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Non-unique and non-bang-bang controls in some linear time-optimal problems

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In this paper, the time-optimal control for multidimensional linear systems is studied in cases when Kalman controllability conditions does hold, but Gamkrelidze generic position condition does not. The main contribution of this study is the demonstration that time-optimal control for multidimensional linear systems is not restricted to the classical discontinuous bang-bang form with a finite number of switching points. Instead, an infinite number of controls, including continuous ones, can be used to take the system to the origin in minimum time. As an illustrative example for this class of problems, the controllable movement of material point in multidimensional space is considered.

Keywords: controllability; generic position conditions; multidimensional linear systems; time-optimal control.

1. Introduction

Optimal control is an important area of control theory, which allows designing more efficient and sophisticated control systems. The theory of optimal control studies the problem of finding a control law for a given system, such that a certain cost function is minimized (see Pontryagin et al., 1962; Bellman & Dreyfus, 1962; Bryson & Ho, 1969; Afanas’ev et al., 1996).

Optimization techniques have been widely studied by many authors as Pontryagin et al. (1962), Bellman & Dreyfus (1962), Bryson & Ho (1969), Afanas’ev et al. (1996), Boltyanski & Poznyak (2011), among many other mathematicians and scientists (see Gamkrelidze, 1958; Kalman, 1961; Filippov, 1959).

However, in those works, the time-optimal control problem is studied only in cases when Kalman controllability and Gamkrelidze generic position conditions hold (see Pontryagin et al., 1962; Bellman & Dreyfus, 1962; Bryson & Ho, 1969; Afanas’ev et al., 1996; Boltyanski & Poznyak, 2011; Gamkrelidze, 1958).

It is important to mention that Gamkrelidze (see Pontryagin et al., 1962; Gamkrelidze, 1958) has shown that for linear systems, generic position conditions are sufficient to obtain, from Pontryagin’s maximum principle, a unique optimal control which takes only limit values. However, what happens when these conditions are not satisfied has not been studied yet.

In the present paper, it will be assumed that the Kalman controllability condition is fulfilled, while the Gamkrelidze generic position condition does not hold. Both these conditions will be briefly recalled in the following section. To the best of authors’ knowledge, this case has not been studied before. But, as it is shown below, very interesting situations may arise.

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In that sense, the goal of this work was not to propose a new control approach but presenting a theoretical analysis of special cases of time-optimal control theory.

The main contribution of this study is the demonstration that the time-optimal control for multidimensional linear systems is not necessarily restricted to the classical discontinuous bang-bang form, with a finite number of switching points; but in some cases, an infinite number of controls, including continuous ones, can be used to take the system to the origin in minimum time.

In order to validate the analysis carried out along this work, a simple but very illustrative mechanical example of this class of linear systems is considered. Such an example is defined as the problem of taking a material point from a given initial state \( X(0) = X_0, X(0) = X_0, \) in space \( \mathbb{R}^n, \) with \( n \geq 2, \)

to the origin \( X(T) = 0, X(T) = 0. \) Using this example, it will be shown that the movements of a material point in \( \mathbb{R}^n \) are controllable only when \( n \) independent controls are applied on the material point. However, such a controllable material point does not fulfill Gamkrelidze generic position conditions in general (see Pontryagin et al., 1962; Bellman & Dreyfus, 1962; Bryson & Ho, 1969; Afanas'ev et al., 1996; Boltianski & Poznyak, 2011; Gamkrelidze, 1958). But most important, it will be shown that there exists an infinity of time-optimal controls, including continuous ones, capable of solving the time-optimal problem. Notice that the problem just described is a very well-known model for control of satellites when gravity forces can be neglected. Besides, similar problems can be found in the analysis and control of the displacement of robotic manipulators.

The rest of the paper is organized as follows. Controllability and generic position conditions are briefly reviewed in Section 2. Section 3 is devoted to the analysis of controllability and generic position conditions for movement of material point in multidimensional space. Some ideas for constructing an infinity time-optimal control are explained in Section 4. Numeric results are presented in Section 5. Finally, in Section 6, some concluding remarks are given.

2. Controllability and generic position conditions for multidimensional linear systems

Consider the classical time-optimal problem for linear systems

\[
X(t) = AX(t) + BU(t), \quad X(t) \in \mathbb{R}^n, \quad U(t) \in \mathbb{R}^m,
\]

\( X(0) = X_0, \quad X(T) = 0, \) 

(2.1)

where \( U(t) = [u_1(t), u_2(t), \ldots, u_m(t)]^T \) is a control vector subjected to the constraints

\[
0 \leq |u_i(t)| \leq 1, \quad 1 \leq i \leq m.
\]

(2.2)

Then, control \( U(t) \) must be chosen such that the elapsed time \( T \) is minimized, i.e.,

\[
\int_0^T \frac{dt}{U} \rightarrow \inf.
\]

(2.3)

The Kalman controllability condition (see Kalman, 1961) for this problem is

\[
\text{rank}(\mathcal{C}) = n.
\]

(2.4)

where the controllability matrix \( \mathcal{C} \) is \( \mathcal{C} = [B \ AB \ A^2B \ \ldots \ A^{n-1}B] \).

If the Kalman controllability condition is satisfied, then, according to Filippov theorem (see Filippov, 1959), it may exist at least one time-optimal control for the previous problem (2.1–2.3).

On the other hand, the so-called generic position conditions, firstly introduced by Gamkrelidze (1958), play a very important role in the theory of time-optimal control.
The Gamkrelidze generic position conditions state that there exists a unique time-optimal control for system (2.1–2.3), if all matrices

\[ G_j = [b_j \ A^j b_j \ A^{j+1} b_j \ ... \ A^{j+m-1} b_j] , \]

where \( b_j \) denotes the \( j \)-th column of matrix \( B \), are nonsingular, i.e.,

\[ \det G_j \neq 0, \ (j = 1, 2, \ldots, m). \]  \hspace{1cm} (2.5)

Also, all the components of such a control have bang-bang form.

In other words, if condition (2.5) is fulfilled, then there exists only one time-optimal control for problem (2.1–2.3), which has bang-bang form, i.e., all components \( u_i(t) \) of this optimal control only take values \( \pm 1 \).

\[ u_i(t) = \pm 1, \ 1 \leq i \leq m. \]  \hspace{1cm} (2.6)

It is clear that for scalar control \( U(t) = u(t) \), the controllability condition (2.4) and the generic position conditions (2.5) both coincide. However, for the case of multidimensional control, where \( m \geq 2 \), conditions (2.4) and (2.5) may impose different restrictions on matrices \( A \) and \( B \).

For example, consider the multidimensional linear system with \( n = m = 2 \) and where

\[ A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} , \]

then the Kalman condition (2.6) is fulfilled independently on the values for elements \( a_{ij} \) of matrix \( A \) because the rank of controllability matrix \( \mathcal{C} \) is \( \text{rank}(\mathcal{C}) = \text{rank}(B) = 2 \).

However, the Gamkrelidze generic position conditions lead to following restrictions:

\[ \det(G_1) = \det \begin{bmatrix} 1 & a_{11} \\ 0 & a_{21} \end{bmatrix} \neq 0 \Rightarrow a_{21} \neq 0, \]

\[ \det(G_2) = \det \begin{bmatrix} 0 & a_{11} \\ 1 & a_{21} \end{bmatrix} \neq 0 \Rightarrow a_{11} \neq 0. \]  \hspace{1cm} (2.7)

Therefore, it becomes clear that for the general case, conditions (2.4) and (2.5) do not coincide.

If condition (2.4) holds, but condition (2.5) is not satisfied, then, as it is shown below, time-optimal control exists but may not be unique, or it may be different from bang-bang control, i.e., different from (2.6). In following section, one example is considered to show both analytically and numerically, the existence of such situations.

3. Controllability and generic position conditions for material point motion in multidimensional space

As a simple example, consider the problem of moving a material point from an arbitrary initial condition \( (X(0) = x_0, X(0) = y_0) \) to the origin, \( (X(T) = 0, Y(T) = 0) \), in minimum time \( T \).
Equations describing this problem appear next

\[ \hat{X}(t) = BU(t), \]  
\[ X(0) = X_0, \quad \hat{X}(0) = \hat{X}_0, \]  
\[ X(T) = 0, \quad \hat{X}(T) = 0, \]  
\[ 0 \leq |y_i(t)| \leq 1, \quad 1 \leq i \leq m, \]  
\[ \int_0^T dt \to \inf. \]  

Here, \( X(t) \in \mathbb{R}^n, U(t) \in \mathbb{R}^m \) and \( B = [b_{ij}] \), with \( i = 1, 2, \ldots, n \) and \( j = 1, 2, \ldots, m \).

Hence, without loss of generality, consider now the \( 2n \) dimension phase vector \( Y(t) \in \mathbb{R}^{2n} \) defined by \( y_1(t) = x_1(t), \ldots, y_n(t) = x_n(t), y_{n+1}(t) = \dot{x}_1(t), \ldots, y_{2n}(t) = x_n(t) \), which can be rearranged as

\[ Y(t) = AY(t) + B_1(U(t)), \]
\[ Y(0) = Y_0, \]  

with \( A \) and \( B_1 \) having dimensions \( 2n \times 2n \) and \( 2n \times m \), respectively, i.e.,

\[ Y(t) \in \mathbb{R}^{2n}, \quad A = \begin{bmatrix} 0_{n \times m} & I_n \\ 0_{n \times n} & 0_{n \times n} \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0_{n \times m} \\ B \end{bmatrix}, \]

and where \( I_n \) is the identity matrix of dimension \( n \times n \).

Consequently, the controllability matrix \( \mathcal{C} \) turns out to be

\[ \mathcal{C} = \begin{bmatrix} B_1 & AB_1 & A^2B_1 & \ldots & A^{2n-1}B_1 \end{bmatrix} \]
\[ = \begin{bmatrix} 0_{n \times m} & B & 0_{n \times m} & \ldots & 0_{n \times m} \\ B & 0_{n \times m} & 0_{n \times m} & \ldots & 0_{n \times m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{n \times m} & 0_{n \times m} & 0_{n \times m} & \ldots & 0_{n \times m} \end{bmatrix}_{2n \times 2n}. \]

For \( m < n \), the rank of \( \mathcal{C} \) is less than or equal to \( 2m \); hence, such a rank becomes less than \( 2n \) and the augmented system (3.6) is not controllable for \( m < n \); \( \text{rank}(\mathcal{C}) \leq 2\text{rank}(B) \leq 2m < 2n \). In this case, the time-optimal control for system (2.1) does not exist.

On the other hand, when \( m = n \), the rank of \( \mathcal{C} \) may be equal to \( 2n \) if \( \text{rank}(B) = n \), and the augmented system (3.6) may be controllable.

Therefore, necessary and sufficient conditions for controllability of system (3.1–3.5) are: (1) \( m = n \) and (2) the rank of matrix \( B \) is equal to \( n \), i.e., \( B \) is nonsingular.

When previous conditions are fulfilled, the rank of matrix \( \mathcal{C} \) equals exactly to \( 2n \), and the systems (3.1–3.5) and (3.6) are controllable. In this case, there exists a time-optimal solution for system (3.1–3.5).

For the controllable system (3.6), generic position conditions are not satisfied because...
and as consequence
\[ \det(G_j) = 0, \quad j = 1, 2, \ldots, n. \] (3.7)

So, equations of controllable motion for a material point in \( \mathbb{R}^n \) are:
\[ \ddot{X}(t) = BU(t), \quad 0 \leq |u_j(t)| \leq 1, \quad 1 \leq j \leq n, \] (3.8)
where \( B \) is a \( n \times n \) nonsingular matrix.

Now, a new control vector is introduced, which is \( W(t) = [w_1(t), w_2(t), \ldots, w_n(t)]^T \), satisfying
\[ W(t) = BU(t), \quad 0 \leq |w_j(t)| \leq \gamma_i = \sum_{j=1}^{n} |b_{ij}|. \] (3.9)

Therefore, as matrix \( B \) is nonsingular, then for any vector \( U(t) \in \mathbb{R}^n \), it corresponds one vector \( W(t) \in \mathbb{R}^n \) and vice versa. Thus, by using control (3.9), the arriving time-optimal problem for controllable motion of material point in multidimensional space becomes into
\[ \ddot{X}(t) = W(t), \quad X(t) \in \mathbb{R}^n, \quad W(t) \in \mathbb{R}^n, \] (3.10)
subject to (3.2), (3.3) and
\[ \int_0^T dt \to \inf_{W}. \] (3.11)

It will be shown below that problem (3.10–3.11) may have an infinite number of time-optimal controls, and some of these controls may have continuous components and/or they may take values different from \( \pm 1 \).

4. Construction of time-optimal controls

Rewrite problem (3.10) as a set of \( n \)-independent one-dimensional sub-problems, as follows:
\[ \begin{align*}
\dot{x}_1(t) &= w_1(t), \quad 0 \leq |w_1(t)| \leq \gamma_i, \\
\ddot{x}_1(t) &= w_1(t), \quad 0 \leq |w_n(t)| \leq \gamma_i, \\
x_1(0) &= x_{10}, \quad \dot{x}_1(0) = \dot{x}_{10}, \\
&\vdots, \quad x_n(0) = x_{n0}, \quad \dot{x}_n(0) = \dot{x}_{n0}, \\
x_1(T) &= 0, \quad \dot{x}_1(T) = 0, \\
x_n(T) &= 0, \quad \dot{x}_n(T) = 0.
\end{align*} \] (4.1)

subject to
\[ \int_0^T dt \to \inf_{w_i, \ldots, w_n}. \] (4.2)

It can be easily observed that each sub-problem in (4.1–4.2) has its unique (independent) optimal control \( w_1(t), \ldots, w_n(t) \), corresponding to individual optimal times \( T_i(x_{i0}, \dot{x}_{i0}, \gamma_i), i = 1, \ldots, n \).

Optimal time of arriving to the origin for the overall problem described by (4.1–4.2) is now defined as the largest arriving time
\[ T(X_0, \dot{X}_0) = \max_{1 \leq i \leq n} T_i(x_{i0}, \dot{x}_{i0}, \gamma_i) = T_i(x_{i0}, \dot{x}_{i0}, \gamma_i). \] (4.3)
It can be readily deduced that it is possible to relocate every coordinate \((x_i(t)), \dot{x}_i(t))\) or the whole vector \((X(t)), \dot{X}(t))\) to the origin at time \(T(X_0, X_0)\). Nevertheless, it is not possible to make it at any time \(T < T(X_0, X_0)\) because at least coordinate \(j\) cannot be transferred to the origin earlier than \(T = T_j(x_{in}, \dot{x}_{in}, y_j)\).

Therefore, \(T(X_0, X_0)\) is the optimal arriving time to the origin for the whole vector \((X(t)), \dot{X}(t))\). With this in mind, the arriving time-optimal control for system (3.1–3.5) can be constructed as follows. The \(j\)-component \(w_j(t)\) is a classical one-dimensional time-optimal control, which only takes limit values \(\pm \gamma_j\), and it only has one switching point (Fig. 1(a)).

Now, suppose condition (4.3) holds only for one control component, namely, \(w_j(t)\), then all other components \(w_i(t), i \neq j\), may be taken as shown in Fig. 1(b).

After an analysis of Fig. 1(b), it can be concluded that by moving points \(i_{j}^{\text{min}}, s_i, i \neq j\), the infinite number of time-optimal controls for problem (3.1–3.5) may be obtained. For example, if \(X(0) = 0\), then optimal control components \(w_j(t), i \neq j\) may take values \(0, \pm \gamma_j\), exclusively, and such controls may be defined by

\[
  \begin{align*}
    w_j(t) &= \begin{cases} 
      0 & 0 \leq t \leq i_{j}^{\text{min}}, \\
      \pm \gamma_j & i_{j}^{\text{min}} \leq t \leq i_{j}^{\text{fin}}, \\
      0 & i_{j}^{\text{fin}} \leq t \leq T,
    \end{cases}
  \end{align*}
\]

where there exists an infinity of initial, switching and final times, \(i_{j}^{\text{min}}, s_i, i \neq j\), such that \(i_{j}^{\text{fin}} - i_{j}^{\text{min}} = T_j(x_{in}, \dot{x}_{in}, \gamma_j)\) and \(x_j(t_{j}^{\text{fin}}) = \dot{x}_j(t_{j}^{\text{fin}}) = 0\).

In exceptional situations, all arriving times \(T_i, i = 1, \ldots, n\), may be equal. As consequence, it makes no difference what arriving time is selected as \(T_j\) because every subsystem arrives to the origin.
at the same time. Therefore, in this particular case, each sub-problem has only one (unique) time-optimal control, which has the classical discontinuous bang-bang form with finite number of switching points.

Another kind of optimal control for system (3.1–3.5) may be designed in the following way. The optimal control $u_j(t)$ is the same control presented in Fig. 1(a); however, controls $u_i(t)$, $i \neq j$ may be chosen as sketched in Fig. 2.

In this case, the time-optimal control for coordinate $i$, $i \neq j$, takes values $\delta^+_{\gamma_i}$ and $\delta^-_{\gamma_i}$, which are located between limit values $\gamma_i$ and $-\gamma_i$, guaranteeing the arrival to the origin for $i$-coordinate in time $T$, i.e., $-\gamma_i \leq \delta^+_{\gamma_i} \leq \delta^-_{\gamma_i} \leq \gamma_i$. Clearly, the time-optimal controls may have much more complicated behaviour, which may include continuous forms as the one shown in Fig. 2.

Thus, the only condition upon the arriving time-optimal control for coordinate $i$, $i \neq j$ is that $x_i(T) = x_i(T) = 0$.

**Remark 4.1** If there exists a time-optimal control for system (3.1–3.5), and it only has one control $u_j(t)$ with the classical form prescribed by Gramkrelidze theorem having one switching point, and limit values $\pm \gamma_j$, then controls for all other coordinates $i$, $i \neq j$ have an infinity of arbitrary forms.

5. Example

Consider two-dimensional space $\mathbb{R}^2$ and the controlled motion of a material point on plane $(x_1, x_2)$ is described by the following equations:

\[
\begin{align*}
\dot{x}_1(t) &= u_1(t), \\
\dot{x}_2(t) &= u_2(t), \\
x_1(0) &= 1, \quad x_1(0) = 0, \quad x_2(0) = -3, \quad \dot{x}_1(0) = 1, \quad (5.1) \\
0 &\leq |u_1(t)| \leq 1, \quad 0 &\leq |u_2(t)| \leq 1.
\end{align*}
\]
The optimal arriving time for the first subsystem \((x_1, \dot{x}_1)\) is \(T_1(1, 0, 1) = 2\), while the optimal arriving time for the second one \((x_2, \dot{x}_2)\) is \(T_2(-3, 1, 1) = 2.7417\). Consequently, the overall system (5.1) cannot be taken to the origin earlier than \(t = 2.7417\), and \(T = 2.7417\) is the optimal time.

Figure 3 shows some possible controls for the first subsystem \((x_1, \dot{x}_1)\) and the resulting trajectories under such conditions. As mentioned before, an infinite number of controls can be used to take \((x_1, x_1)\) to the origin, while only one control, a classical one, can be used to take \((x_2, \dot{x}_2)\) to the origin at \(t = T\).

In this case, Control 1 has been computed in the classical way, considering that the system \((x_1, \dot{x}_1)\) must reach the origin at \(t = 2\) (see Pontryagin et al., 1962). Control 2 has been obtained by adjusting a sine signal after the first switching point on the basis of the trial and error approach. In the same way, Control 3 has been obtained by adjusting a cosine signal before the first switching point. In both cases, it was considered that the system \((x_1, \dot{x}_1)\) reaches the origin a little later than \(t = 2\). Below, an analytical approach to obtain continuous (linear) control is given.

Notice that system \((x_1, \dot{x}_1)\) cannot be taken to the origin before \(t = 2\), and the classical controller is the only one capable of taking the system to the origin in such a minimum time. However, the three controls take the system \((x_1, \dot{x}_1)\) to the origin before \(t = 2.7417\). Therefore, the optimal time condition is maintained. Besides, this example is useful to verify the existence of an infinity of controls for \((x_1, \dot{x}_1)\), which can be linear, switched, continuous, etc. The only condition is to fulfill \(x_1 (T) = 0\) and \(\dot{x}_1 (T) = 0\). Nevertheless, the computation of such controllers cannot be defined by a general method.

Figure 4 shows such a unique optimal control for the subsystem \((x_2, \dot{x}_2)\) together with its corresponding optimal trajectories for position and velocity (see Pontryagin et al., 1962).

Now, consider the same problem with the exception that a 'linear' control \(w_1 (t) = at + b\) is requested for subsystem \((x_1, \dot{x}_1)\). It is clear that linear controls, in many cases, are more simple to be realized, and they have some other advantages. In this case, as well as in the previous example, \(T = 2.7417\) is defined by the second subsystem \((x_2, \dot{x}_2)\); however, the problem requires the control for the subsystem \((x_1, \dot{x}_1)\) to obey a linear rule. This example is intended to show how different types of controls can be obtained
from a prescribed form for the control signal. On the other hand, linear control has been widely studied, for instance in Linear Quadratic problem, $H_{\infty}$ control, etc.

To find the solution, it is only necessary to solve the first subsystem by knowing that the aggregate system (5.1) has to be at the origin when $t = T = 2.7417$, namely,

\[ \int_0^T \ddot{x}_1 \, dt = \ddot{x}_1(T) = \frac{aT^2}{2} + bT + \dot{x}_1(0) = 0, \quad (5.2) \]

\[ \int_0^T \ddot{x}_1 \, dt = \ddot{x}_1(T) = \frac{aT^3}{6} + \frac{bT^2}{2} + \dot{x}_1(0)T + x_1(0) = 0. \quad (5.3) \]

Therefore, by replacing initial conditions, it results:

\[ \frac{aT^3}{2} + bT = 0, \quad \frac{aT^3}{6} + \frac{bT^2}{2} + 1 = 0. \quad (5.4) \]

Finally, (5.4) must be simultaneously solve for $a$ and $b$, yielding:

\[ a = \frac{12}{T^3} \approx 0.5822, \quad b = -\frac{6}{T^2} \approx -0.7981. \]

The numerical results are presented in Fig. 5. As it can be seen, the linear controller is capable of taking subsystem $(x_1, \dot{x}_1)$ to the origin at $t = T = 2.7417$ (optimal time).

Notice that in this particular case, control $u_1(t)$ is not only linear, but it is symmetric as well, i.e.,

\[ u_1(0) = -\frac{6}{T^2} \approx 0.7981, \quad u_1(T) = \frac{6}{T^2} \approx 0.7981, \]

with $T = 2.7417$. Obviously, this feature depends on the initial conditions, and it can be useful when the actuator must to replicate this kind of behaviour.
Finally, it can be readily observed that, in a similar way, an infinite number of control laws capable of reaching the origin at $t = T$ (or earlier) can be easily designed for subsystem $(x_1, \dot{x}_1)$.

6. Conclusions

In this paper, the time-optimal control for multidimensional linear systems when Kalman controllability conditions hold, but Gramkrelidze generic position conditions are not fulfilled, has been studied.

As mentioned before, the goal of this work was not to develop a new control approach. Instead, it consists in presenting a theoretical analysis of the time-optimal control theory, in some special cases.

In that sense, it is shown that time-optimal control for multidimensional linear systems is not restricted to the classical discontinuous bang-bang form with a finite number of switching points, but an infinite number of controls, including continuous ones, can be used to take the system to the origin in minimum time.

As a simple but illustrative example, a material point moving in multidimensional space was considered to validate the accuracy of the analysis. It is also shown that the arriving time-optimal control problem for a controllable material point in multidimensional space has an infinite number of optimal solutions, and as consequence, the material point can have an infinite number of optimal trajectories. Finally, some hints for construction of such controls were given.

This work suggests the study of more general situations involving non-trivial matrices $A$ and finding other conditions instead of Gramkrelidze generic position conditions, which may conduct to unique optimal control.

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