimited extrapolation is based on signal expansion into a series of generalized prolate spheroidal wave functions (GPSWFs). However, the requirement to compute and store the GPSWFs and the errors due to the series truncation render this algorithm ill-suited for a practical implementation. In this correspondence, we first propose a new formulation of the Gerchberg–Papoulis (GP) algorithm for LCT bandlimited extrapolation. Then, we present a fast convergence algorithm for the new formulation. The classical GP algorithm related to Fourier bandlimited signals is noted as a special case. Moreover, the comparison between the proposed algorithm and the one based on GPSWFs expansion is provided, and the validity of the theoretical derivations is demonstrated via simulations. Several potential applications of the achieved theory are also presented.

Index Terms—Bandlimited signals, linear canonical transform, prolate spheroidal wave functions, signal extrapolation.

I. INTRODUCTION

The linear canonical transform (LCT) is a four-parameter family of linear integral transform [1], [2]. Since the LCT has extra degrees of freedom, it is flexible and has received much attention in optics, quantum mechanics, communications, signal processing, and many other applications [2]–[5].

The LCT with parameter \((a, b, c, d)\) of a signal \(f(t)\), denoted by \(F_{(a,b,c,d)}(u)\), is defined as [2]

\[
F_{(a,b,c,d)}(u) = \mathcal{L}_{(a,b,c,d)}[f(t)](u) = \begin{cases} \int_{-\infty}^{\infty} f(t) \hat{K}_{(d,a-b,c,d)}(u, t) dt, & b \neq 0 \\ \sqrt{4\pi} \int_{-\infty}^{\infty} f(t) \hat{K}_{(d,a-b,c,d)}(u, t) dt, & b = 0 \end{cases}
\]

where \(a, b, c, d\) are real numbers satisfying \(ad - bc = 1\), and the transform kernel \(
\hat{K}_{(a,b,c,d)}(u, t) = \frac{1}{\sqrt{2\pi b}} e^{i\frac{at^2+bt}{b}} e^{-\frac{|u|^2}{2}} \frac{1}{\sqrt{2\pi i}}
\)

In general, we only consider the case of \(b \neq 0\), since the LCT with \(b = 0\) is just a chirp multiplication operation. Correspondingly, the inverse LCT is given by

\[
f(t) = \mathcal{L}^{-1}_{(a,b,c,d)}[F_{(a,b,c,d)}(u)](t) = \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} F_{(a,b,c,d)}(u) \hat{K}_{(d,a-b,c,d)}(u, t) du,
\]

where the inverse transform kernel satisfies \(\hat{K}_{(d,a-b,c,d)}(t, u) = \hat{K}_{(a,b,c,d)}(u, t)\) where * in the superscript denotes complex conjugate. Whenever \((a, b, c, d) = (0, 1, -1, 0)\), (1) reduces to the Fourier transform (FT), and if \((a, b, c, d) = (\cos \alpha, \sin \alpha, -\sin \alpha, \cos \alpha)\), (1) reduces to the fractional Fourier transform (FRFT).

Many results in the Fourier domain including bandlimited extrapolation have currently been extended to the LCT domain [4]–[11]. The bandlimited extrapolation whose goal is to determine a bandlimited function from a known finite segment of the function is encountered in many applications such as radio astronomy, synthetic aperture radar, and communication theory [12]–[15]. Various algorithms were introduced for solving this problem [12]–[14], [16]–[21]. The Gerchberg–Papoulis (GP) algorithm [13], [14] is more attractive because of its relative simplicity (its core computation is a DFT) and is based on successive reduction of error energy. There exist many extensions and modifications of the GP algorithm [17]–[23]. However, those extensions and modifications were derived from the Fourier bandlimited signal viewpoint. In [24], although Sharma and Joshi generalized the GP algorithm to fractional bandlimited signals by projecting the iterations alternately in the time domain and the fractional domain, theoretical derivations such as the convergence issue and iteration numbers were not addressed. Moreover, since the FRFT is a special case of the LCT, and it has been shown that a signal bandlimited in one LCT domain can be also bandlimited in some other LCT domains [9], theoretically, the fractional bandlimited signals to be extrapolated can be bandlimited in one LCT domain also. In this case, the signal should be projected alternately in the time domain and the LCT domain corresponding to the least bandwidth. Thus, it is desirable to derive extrapolation of signals bandlimited in the LCT domain.

In [25], Zhao et al. introduced an algorithm for LCT bandlimited signal extrapolation by expanding the signal into a series of generalized prolate spheroidal wave functions (GPSWFs) [26]. The algorithm, however, is ill-suited for a practical implementation due to its requirement to compute and store the GPSWFs, and it is also limited by series truncation errors.

In this correspondence, we first propose a new formulation of the GP algorithm for extrapolation of LCT bandlimited signals. Then, we present a fast convergence algorithm for the new formulation. The classical GP algorithm related to Fourier bandlimited signals is noted as a special case. Moreover, the comparison between the proposed algorithm and the one based on GPSWFs expansion is provided, and the validity of the theoretical derivations is demonstrated via simulations. Several potential applications of the achieved theory are also presented.

The outline of this correspondence is organized as follows. In the next section, notations and properties of LCT bandlimited signals are briefly introduced. Then, some useful formulas and the concept of GP-SWFs are given. In Section III, a new formulation of the GP algorithm...
associated with the LCT is proposed, and further, a fast convergence algorithm for the new formulation is presented. The comparison between the proposed algorithm and the one based on GPSWFs expansion [25] is provided, and numerical results and several potential applications are given in Section IV. Conclusions appear at the end of the correspondence.

II. PRELIMINARIES

A. Notation

Let $\mathcal{H}$ be a Hilbert space with inner product $(\cdot, \cdot)$, and norm $\| \cdot \| = (\cdot, \cdot)^{\frac{1}{2}}$. Further, let $\mathcal{D}$ be a linear operator with domain $\mathcal{D}(\mathcal{D}) \subseteq \mathcal{H}$ and range in $\mathcal{H}$. The adjoint of the operator $\mathcal{D}$, $\mathcal{D}^*$ is defined using the following equation [16]:

$$
(\mathcal{D}x, y) = (x, \mathcal{D}^*y) \forall x \in \mathcal{D}(\mathcal{D}), y \in \mathcal{D}(\mathcal{D}^*)
$$

(4)

where the inner product $(\cdot, \cdot)$ is given by

$$
(x, y) = \int_{-\infty}^{+\infty} x(t)y^*(t)dt.
$$

(5)

Furthermore, $\mathcal{D}$ is said to be self-adjoint if $\mathcal{D} = \mathcal{D}^*$.

B. Properties of LCT Bandlimited Signals

A signal $f(t)$ is said to be $\Omega$-LCT bandlimited if its energy is finite and its LCT $F_{a,b,c,d}(u)$ vanishes outside the region $(-\Omega, +\Omega)$ [6]

$$
F_{a,b,c,d}(u) = 0 \text{ for } |u| \geq \Omega
$$

(6)

$$
\int_{-\infty}^{+\infty} |f(t)|^2 dt = \int_{-\infty}^{+\infty} |F_{a,b,c,d}(u)|^2 du < +\infty
$$

(7)

where $\Omega > 0$. Let $\mathcal{B}_{a,b,c,d}^\Omega$ be the subspace of $\Omega$-LCT bandlimited functions. Then, for any function $g(t) \in \mathcal{B}_{a,b,c,d}^\Omega$, it follows from (3) and the Cauchy–Schwarz inequality that

$$
|g(t)|^2 \leq \frac{1}{\Omega} \int_{-\Omega}^{+\Omega} G_{a,b,c,d}(u)K_{a,b,c,d}(u,t)du
$$

$$
\leq \frac{1}{\Omega} \int_{-\Omega}^{+\Omega} |G_{a,b,c,d}(u)|^2 du \cdot \int_{-\Omega}^{+\Omega} |K_{a,b,c,d}(u,t)|^2 du
$$

$$
= \frac{\mathcal{E}_G\Omega}{\pi|b|}
$$

(8)

where $G_{a,b,c,d}(u)$ and $\mathcal{E}_G$ denote the LCT and the energy of the function $g(t)$, respectively.

C. Some Useful Formulas

The relationship between the LCT and the FT is given below, which will be used in this subsection

$$
\mathcal{L}^{(a,b,c,d)}[f(t)](u) = \frac{1}{\sqrt{b}} e^{i\frac{ab}{b^2}t^2} \cdot \mathcal{F} \left[ f(t)e^{i\frac{ab}{b^2}t^2} \right] \left( \frac{u}{b} \right)
$$

(9)

where $\mathcal{F}$ denotes the FT operator. (9) can be easily derived using (1). The following formula will be used in this correspondence:

$$
x(t) \ast \partial h(t) = e^{-i\frac{ab}{b^2}t^2} \ast \left[ \left( x(t)e^{i\frac{ab}{b^2}t^2} \right) \ast h(t) \right]
$$

(10)

where $\ast$ denotes the ordinary convolution operator. We refer to $\partial$ as the modified ordinary convolution operator associated with the LCT. (10) can be easily deduced using the method of defining fractional operations proposed in [27]. It is easy to verify that

$$
\mathcal{L}^{(a,b,c,d)}[x(t) \ast \partial h(t)](u) = \sqrt{2\pi} X_{a,b,c,d}(u) H \left( \frac{u}{b} \right)
$$

(11)

where $X_{a,b,c,d}(u)$ denotes the LCT of $x(t)$, and $H(u)$ indicates the FT of $h(t)$. The proof of (11) is as follows.

Proof: It follows from (10), (1), and (9) that

$$
\mathcal{L}^{(a,b,c,d)}[x(t) \ast \partial h(t)](u) = \frac{1}{\sqrt{b}} e^{i\frac{ab}{b^2}t^2} \mathcal{F} \left[ \left( x(t)e^{i\frac{ab}{b^2}t^2} \right) \ast h(t) \right] \left( \frac{u}{b} \right)
$$

(12)

Then, by utilizing the ordinary convolution theorem [2] and (9), we have from (12)

$$
\mathcal{L}^{(a,b,c,d)}[x(t) \ast \partial h(t)](u) = \frac{1}{\sqrt{b}} e^{i\frac{ab}{b^2}t^2} \sqrt{2\pi} \mathcal{F} \left[ \left( x(t)e^{i\frac{ab}{b^2}t^2} \right) \ast h(t) \right] \left( \frac{u}{b} \right)
$$

$$
\mathcal{L}^{(a,b,c,d)}[x(t) \ast \partial h(t)](u) = \sqrt{2\pi} X_{a,b,c,d}(u) H \left( \frac{u}{b} \right)
$$

(13)

This completes the proof of (11).

Note that whenever $(a, b, c, d) = (0, 1, -1, 0)$, (10) reduces to the ordinary convolution of the FT [2]. If $(a, b, c, d) = (\cos \alpha, \sin \alpha, -\sin \alpha, \cos \alpha)$, (10) reduces to the fractional convolution of the FRFT introduced in [28]. Likewise, we conclude that

$$
\mathcal{L}^{(a,b,c,d)}\left[ \sqrt{2\pi} x(t) h(t) \right](u) = \frac{1}{|b|} e^{i\frac{\pi}{b^2}t^2} \mathcal{F} \left[ \left( X_{a,b,c,d}(u) e^{i\frac{\pi}{b^2}t^2} \right) \ast H \left( \frac{u}{b} \right) \right]
$$

(14)

The proof of (14) is similar to that of (11) and is omitted.

D. Generalized Prolate Spheroidal Wave Functions

Let a signal $f(t)$ be $\Omega$-LCT bandlimited and $f(t)$ be given for $t \in (-T, +T)$ with $T > 0$. Define two projection operators $\mathcal{D}_T$ and $\mathcal{B}_T$ as follows:

$$
\mathcal{D}_T f(t) = \begin{cases} f(t), & |t| < T \\ 0, & |t| \geq T \end{cases}
$$

(15)

$$
\mathcal{B}_T f(t) = L^{(d,-b,-c,a)} \left[ \mathcal{D}_T F_{a,b,c,d}(u) \right] (t) = f(t) \Theta \left( \frac{\sin b^{-1} \Omega t}{\pi T} \right)
$$

(16)

Note from (4) that these projection operators $\mathcal{D}_T$ and $\mathcal{B}_T$ are self-adjoint satisfying

$$
\mathcal{D}_T \mathcal{D}_T = \mathcal{D}_T, \quad \mathcal{B}_T \mathcal{B}_T = \mathcal{B}_T.
$$

(17)

Then, by using the modified ordinary convolution operator given in (10), it follows that the GPSWFs $\phi_k(t)$ are the eigenfunctions of the following equation [25, 26].

$$
\left( \mathcal{D}_T \Theta \left( \frac{\sin b^{-1} \Omega t}{\pi T} \right) \right) = \lambda \phi_k(t).
$$

(18)

For more details of GPSWFs, see [25] and [26]. $\phi_k(t)$ are orthogonal in $(-\infty, +\infty)$ and $(-T, T)$, i.e.,

$$
\int_{-\infty}^{+\infty} \phi_k(t)\phi_l(t)dt = \delta_{k,l}, \quad \int_{-T}^{+T} \phi_k(t)\phi_l(t)dt = \lambda_k \delta_{k,l}.
$$

(19)

where the eigenvalues $\lambda_k$ corresponding to $\phi_k(t)$ are real, positive, such that

$$
1 > \lambda_0 > \cdots > \lambda_k > \cdots > 0 \text{ and } \lambda_k \to 0 \text{ as } k \to \infty.
$$

(20)
Since \( \{ \phi_k(t) \mid k \in \mathbb{Z}^+ \} \) forms a complete eigenfunction set over both \( \mathbb{L}^2_{\mathbb{R},\mathbb{C},\mathbb{D}}(t) \) and \( \mathbb{L}^2(-T, T) \) [26], any \( \Omega \)-LCT bandlimited signal \( f(t) \in \mathbb{L}^2_{\mathbb{R},\mathbb{C},\mathbb{D}} \) can be written as a sum

\[
f(t) = \sum_{k=0}^{\infty} \eta_k \phi_k(t), \quad t \in \mathbb{R}
\]

\[
\eta_k = \int_{-\infty}^{\infty} f(t) \phi_k^*(t) dt.
\]

Similarly, an arbitrary signal \( g(t) \in \mathbb{L}^2(-T, T) \) can be expanded into a series in the interval \( (-T, T) \)

\[
g(t) = \sum_{k=0}^{\infty} \zeta_k \phi_k(t), \quad t \in (-T, T)
\]

\[
\zeta_k = \frac{1}{\lambda_k} \int_{-T}^{T} f(t) \phi_k^*(t) dt.
\]

It is easy to verify that

\[
\sum_{k=0}^{\infty} |\eta_k|^2 < +\infty, \quad \sum_{k=0}^{\infty} |\lambda_k|^2 < +\infty.
\]

III. EXTRAPOLATION OF BANDLIMITED SIGNALS IN THE LINEAR
CANONICAL TRANSFORM DOMAIN

A. A New Formulation of the Gerchberg–Papoulis Algorithm
Associated With the LCT

The problem of LCT bandlimited extrapolation can be formulated as: given only a finite segment \( x(t), t \in (-T, +T) \) of an \( \Omega \)-LCT bandlimited signal \( f(t) \), find \( f(t) \) for \( t \) not in \( (-T, +T) \). For this purpose, we first introduce the following theorem.

Theorem 1: The GPSWFs \( \phi_k(t) \) are \( \Omega \)-LCT bandlimited satisfying

\[
\Phi_{\mathbb{L}_{\mathbb{R},\mathbb{C},\mathbb{D}}}(u) = 0 \quad \text{for} \quad |u| \geq \Omega
\]

where \( \Phi_{\mathbb{L}_{\mathbb{R},\mathbb{C},\mathbb{D}}}(u) \) denotes the LCT of \( \phi_k(t) \).

Proof: Let \( \Phi_{\mathbb{L}_{\mathbb{R},\mathbb{C},\mathbb{D}}}(u) \) and \( \Phi_{\mathbb{L}_{\mathbb{R},\mathbb{C},\mathbb{D}}}(u) \) denote the LCT of \( \mathcal{D}_T \phi_k(t) \) and \( \phi_k(t) \), respectively. Then, by applying (10), (11), and (15), we have from (18)

\[
\mathcal{D}_T \Phi_{\mathbb{L}_{\mathbb{R},\mathbb{C},\mathbb{D}}}(u) = \lambda_k \Phi_{\mathbb{L}_{\mathbb{R},\mathbb{C},\mathbb{D}}}(u)
\]

Next, it follows from (27) and (15) that

\[
\mathcal{D}_T \Phi_{\mathbb{L}_{\mathbb{R},\mathbb{C},\mathbb{D}}}(u) = \Phi_{\mathbb{L}_{\mathbb{R},\mathbb{C},\mathbb{D}}}(u).
\]

Then, applying the inverse LCT and (29) yields

\[
\phi_k(t) \Theta \frac{\sin(h^{-1}\Omega t)}{\pi t} = \phi_k(t).
\]

Theorem 2: For an arbitrary \( \Omega \)-LCT bandlimited signal \( f(t) \), if a finite segment \( \mathcal{D}_T f(t) \) is given, we form a new formulation of the GP algorithm as follows:

\[
\begin{cases}
    f_{n+1}(t) &= [f_n(t) + \mathcal{D}_T (f(t) - f_n(t))] \Theta \frac{\sin(h^{-1}\Omega t)}{\pi t} \\
    f_0(t) &= 0
\end{cases}
\]

where \( n \geq 0 \), then, for any \( t \),

\[
f_n(t) \rightarrow f(t) \quad \text{as} \quad n \rightarrow \infty.
\]

Proof: Since \( f(t) \) is \( \Omega \)-LCT bandlimited, we have from (21)

\[
f(t) = \sum_{k=0}^{\infty} \eta_k \phi_k(t), \quad t \in \mathbb{R}
\]

Clearly, \( f_n(t) \) is also \( \Omega \)-LCT bandlimited, it follows that

\[
f_n(t) = \sum_{k=0}^{\infty} \eta_{n,k} \phi_k(t), \quad t \in \mathbb{R}
\]

Next, by applying (18), we obtain

\[
(\mathcal{D}_T \phi_k(t)) \Theta \frac{\sin(h^{-1}\Omega t)}{\pi t} = \lambda_k \phi_k(t).
\]

Inserting (33), (34), (35), and (36) into (31) yields

\[
\begin{cases}
    \sum_{k=0}^{\infty} \eta_{n+1,k} \phi_k(t) &= \left( \sum_{k=0}^{\infty} (\eta_{n,k} + \lambda_k (\eta_n - \eta_{n,k})) \phi_k(t) \right) \\
    \sum_{k=0}^{\infty} \eta_{0,k} \phi_k(t) &= 0.
\end{cases}
\]

Then, in view of (19), (36) becomes

\[
\begin{cases}
    \eta_{n+1,k} &= (1 - \lambda_k) \eta_{n,k} + \lambda_k \eta_k \\
    \eta_{0,k} &= 0.
\end{cases}
\]

Solving for \( \eta_{n,k} \), we obtain

\[
\eta_{n,k} = \eta_k (1 - (1 - \lambda_k)^n).
\]

Next, substituting (38) into (34) results in

\[
f_n(t) = \sum_{k=0}^{\infty} \eta_k (1 - (1 - \lambda_k)^n) \phi_k(t)
\]

for \( t \in \mathbb{R} \). From (39), the error \( e_n(t) = f(t) - f_n(t) \) is given by

\[
e_n(t) = \sum_{k=0}^{\infty} \eta_k (1 - \lambda_k)^n \phi_k(t),
\]

and the energy of \( e_n(t) \) can be written as

\[
\mathcal{E}_n = \int_{-\infty}^{+\infty} |e_n(t)|^2 dt = \sum_{k=0}^{\infty} |\eta_k|^2 (1 - \lambda_k)^{2n}.
\]

Then, it follows from (25) that

\[
\mathcal{E}_f = \int_{-\infty}^{+\infty} |f(t)|^2 dt = \sum_{k=0}^{\infty} |\eta_k|^2 < +\infty.
\]

Hence, given \( \epsilon > 0 \), one can find an integer \( N \) such that

\[
\sum_{k>N} |\eta_k|^2 < \epsilon.
\]
Next, (20) implies that
\[ 1 - \lambda_k < 1, \quad \text{and} \quad 1 - \lambda_k \leq 1 - \lambda_N \quad \text{for} \quad k \leq N. \tag{44} \]

By applying (42), (43), and (44), we have from (41)
\[
\begin{align*}
E_{en} &= \sum_{k=0}^{N} \eta_k^2 (1 - \lambda_k)^{2n} + \sum_{k>N} \eta_k^2 (1 - \lambda_k)^{2n} \\
&< (1 - \lambda_N)^{2n} \sum_{k=0}^{N} \eta_k^2 + \sum_{k>N} \eta_k^2 \\
&< (1 - \lambda_N)^{2n} E_f + \epsilon. \tag{45}
\end{align*}
\]

Since \( \epsilon \) is arbitrary and
\[ (1 - \lambda_N)^{2n} \to 0 \quad \text{with} \quad n \to \infty, \tag{46} \]

it follows that
\[ E_{en} \to 0 \quad \text{with} \quad n \to \infty. \tag{47} \]

Moreover, let \( F_{(a,b,c,d)}(u) \) and \( F_{n,(a,b,c,d)}(u) \) denote the LCT of \( f(t) \) and \( f_n(t) \) respectively. Since \( f(t) \) and \( f_n(t) \) are \( \Omega \)-LCT bandlimited, the LCT of \( e_n(t) \), denoted by \( F_{n,(a,b,c,d)}(u) \), satisfies
\[ F_{n,(a,b,c,d)}(u) = F_{(a,b,c,d)}(u) - F_{n,(a,b,c,d)}(u) \tag{48} \]

which is zero for \(|u| \geq \Omega \). Then, it follows from (8) that
\[ |e_n(t)| \leq \sqrt{\frac{E_{en}}{\pi |b|}}. \tag{49} \]

Thus, \( e_n(t) \to 0 \) with \( n \to \infty \) and (32) is established. This completes the proof of Theorem 2.

Since the LCT is a generalization of the FT and the FRFT [2], we can easily derive the following corollaries.

**Corollary 1:** When \((a, b, c, d) = (0, 1, -1, 0)\), Theorem 2 reduces to the classical GP algorithm related to Fourier bandlimited signals introduced in [13] and [14].

**Corollary 2:** If \((a, b, c, d) = (\cos \alpha, \sin \alpha, -\sin \alpha, \cos \alpha)\), Theorem 2 provides a new formulation of the GP algorithm for extrapolation of fractional bandlimited signals [28].

The proof of Corollary 1 and Corollary 2 is easy and is omitted. Moreover, by choosing different parameter \((a, b, c, d)\), some other extrapolation strategies constructed by other special cases of the LCT can be also derived using the achieved results.

**B. A Fast Convergence Algorithm for the Proposed Formulation**

For simplification, we introduce some notations
\[ x(t) = \mathcal{D}_f f(t), \quad \mathcal{M}_f f(t) = (1 - \mathcal{D}_f) f(t). \tag{50} \]

The proposed formulation in Theorem 2 provides a simple algorithm for determining an \( \Omega \)-LCT bandlimited signal \( f(t) \) with only a finite segment \( x(t) \) being given. The algorithm is inspired by the identity
\[ f(t) = x(t) + \mathcal{M}_f f(t) \tag{51} \]

which immediately suggests the fundamental iterations
\[
\begin{align*}
x_n(t) &= x(t) + \mathcal{M}_f \mathcal{M}_f x_{n-1}(t), \quad n \geq 1 \\
x_0(t) &= x(t).
\end{align*}
\]

In each iteration of (52), we need to compute a LCT and inverse LCT. Assuming \( M_1 \) iterations of (52) with LCT implementations on samples \( N \) we get complexity of order \( O(2M_1 N \log N) \). Moreover, if the iteration is terminated at the \( n \)th step of (52), and the recovery of \( f(t) \) can be derived, from (41) the resulting mean-square error is given by
\[ E_{en} = \sum_{k=0}^{\infty} \eta_k^2 (1 - \lambda_k)^{2n} \tag{53} \]

where \( \lambda_k \) are the eigenvalues of (18). The \( \lambda_k \)'s increase as the product \( \Omega T \) increases and \( \lambda_k \to 1 \) with \( \Omega T \to \infty \) [26]. Thus, if \( \Omega T \) is large, the required number of iterations is small. Moreover, the error \( E_{en} \) depends also on the coefficients \( \eta_k \). If \( n \) is small, then the contribution of the high-order terms in (53) is small. However, as \( n \) increases, their effect becomes relatively more significant since \((1 - \lambda_k) \to 1 \) with \( k \to \infty \). Thus, the iteration in (52) converges usually slowly. In the following, we will introduce a fast convergence algorithm for the proposed formulation in Theorem 2.

The \( n \)th iteration of (51) can be expressed as
\[ x_n(t) = x(t) + \sum_{i=1}^{n} (\mathcal{D}_f^{-1} \mathcal{M}_f) x(t) \tag{54} \]

from which along with (51) it follows that in each iteration, the proposed formulation in Theorem 2 approximates the missing segment \( \mathcal{M}_f f(t) \) of the \( \Omega \)-LCT bandlimited signal \( f(t) \) with a linear combination of the functions \( (\mathcal{D}_f^{-1} \mathcal{M}_f) x(t), r = 1, \ldots, n \), where all coefficients are equal to one. Therefore, choosing a better approximation to \( \mathcal{M}_f f(t) \) with these same functions will yield fast convergence, i.e.,
\[ \arg \min_{(p_1, p_2, \ldots, p_n)} \left\| \mathcal{M}_f f(t) - \sum_{i=1}^{n} p_i (\mathcal{D}_f^{-1} \mathcal{M}_f) x(t) \right\|^2 \tag{55} \]

where the coefficients \( p_i, r = 1, \ldots, n \), achieved by the orthogonal projection of \( \mathcal{M}_f f(t) \) onto the following subspace:
\[ \mathcal{S} = \text{span} \left\{ (\mathcal{D}_f^{-1} \mathcal{M}_f) x(t), (\mathcal{D}_f^{-1} \mathcal{M}_f)^2 x(t), \ldots, (\mathcal{D}_f^{-1} \mathcal{M}_f)^n x(t) \right\}. \tag{56} \]

It follows that (55) can be achieved by projecting \( \mathcal{M}_f f(t) \) on an orthonormal basis of the subspace \( \mathcal{S} \), which can be obtained using the modified Gram–Schmidt (MGS) algorithm [29]. For this purpose, the inner products between \( \mathcal{M}_f f(t) \) and \( (\mathcal{D}_f^{-1} \mathcal{M}_f) x(t), r = 1, \ldots, n \), are needed. Although \( \mathcal{M}_f f(t) \) is unknown, we can calculate these inner products by a recursive procedure, i.e.,
\[
\langle \mathcal{M}_f f(t), (\mathcal{D}_f^{-1} \mathcal{M}_f)^r x(t) \rangle = \langle \mathcal{M}_f f(t), (\mathcal{D}_f^{-1} \mathcal{M}_f)^{r-1} x(t) \rangle - \langle x, (\mathcal{D}_f^{-1} \mathcal{M}_f)^r x(t) \rangle \tag{57}
\]

which is deduced using (16) and the self-adjoint and idempotent properties of projection operators. The initialization step of (57) is given by
\[
\langle \mathcal{M}_f f(t), (\mathcal{D}_f^{-1} \mathcal{M}_f)^0 x(t) \rangle = \langle x, (\mathcal{D}_f^{-1} \mathcal{M}_f)^1 x(t) \rangle \tag{58}
\]

Moreover, we also need to know the inner products between the functions \( (\mathcal{D}_f^{-1} \mathcal{M}_f)^r x(t), r = 1, \ldots, n \), when using the MGS algorithm, i.e.,
\[ \langle (\mathcal{D}_f^{-1} \mathcal{M}_f)^r x(t), (\mathcal{D}_f^{-1} \mathcal{M}_f)^s x(t) \rangle = \langle (\mathcal{D}_f^{-1} \mathcal{M}_f)^{r-s} x(t), (\mathcal{D}_f^{-1} \mathcal{M}_f)^{s-1} x(t) \rangle \tag{59} \]

which implies that the inner product between \( (\mathcal{D}_f^{-1} \mathcal{M}_f)^r x(t) \) and \( (\mathcal{D}_f^{-1} \mathcal{M}_f)^s x(t) \) can be calculated over a finite interval and is dependent on \( k + l \) not \( k \) and \( l \) separately. Note that the required computations in (59) can be used in (57). The identity follows from the self-adjoint and idempotent properties of orthogonal projection operators.
In the $n$th step of the fast convergence algorithm only $O(2n)$ inner products need to be calculated, and therefore, the fast convergence algorithm can reduce the amount of computation in (52).

A more computationally attractive approach to deriving the optimization of (55) is bases on (59). Let $y = [y_1, y_2, \cdots, y_n]^T$ where

$$y_l = \left(\mathcal{M}_r f, (\mathcal{M}_r \mathcal{B}_2)^l x\right), \quad l = 1, \cdots, n,$$

and denote by $A$ a vector of functions, written as

$$A = \left[(\mathcal{M}_r \mathcal{B}_3)^l x, (\mathcal{M}_r \mathcal{B}_3)^2 x \cdots (\mathcal{M}_r \mathcal{B}_3)^n x\right].$$

Next, let $H$ denote a matrix with its element satisfying

$$H_{k,l} = \left(\mathcal{M}_r \mathcal{B}_3)^k x, (\mathcal{M}_r \mathcal{B}_3)^l x\right), \quad k, l = 1, \cdots, n.$$ Solving for the optimization of (55), the optimal coefficient vector $\mathbf{p} = [p_1, p_2, \cdots, p_n]^T$ satisfies

$$H \mathbf{p} = \mathbf{y}$$

where $\mathbf{y}$ can be calculated by (57). Then, the best approximation to $\mathcal{M}_r f(t)$ is given by

$$\mathbf{A} = AH^{-1} \mathbf{y}.$$ It follows from (59) that the matrix $H$ is a Hankel matrix [29]. Fast algorithms [29], [30] exist for solving the linear Hankel system of (63), where $y$ and $H$ can be updated during the iterations, and the optimization will be computed once after some iterations. Therefore, the fast convergence algorithm, for $M_2$ steps, using MGS algorithm requires about $O(2M_2 N \log N + N \log M_2)$ operations. Using a fast inverse of a Hankel matrix, we get about $O(2M_2 N \log N + N + M_2^2)$ operations. Usually, $M_2 \ll M_1$, so there is a significant reduction in computational complexity. Moreover, this fast convergence algorithm is very suitable for parallel implementation, i.e., next iterations can be done in parallel to the computation of the improved approximation in the current iteration.

IV. COMPARISON WITH THE EXISTING METHOD AND POTENTIAL APPLICATIONS

A. Theoretical Analysis

It follows from [25] that the existing method using GPSWFs expansion for determining a LCT bandlimited signal $f(t)\forall t$, or finding $F_{(a,b,c,d)}(u)$ and the proposed algorithm hold under the same condition that a segment $\mathcal{D}_T f(t)$ of $f(t)$ is known. In [25], the given segment $\mathcal{D}_T f(t)$ can be expanded into a series shown in (23) where the coefficients $\zeta_k$ are given by

$$\zeta_k = \frac{1}{\lambda_k} \int_{-T}^{+T} (\mathcal{D}_T f(t)) \phi_k(t)dt.$$ Then, the desired $\Omega$-LCT bandlimited signal $f(t)$ can be written as

$$f(t) = \sum_{k=0}^{\infty} \zeta_k (\mathcal{D}_T \phi_k(t)).$$ Taking the LCT with parameter $(a,b,c,d)$ of both sides of (66), we obtain

$$F_{(a,b,c,d)}(u) = \sum_{k=0}^{\infty} \zeta_k \mathcal{L}^{(a,b,c,d)}(\mathcal{D}_T \phi_k(t))(u).$$ Next, it follows from (28) and (14) that

$$\mathcal{L}^{(a,b,c,d)}(\mathcal{D}_T \phi_k(t))(u) = \frac{\hat{\Phi}_{k,(a,b,c,d)}(u)}{\pi u} \cdot \frac{\sin(b^{-1} Tu)}{u}$$

$$= e^{j \frac{a}{\pi u}} \int_{-\Omega}^{\Omega} \Phi_{k,(a,b,c,d)}(v) e^{-j \frac{a}{\pi u}} \sin \left(b^{-1} (u-v)\right)dv.\quad (68)$$

Then, inserting (27) and (28) into (68) results in

$$\lambda_k \Phi_{k,(a,b,c,d)}(u) e^{-j \frac{a}{\pi u}} \sin \left(b^{-1} (u-v)\right)dv$$

$$= \int_{-\Omega}^{+\Omega} \Phi_{k,(a,b,c,d)}(v) e^{-j \frac{a}{\pi u}} \sin \left(b^{-1} (u-v)\right)dv$$

$$\lambda_k \Phi_{k,(a,b,c,d)}(u) e^{-j \frac{a}{\pi u}} \sin \left(b^{-1} (u-v)\right)dv$$

$$= \lambda_k \Phi_{k,(a,b,c,d)}(u) e^{-j \frac{a}{\pi u}} \sin \left(b^{-1} (u-v)\right)dv$$

$$= \int_{-\Omega}^{+\Omega} \Phi_{k,(a,b,c,d)}(v) e^{-j \frac{a}{\pi u}} \sin \left(b^{-1} (u-v)\right)dv.$$ (73)

Combining (69) with (71) yields

$$\Phi_{k,(a,b,c,d)}(u) = \rho \Phi_{k}(\frac{T u}{\Omega}) e^{j \frac{a}{\pi u}} (\frac{\Omega \lambda_k}{T})^2 + j \frac{a}{\pi u} \omega$$

for every $u \in (-\Omega, +\Omega)$, where $\rho \in \mathbb{R}$. To determine $\rho$, we use (19) and the Parseval’s formula of the LCT [2], obtaining

$$\int_{-\infty}^{+\infty} |\hat{\phi}_k(t)|^2 dt = \rho \int_{-\Omega}^{+\Omega} |\hat{\phi}_k(u)|^2 du$$

$$= \frac{\rho^2 \Omega}{T} \int_{-T}^{+T} |\hat{\phi}_k(x)|^2 dx = \frac{\rho^2 \Omega \lambda_k}{T}$$

so that

$$\rho = \sqrt{\frac{T}{\Omega \lambda_k}}.$$ By applying (72), (28), (68), (67), and (74), we obtain

$$F_{(a,b,c,d)}(u) = \sum_{k=0}^{\infty} \sqrt{\frac{T \lambda_k}{\Omega}} \phi_k \left(\frac{T u}{\Omega}\right) e^{j \frac{a}{\pi u}} (\frac{\Omega \lambda_k}{T})^2 + j \frac{a}{\pi u} \omega$$

for every $u \in (-\Omega, +\Omega)$.

Note from (75) that the existing method [25] of LCT bandlimited extrapolation involves the computation and storage of the GPSWFs $\phi_k(t)$, an extremely difficult numerical task, and it is also limited by series truncation errors. By contrast, the proposed algorithm does not need to compute and store the GPSWFs and does not have series truncation errors. It only requires the computation of direct and inverse LCTs and is computationally efficient and easy to implement.
C. Potential Applications

The proposed algorithm for extrapolation of signals bandlimited in the LCT domain states that a LCT bandlimited signal can be recovered exactly from its partial time domain information.

The immediate application of the proposed algorithm can be found in spectral estimation in communication systems constructed by the LCT [3] and limited-angle tomography in imaging reconstruction, where only limited observation data are available. In addition, the proposed algorithm is an efficient solution for missing, lost or misreceived chirp-like signals used in radar, sonar and communications. Since free space propagation in the Fresnel approximation, transmission through thin lenses, propagation through quadratic graded-index media, and their arbitrary combinations fall into the first-order optical systems, which can be described efficiently by the LCT [2], the proposed algorithm can be also useful for increasing the resolution and/or field of view in an optical imaging system and reconstructing wavefields or beam profiles from partial measurements.

V. Conclusion

In this correspondence, we proposed an efficient algorithm for extrapolation of bandlimited signals in the linear canonical transform domain. The classical Gerchberg–Papoulis algorithm related to Fourier bandlimited signals is shown to be a special case of it. Moreover, the comparison between the proposed algorithm and the existing one based on the generalized prolate spheroidal wave functions expansion is provided, and the validity of the theoretical derivations is demonstrated via simulations. Several potential applications of the achieved theory are also presented.

ACKNOWLEDGMENT

The authors would like to thank the associate editor and the anonymous reviewers for their detailed and constructive comments that have helped the presentation of this correspondence.

REFERENCES

The Consistency of MDL for Linear Regression Models
With Increasing Signal-to-Noise Ratio

Daniel F. Schmidt and Enes Makalic

Abstract—Recent work by Ding and Kay has demonstrated that the Bayesian information criterion (BIC) is an inconsistent estimator of model order in nested model selection as the noise variance $\tau^* \rightarrow 0$. Unfortunately, Ding and Kay have erroneously concluded that the minimum description length (MDL) principle also leads to inconsistent estimates of model order in this setting by equating BIC with MDL. This correspondence shows that only the earlier MDL criterion based on asymptotic assumptions has this problem, and proves that the new MDL linear regression criteria based on normalized maximum likelihood and Bayesian mixture codes satisfy the notion of consistency as $\tau^* \rightarrow 0$. The main result may be used as a basis to easily establish similar consistency results for other closely related information theoretic regression criteria.

Index Terms—Consistency, linear models, minimum description length, model selection.

I. INTRODUCTION

A model selection criterion is consistent if it almost surely selects the true generating model as the sample size $n \rightarrow \infty$, assuming the true data generating model is included in the set of candidate models under consideration. A recent paper by Ding and Kay [1] has examined a slightly different notion of model selection consistency in the context of linear regression models, in which the sample size $n$ is fixed and the noise variance $\tau^*$ tends to zero. In the linear regression setting, the data $y = (y_1, \ldots, y_n)'$ is assumed to have been generated from a linear combination of explanatory variables, or covariates, with additive noise, that is

$$y = X\beta^* + \varepsilon \quad (1)$$

where $X = (x_1, \ldots, x_q)$ is an $(n \times q)$ design matrix, $\beta^* \in \mathbb{R}^q$ is a vector of unknown, true, regression coefficients, and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)'$ is a vector of disturbances, distributed as per $\varepsilon_i \sim N(0, \tau^*)$. It is common to assume that the model selection problem is nested, in the sense that the sub-design matrices considered by the model selection criteria are of the form $X_k = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_q)$, for $k \in \{1, \ldots, q\}$, and that $\beta^*_k \equiv 0$ for all $i > k^*$. The task of a model selection criterion is to infer the true order $k^* > 0$.

A standard approach to estimating $k^*$ is by minimization of a penalized negative log-likelihood. These methods advocate choosing the model that minimizes the sum of the negative log-likelihood and a suitable complexity penalty, that is,

$$\hat{k} = \arg\min_{k \in \{1, \ldots, q\}} \left\{ \frac{n}{2} \log \hat{\tau}_k + \alpha_k \right\}$$

where $\hat{\tau}_k$ is the maximum likelihood estimate of $\tau^*$ for model order $k$ given by

$$\hat{\tau}_k = \frac{y' \left( I_n - X_k (X_k' X_k)^{-1} X_k' \right) y}{n}.$$