Abstract. Soft glassy materials exhibit the so-called glassy transition which means that the behavior of the model at low shear rate changes when a certain parameter (which we call the glass parameter) crosses a critical value. This behavior goes from a Newtonian behavior to a Herschel-Bulkley behavior through a power-law-type behavior at the transition point. In a previous paper we rigorously proved that the Hébraud-Lequeux model, a Fokker-Planck-like description of soft glassy material, exhibits such a glass transition. But the method we used was very specific to the one dimensional setting of the model and as a preparation for generalizing this model to take into account multidimensional situations, we look for another technique to study the glass transition of this type of model. In this paper we shall use matched asymptotic expansions for such a study.

The difficulties encountered when using asymptotic expansions for the Hébraud-Lequeux are that multiple ansaetze have to be used even though the initial model is unique, due to the glass transition. We shall delineate the various regimes and give a rigorous justification of the expansion by means of an implicit function argument. The use of a two parameter expansion plays a crucial role in elucidating the reasons for the scalings which occur.

Key words. glass transition, boundary layers, implicit functions, linear perturbation

AMS subject classifications. 34E10, 76A05

1. Introduction. Soft glassy materials are complex fluids which by definition exhibit a transition when a parameter goes through a critical value. Soft glassy materials include for instance classical glasses in which case the parameter is simply the temperature. In this case we talk of molecular glasses. But another class of materials that exhibit similar properties is what is called granular glasses and is formed by “particles” dispersed in a Newtonian fluid: this is the case for suspensions of solid particles and for emulsions which are “particles” of a fluid dispersed in another immiscible fluid. In this setting the parameter is related to the concentration of particles: as long as the particles are not too concentrated the behavior of the material is Newtonian. But when the concentration reaches a certain value, the particles are so packed that the structure has to break before the material can flow. This leads to a stress threshold behavior similar to Herschel-Bulkley models.

One model available in the literature to describe this kind of phenomenon is the one introduced by HÉBRAUD and LEQUEUX in [10]. In a previous paper [12], we mathematically proved that the Hébraud-Lequeux model (referred to as HL for the sake of brevity) describes the glass transition as was announced in HÉBRAUD’s PHD thesis [9]. This work relied on analytical computations of the solutions of the model which was possible because HL was designed to describe simple shear flow and is thus, from a mathematical point of view, a 1d problem governed by an ODE. But in an upcoming work we intend to extend the model to multidimensional situations and thus study a model similar to HL but set on several dimension of space. This means that analytical solutions might not be available anymore. This led us to search for
other tools to prove the same results, techniques that would be more easily generalized to multidimensional situations. The tool we found to be appropriate are asymptotic expansions.

Let us introduce the model and our notations. We are interested in the stationary dimensionless version of the HL model. Here the unknown is \( p \) which is a probability density on the stress space. The stress variable will always be denoted \( \sigma \). The small parameter is \( y \). It is a dimensionless shear rate and is chosen to be nonnegative. The glass parameter is denoted by \( \mu \). Finally, the model uses a function \( \phi \) of \( y \) called the fluidity. Then the model reads:

\[
\begin{cases}
- \phi(y) \partial_{\sigma}^2 p + y \partial_{\sigma} p + 1_{[-1,1]} p = \frac{\phi(y)}{\mu} \delta_0, \\
p \in H^1(\mathbb{R}), \\
p(\pm\infty) = 0, \\
p \geq 0, \\
\int_{\mathbb{R}} p(\sigma) d\sigma = 1.
\end{cases}
\]  

Here \( 1_{[-1,1]} \) denotes the characteristic function of the complement of the interval \([-1,1]\) and \( \delta_0 \) is the delta function located at the origin. The fluidity is not explicitly defined. It comes from the constraint on the integral of \( p \). Indeed, if instead of \( \phi(y) \) in the PDE we had a given constant \( \Gamma \), then solving the boundary value problem is a simple matter. But since we have in addition the integral constraint, there is only one \( \Gamma \) that allows for solving the equation and the integral constraint simultaneously when \( y > 0 \). See CANCES, CATTO and GATI [3] or [12] for further details on this question.

On the other hand, by integrating the equation one has the following connection between \( p \) and \( \phi \):

\[
\frac{\phi}{\mu} = \int_{|\sigma|>1} p(\sigma) d\sigma.
\]

Finally, to connect more specifically to the physics of the phenomenon we are trying to model, we introduce the macroscopic stress \( \tau = \int_{\mathbb{R}} \sigma p(\sigma) d\sigma \). Then \( \tau \) is only a function of \( y \). In this setting the glass transition occurs at the critical value 1/2: when \( \mu > 1/2 \) the behavior is Newtonian \( \tau \sim \eta y \), when \( \mu = 1/2 \) we will have a power-law fluid with exponent 1/5, that is \( \tau \sim y^{1/5} \), and finally when \( \mu < 1/2 \) we obtain a Herschel-Bulkley fluid with exponent 1/2, that is to say \( \tau \sim \tau_0 + A \sqrt{y} \), where \( \tau_0 > 0 \) is called the dynamic yield stress. We note that the asymptotic expansions for \( p \) translate into a straightforward manner into expansions for \( \tau \), as long as we show convergence in a function space that embeds into a space for which \( p \rightarrow \int \sigma p \) is a continuous linear form.

The paper is organized as follows: §3 is devoted to the presentation of the model and the derivation of the asymptotic hierarchy. We explain how to find the appropriate scales of expansions and the sizes of the relevant boundary layers. In §4 we justify the formal expansions by an implicit function argument. In doing so, we find it advantageous to introduce a priori two small parameters \( a = y/\phi(y) \) and \( b = \sqrt{\phi(y)} \). This allows us to reduce our problem to a single equation of the form \( F(\mu, a, b) = 0 \). The behavior of \( \phi \) as a function of \( y \) arises naturally from the analysis of this function.
2. Reformulation of the Problem and Main Result. At the start of this study we were inspired by the similarity of the problem given by the system (1.1) with the problem of stationary Navier-Stokes equations with a penalization term to take into account an obstacle in viscous flows studied by Angot, Bruneau and Fabrie, [1], Carbou and Fabrie [5], or Carbou [4] in the context of porous materials, or the problem of wave equations with a penalization term studied by Fornet and Gues [8] and Fornet [7, 6]. This encouraged us to see if the same methods could be adapted to our problem. In the HL setting, the obstacle would be the exterior of the $[-1,1]$ interval.

The first thing to do is to separate what happens inside and outside the obstacle. This means we rewrite (1.1) with unknowns $q = p_{|1-1|}$, $r = p_{[1-1]}$, and obtain

\begin{alignat}{2}
- \phi(y) \partial_y^2 q + y \partial_y q &= \frac{\phi(y)}{\mu} \delta_0 & \quad & \text{in } [-1,1], \\
- \phi(y) \partial_y^2 r + y \partial_y r + r &= 0 & \quad & \text{in } [-1,1], \\
q &\geq 0 & \quad & (2.1a) \\
r &\geq 0 & \quad & (2.1b) \\
r(\pm1) &= q(\pm1) & \quad & (2.1c) \\
\partial_y r(\pm1) &= \partial_y q(\pm1) & \quad & (2.1d) \\
\int_{|\sigma|>1} r(\sigma) d\sigma + \int_{-1}^{1} q(\sigma) d\sigma &= 1. & \quad & (2.1e)
\end{alignat}

We refer to (2.1f) and (2.1g) as the transmission conditions and (2.1h) as the integral constraint.

Now our main result can be stated:

**Theorem 2.1.** The solution of (2.1a)–(2.1h), for small $y$, can be expanded in a convergent series whose terms can be described in terms of boundary layers. Moreover the expansion changes if the parameter $\mu$ changes, which leads more precisely to the following discussion:

If $\mu > 1/2$, there is no boundary layer terms ($Q$ and $R$ are functions of $\sigma$) and we have

\begin{alignat*}{2}
q &= Q^0 + yQ^1 + y^2Q^2 + \cdots, \\
r &= R^0 + yR^1 + y^2R^2 + \cdots,
\end{alignat*}

and consequently

\[ \frac{\phi}{\mu} = \tau_0 + \tau_1 y + \tau_2 y^2 + \cdots. \]

If $\mu < 1/2$, the boundary layer is of size $y^{1/2}$ and

\begin{alignat*}{2}
q &= Q^0 + y^{1/2}Q^1 + yQ^2 + \cdots, \\
r &= \sqrt{y}R^1 + yR^2 + \cdots,
\end{alignat*}
and consequently
\[
\frac{\phi}{\mu} = c_1 y + c_2 y^{3/2} + \cdots ,
\]
with \(R_k\) depending on \((|\sigma| - 1)/y^{1/2}\).

If \(\mu = 1/2\), the boundary layer is of size \(y^{1/5}\) and
\[
q = Q^0 + y^{1/5}Q^1 + y^{2/5}Q^2 + \cdots ,
\]
\[
r = y^{2/5}R^2 + y^{3/5}R^3 + \cdots ,
\]
and consequently
\[
\frac{\phi}{\mu} = c_2 y^{4/5} + c_2 y + \cdots ,
\]
with \(R_k\) depending on \((|\sigma| - 1)/y^{2/5}\).

Using the previous theorem, the following result related to the macroscopic stress may be deduced:

**Corollary 2.2.** The previous expansions converge in a space in which the application \(p \mapsto \int_{\mathbb{R}} \sigma p\) is a continuous linear form. Thus by a simple integration we have the following expansions:

- **If \(\mu > 1/2\), the stress expands as**
  \[
  \tau = \eta_0 y + \eta_1 y^2 + \cdots
  \]
- **If \(\mu < 1/2\), the stress expands as**
  \[
  \tau = \tau_0 + A_0 \sqrt{y} + \cdots
  \]
- **If \(\mu = 1/2\), the stress expands as**
  \[
  \tau = B_0 y^{1/5} + B_1 y^{2/5} + \cdots
  \]

The various constants (which may be 0 except for the first one of those we have indicated) can be computed in terms of the profiles of the expansion of \(q\) and \(r\).

Section 3 is devoted to understanding from a formal point of view where the boundary layers come from, starting from an *a priori* unknown boundary layer expansion (meaning that we will not prescribe the size of the boundary layer nor the scale of the expansion). We will then obtain the equations of the profile in each case. This method is fairly general and may be applied in multi dimensional generalizations and the equations of profile can be useful for numerical purposes.

Even though this section does not contain any proof of convergence, one could follow up with a convergence proof based on existence and uniqueness of the profiles and estimation of the remainder at a given order, in a manner similar to Fabrie and Boyer [2]). However in §4 we shall give a simpler proof which exploits the fact that the problem is more easily analyzed in a two parameter setting. In this setting, \(a = y/\phi(y)\) and \(b = \sqrt{\phi(y)}\) are treated a priori as independent parameters. It turns out that the solution of the differential equation is actually an analytic function of \(a\) and \(b\). The proof uses perturbation theory. One of the essential differences between the one-dimensional and multi-dimensional case is that the limit \(b \to 0\)
becomes a singular perturbation problem in several dimensions. The perturbation argument would therefore become more complicated, and the solution would depend only smoothly, but not analytically, on $b$. In a second step, we analyze the remaining equation resulting from the integral constraint. This is a finite dimensional problem of the form $F(\mu, a, b) = 0$. The implicit function theorem can be used to establish a relationship between $a$ and $b$ which naturally yields the expansions of Theorem 2.1.

3. Derivation of the Asymptotic Hierarchy. In this section we shall show the formal computations that will be justified by the next section. The formal computations are of interest in their own right, since the appropriate ansatz is not obvious a priori. Of course the results described by HÉBRAUD and LEQUEUX in [10] and proved in [12] were a powerful guide. Yet they do not give the actual ansatz. The transitional case $\mu = 1/2$ was especially hard to devise.

3.1. Ansatz. To describe the boundary layer which lies in the exterior domain $[-1, 1]^c$, we need the distance to the boundary $\{ -1, 1 \}$ and we call this distance $\theta_e$, which is simply

$$\theta_e(\sigma) = |\sigma| - 1.$$  

We make the following ansatz for $q$ and $r$:

$$q(\sigma) = \sum_{k=0}^{+\infty} y^{k/s} Q^k(\sigma), \quad (3.1a)$$

$$r(\sigma) = \sum_{k=0}^{+\infty} y^{k/s} R^k(\sigma) + y^{k/s} R(\text{sign}(\sigma), \frac{\theta_e(\sigma)}{y^{l/s}}), \quad (3.1b)$$

which implies that $\phi/\mu$ has the following expansion in view of (1.2):

$$\frac{\phi(y)}{\mu} = \sum_{k=0}^{+\infty} \tilde{c}_k y^{k/s}. \quad (3.1c)$$

Here $l$ and $s$ are two integers satisfying $1 \leq l \leq s$. We have also introduced:

$$\tilde{c}_k = \begin{cases} c_k, & \text{if } 0 \leq k \leq l - 1, \\ c_k + c_{k-l}, & \text{if } k \geq l, \end{cases}$$

with

$$c_k = \int_{|\sigma|>1} \tilde{R}(\sigma) d\sigma, \quad \tilde{c}_k = \int_0^{+\infty} y^{(k+1)/s} [R^k(-1, z) + R^k(1, z)] dz. \quad (3.2)$$

We recall that in a boundary layer setting we have, from a formal point of view, the property:

$$\forall k \forall m \lim_{z \to \infty} |\partial_z^m R^k(\pm 1, z)| = 0. \quad (3.3)$$

In more details we have the following:

PROPOSITION 3.1 (Necessary form of the ansatz). The parameters in the previous ansatz in each case can only be:
if \( \mu > 1/2 \): we have \( s = 1 \) and no boundary layer, which means \( l \) is undefined and all the \( R^k \) are zero;

if \( \mu < 1/2 \): we have \( s = 2 \) and \( l = 1 \), which means that the boundary layer is of size \( y^{1/2} \). In the exterior all the \( R^k \) are zero;

if \( \mu = 1/2 \): we have \( s = 5 \) and \( l = 2 \) which means that the boundary layer is of size \( y^{2/5} \) while the expansion is in powers of \( y^{1/5} \). In the exterior all the \( R^k \) are again zero.

What we imply by necessary conditions is that taking other parameters will rapidly lead to ill-posed problems for the profiles, even at the leading order. Another detail to note is that for \( \mu < 1/2 \) for instance one could take \( s = 4 \) and \( l = 2 \) but this would lead to the same expansion with a lot of coefficients (half in fact) simply vanishing. We have indicated the choices of parameters that lead to the minimum of “trivially zero” terms.

The interest of this proposition is that its proof gives a methodology to find the size of the boundary layer when you have no \textit{a priori} knowledge (from physics or elsewhere) to guide you. We have of course in mind our multidimensional generalization of the HL model for which we lack this kind of information.

Before we can prove this proposition, we need to derive the equations solved by the profiles.

### 3.2. Equations of Profile

We now put these ansaetze in (2.1a)-(2.1h) and assemble the terms of the same formal order. We obtain the following hierarchy of equations.

**Equation (2.1a):** We put the ansatz (3.1a) and the ansatz (3.1c) in (2.1a):

\[
0 \leq k \leq s - 1 : \quad - \mu \sum_{k' = 0}^{k} \tilde{c}_{k'} \partial_z^2 Q^{k-k'} = \tilde{c}_k \delta_0, \\
\]

\[
s \leq k : \quad - \mu \sum_{k' = 0}^{k} \tilde{c}_{k'} \partial_z^2 Q^{k-k'} + \partial_z Q^{k-s} = \tilde{c}_k \delta_0.
\]

**Equation (2.1b):** We put the ansatz (3.1b) and the ansatz (3.1c) in (2.1b). We can then separate in these equations the equations obeyed by the \( R^k \) and those obeyed by the \( R^k \) by using the property stated in Eq. (3.3). We then obtain:

\[
-2l \leq k \leq -1 : \quad - \mu \sum_{k' = 0}^{k+2l} \tilde{c}_{k'} \partial_z^2 R^{k+2l-k'} = 0,
\]

\[
0 \leq k \leq s - l - 1 : \quad - \mu \sum_{k' = 0}^{k+2l} \tilde{c}_{k'} \partial_z^2 R^{k+2l-k'} + R^k = 0, \\
\]

\[
s - l \leq k : \quad - \mu \sum_{k' = 0}^{k+2l} \tilde{c}_{k'} \partial_z^2 R^{k+2l-k'} + R^k = -\theta' \partial_z R^{k-s+l}.
\]
and

\[ 0 \leq k \leq s - 1 : \quad - \mu \sum_{k'=0}^{k} \tilde{c}_{k'} \partial^2 R^{k-k'} + R^{k} = 0, \tag{3.4c} \]

\[ s \leq k : \quad - \mu \sum_{k'=0}^{k} \tilde{c}_{k'} \partial^2 R^{k-k'} + R^{k} = - \partial_z R^{k-s}. \]

**Equation (2.1c):** With (3.3) in mind, Eq. (2.1c) only tells us

\[ 0 \leq k : \quad \bar{R}^{k}(\pm \infty) = 0. \tag{3.4d} \]

**Equation (2.1f):** The continuity relation translates into

\[ 0 \leq k : \quad \overline{Q}^{k}(\pm 1) = \bar{R}^{k}(\pm 1) + R^{k}(\pm 1, 0). \tag{3.4e} \]

**Equation (2.1g):** The continuity of the derivative translates into (given that \( \theta'_e \) is non zero):

\[ 0 \leq k \leq l - 1 : \quad 0 = \partial_z R^{k}(\pm 1, 0), \tag{3.4f} \]

\[ 0 \leq k : \quad \partial_{\sigma} Q^{k}(\pm 1) = \partial_{\sigma} \bar{R}^{k}(\pm 1) + \theta'_{e}(\pm 1) \partial_z R^{k+l}(\pm 1, 0). \tag{3.4g} \]

**Equation (2.1h):** Finally we get from Eq. (2.1h) the following constraints for the profile:

\[ k = 0 : \quad \int_{-1}^{1} \overline{Q}^{0}(\sigma) d\sigma + \frac{\tilde{c}_{0}}{\mu} = 1, \]

\[ k > 0 : \quad \int_{-1}^{1} \overline{Q}^{k}(\sigma) d\sigma + \frac{\tilde{c}_{k}}{\mu} = 0. \tag{3.4h} \]

**Influence of Eqs. (2.1d) and (2.1e):** The two conditions (2.1d) and (2.1e) translate into the positivity of the lowest order term of the expansions (3.1a) and (3.1b). Moreover, by taking into account (3.2), this also means that the first nonzero \( \tilde{c}_{k} \) must be positive.

### 3.3. Proof of Proposition 3.1.

Now that we have the equations solved by the profiles, we can turn to the proof of Prop. 3.1. We break it down into several steps.

**Lemma 3.2.** If \( \tilde{c}_{0} \neq 0 \) then all the \( R^{k} \) are necessarily zero (thus \( l \) is undefined).

**Proof.** We argue by induction. Take \( k = -2l \) in (3.4b) and \( k = 0 \) in (3.4f) and in (3.3). Then the problem solved by \( R^{0} \) is

\[
\begin{cases}
\forall z > 0 & -\mu \tilde{c}_{0} \partial_{z}^{2} R^{0}(\pm 1, z) = 0, \\
 & -\partial_z R^{0}(\pm 1, 0) = 0, \\
 & R^{0}(\pm 1, +\infty) = 0.
\end{cases}
\]

whose only solution is \( R^{0} = 0 \). Now suppose \( R^{0}, \cdots, R^{p-1} \) are identically 0 for \( 1 \leq p \). Then take \( k = -2l + p \) in (3.4b). We find that \( R^{p} \) satisfies the equation

\[ -\mu \tilde{c}_{0} \partial_{z}^{2} R^{p} = 0, \]
i.e. $\partial^2_z R^p = 0$. But since from (3.3) $\partial_z R^p \to 0$ as $z$ tends to infinity then we find $\partial_z R^p = 0$ for all $z$. Then again $R^p \to 0$ at infinity and $R^p$ is identically 0. □

Now we look at the other case when $\tilde{c}_0 = 0$.

**Lemma 3.3.** If $\tilde{c}_0 = 0$ then necessarily

1. $\tilde{R}^k$ is identically 0 (and thus so is $\tilde{c}_k$) for all $k$,
2. $c_1, \ldots, c_{l-1}$ are 0 and $c_l$ cannot be 0,
3. $R^0, \ldots, R^{l-1}$ are all identically zero.

**Proof.** The first point is proved by induction. Taking (3.4c) for $k = 0$ gives

$$-\mu \tilde{c}_0 \partial^2_z \tilde{R}^0 + \tilde{R}^0 = 0,$$

and since $\tilde{c}_0 = 0$ we have $\tilde{R}^0 = 0$. The induction is then clear since all profile $\tilde{R}^k$ will satisfy the same equation.

To prove the second point, introduce $m$ the index of the first non zero $c_k$ and suppose $0 \leq m \leq l-1$. Then take (3.4b) with $k = m - l$ which is the first non trivially satisfied equation for the $\tilde{R}^k$. Since $m \leq l - 1$ then we have $m - l \leq -1$. Provide for this equation the boundary condition given by (3.4f) with $k = 0$ and the property (3.3), and this leads to the following problem for $\tilde{R}^0$:

$$\begin{cases}
\forall z > 0 & -\mu c_m \partial^2_z \tilde{R}^0(\pm 1, z) = 0, \\
& -\partial_z \tilde{R}^0(\pm 1, 0) = 0, \\
& \tilde{R}^0(\pm 1, +\infty) = 0,
\end{cases}$$

which can only be satisfied by the null function. All following $\tilde{R}^k$ will then satisfy the same problem and be consequently 0 and thus so is $\tilde{R}^m$. This is contradictory since $c_m$ is supposed to be nonzero and the integral of $\tilde{R}^m$.

Suppose now $m \geq l + 1$. Once again we argue by induction that all $\tilde{R}^k$ are identically 0 which leads to a contradiction with $c_m \neq 0$. Take $k = 0$ in (3.4b) and you find that $\tilde{R}^0 = 0$ because all the coefficients in the sum are 0 due to the fact that the first index for which $\tilde{c}_k$ is non zero is $l + m$. Then suppose $\tilde{R}^0, \ldots, \tilde{R}^{p-1}$ are all identically zero and take $k = p$ in (3.4b). When we look at

$$\sum_{k' = 0}^{p+2l} \tilde{c}_{k'} \partial^2_z \tilde{R}^{p+2l-k'},$$

we see that if $k' \leq m+l-1$, then $\tilde{c}_{k'} = 0$, and if $k' \geq m+l$, then $p+2l-k' \leq p-(m-l)$, and thus $\partial^2_z \tilde{R}^{p+2l-k'} = 0$, so that the sum is in fact 0. And if $p \geq 0$, then we can conclude $\tilde{R}^p = 0$.

Finally suppose $m = l$. By taking $k = 0$ in (3.4b) and in (3.4f) and then using the fact that we have $\tilde{c}_0 = \cdots = \tilde{c}_{2l-1} = 0$ we find that the problem solved by $\tilde{R}^0$ is

$$\begin{cases}
\forall z > 0 & -\mu c_l \partial^2_z \tilde{R}^0(\pm 1, z) + \tilde{R}^0(\pm 1, z) = 0, \\
& -\partial_z \tilde{R}^0(\pm 1, 0) = 0, \\
& \tilde{R}^0(\pm 1, +\infty) = 0,
\end{cases}$$

which once again can only be solved by the null function. Argue by induction to find that $\tilde{R}^1, \ldots, \tilde{R}^{l-1}$ all satisfy the same problem as $\tilde{R}^0$ and are then identically 0 □

**Remark 1.** The fact that $\tilde{c}_2l$ is the first nonzero term of the expansion of $\tilde{\phi}$ is not surprising since in the equation of $r$ you would want to balance the second order
derivative with the zeroth order term to have an exponentially fast decay at infinity
and when the coefficient in front of the second order derivative is of order $\gamma^\alpha$ this can
only be achieved if the boundary layer size is $\gamma^{\alpha/2}$.

Now we complete the proof of the proposition.

**Lemma 3.4.** If $\tilde{c}_0 \neq 0$ then the fundamental problem is well-posed if and only if
$\mu > 1/2$. In this case $s$ can be taken as one.

**Proof.** Now we can look at the problem satisfied by $Q^0$, $R^0$ and $\tau_0 = \tilde{c}_0$. We take
$k = 0$ in (3.4a) and in (3.4c) and $k = -2l$ in (3.4b) for the equations and complete
with the transmission condition from (3.4e) and (3.4g) with $k = 0$ and finally $k = 0$
in (3.4h). This leads to

$$
\begin{align*}
- \mu \tau_0 \partial_\sigma^2 Q^0 &= \tau_0 \delta_0, \\
- \mu \tau_0 \partial_\sigma^2 R^0 + R^0 &= 0, \\
Q^0(\pm 1) &= R^0(\pm 1), \\
\partial_\sigma Q^0(\pm 1) &= \partial_\sigma R^0(\pm 1), \\
\int_{-1}^{1} Q^0(\sigma) d\sigma + \frac{\tau_0}{\mu} &= 1, \\
\tau_0 &\neq 0.
\end{align*}
$$

We can compute analytically the solution to this problem which exists (and is unique)
if, and only if, $\mu > 1/2$ (see Cances et al [3]).

Now we show that for $k = 1 + ns$ to $k = s - 1 + ns$ and for $n \geq 0$ the problem
solved by $Q^k$, $R^k$, and $\tau_k = \tilde{c}_k$ is solved by the trivial solution $(0, 0, 0)$. We argue
by induction on $n$. Let us look at the case $n = 0$. We argue by induction on $k$. The
problem solved by $Q^1$, $R^1$, $\tau_1$, is

$$
\begin{align*}
- \mu \tau_1 \partial_\sigma^2 Q^1 &= \tau_1 \delta_0 + \mu \tau_1 \partial_\sigma^2 Q^0, \\
- \mu \tau_1 \partial_\sigma^2 R^1 + R^1 &= \mu \tau_1 \partial_\sigma^2 R^0, \\
Q^1(\pm 1) &= R^1(\pm 1), \\
\partial_\sigma Q^1(\pm 1) &= \partial_\sigma R^1(\pm 1), \\
\int_{-1}^{1} Q^1(\sigma) d\sigma + \frac{\tau_1}{\mu} &= 0.
\end{align*}
$$

Now using the fundamental problem we see that $\tau_1 \delta_0 + \mu \tau_1 \partial_\sigma^2 Q^0 = 0$. It is now clear
that the trivial solution satisfies this problem. We can see now that for $k = 2, \cdots, s-1,$
$Q^k$, $R^k$, $\tau_k$ satisfies

$$
\begin{align*}
- \mu \tau_k \partial_\sigma^2 Q^k &= \tau_k \delta_0 + \mu \tau_k \partial_\sigma^2 Q^0 = 0, \\
- \mu \tau_k \partial_\sigma^2 R^k + R^k &= \mu \tau_k \partial_\sigma^2 R^0, \\
Q^k(\pm 1) &= R^k(\pm 1), \\
\partial_\sigma Q^k(\pm 1) &= \partial_\sigma R^k(\pm 1), \\
\int_{-1}^{1} Q^k(\sigma) d\sigma + \frac{\tau_k}{\mu} &= 0.
\end{align*}
$$
which is again satisfied by the trivial \((0, 0, 0)\) solution. Now let us assume that we have proved our result for \(n = 0, \cdots, p - 1\). We prove that \(Q_{1+ps}, R_{1+ps}, \tau_{1+ps}\) is trivial. The other cases can be deduced by induction in the same fashion that in the \(n = 0\) case. The equation solved by \(Q_{1+ps}\) is (3.4a) with \(k = 1 + ps \geq s\) which is:

\[
- \mu \sum_{k'} \bar{c}_{k'} \partial_{s}^{2} Q_{1+ps-k'} + \partial_{s} Q_{1+ps-s} = \tau_{1+ps} \delta_{0}.
\]

Now \(Q_{1+(p-1)s}\) is identically 0 by hypothesis so \(\partial_{s} Q_{1+ps-s} = 0\). And if we consider the sum, the potentially non zero coefficients are \(\bar{c}_0, \bar{r}_s, \ldots, \bar{r}_{ps}\) and also \(\tau_{1+ps}\). But for \(k \geq 1\), \(\bar{c}_{ks}\) is multiplied by \(\partial_{s}^{2} Q_{1+(p-k)s}\) which is zero by hypothesis. So the equation on \(Q_{1+ps}\) is really only:

\[
- \mu \bar{c}_0 \partial_{s}^{2} Q_{1+ps} = \tau_{1+ps} \delta_{0} + \mu \bar{c}_{1+ps} \partial_{s}^{2} R_{0} = 0.
\]

when using the fundamental problem. In the same way the equation in \(R_{1+ps}\) reduces to:

\[
- \mu \bar{c}_0 \partial_{s}^{2} R_{1+ps} + R_{1+ps} = \mu \bar{c}_{1+ps} \partial_{s}^{2} R_{0},
\]

which means that once again we can take \(Q_{1+ps} = 0\), \(R_{1+ps} = 0\) and thus \(\tau_{1+ps} = 0\). Finally only the multiples of \(s\) are potentially nonzero and we can minimize the number of equations by taking \(s = 1\).

**Remark 2.** The fact that we use the analytical solution of the fundamental problem seems contradictory to our intent not to compute solutions explicitly. But what we want to avoid is computing the analytical solution of the exact initial problem. We believe that it will be in our reach to do exact computations on the fundamental problem even in several space dimensions.

**Lemma 3.5.** If \(\tilde{c}_0 = 0\) and \(2l \geq s\), then \(\mu < 1/2\), and one can take \(l = 1\) and \(s = 2\).

**Proof.** Since \(\tilde{c}_0 = 0\), we know that we must have \(c_l \neq 0\). Once again we look at the fundamental problem. Since \(\widetilde{R}_0\) and \(R_0\) are both zero, \(\widetilde{Q}_0\) must vanish on the boundary of \([-1, 1]\). Finally with \(\tilde{c}_0 = 0\), \(\widetilde{Q}_0\) must be of integral 1. Now there are two possibilities. If \(2l > s\) then the first non trivial equation featuring \(\widetilde{Q}_0\) is for (3.4a) with \(k = s\) and is simply \(\partial_{s} \widetilde{Q}_0 = 0\). This leads to a contradiction because \(\widetilde{Q}_0\) would then be a constant which, given the boundary conditions, is 0, which is incompatible with the constraint. Thus we must have \(2l = s\).

Then the equation comes from (3.4a) \(k = 2l\) and, in summary, \(\widetilde{Q}_0\) satisfies:

\[
\begin{cases}
- \mu c_l \partial_{s}^{2} \widetilde{Q}_0 + \partial_{s} \widetilde{Q}_0 = c_l \delta_{0}, \\
\widetilde{Q}_0 (\pm 1) = 0, \\
\int_{-1}^{1} \widetilde{Q}_0 (\sigma) d\sigma = 1, \\
c_l \neq 0.
\end{cases}
\]

One can see that this problem has a solution only if \(\mu < 1/2\). We now have to see what \(l\) or \(s\) is. The first boundary profile to be non zero is \(R_l\) and it gives boundary
conditions to the problem of $\tilde{Q}^j$. It is then easy to see that to minimize the number of non trivial problems one can take $l = 1$ and thus $s = 2$. □

**Lemma 3.6.** If $\tilde{c}_0 = 0$ and $2l + 1 \leq s$ then
1. $\mu = 1/2$,
2. $l \geq 2$ and $2s = 5l$,
3. one can take $l = 2$ and $s = 5$.

**Proof.** If we are in the conditions of the lemma then the fundamental problem is

$$
\begin{cases}
- \mu c_l \partial_x^2 \tilde{Q} = c_l \delta_0, \\
\tilde{Q}^0(\pm 1) = 0, \\
\int_{-1}^{1} \tilde{Q}^0(\sigma) d\sigma = 1, \\
c_l \neq 0.
\end{cases}
$$

It is easily seen that this problem can only be solved when $\mu = 1/2$ by the function

$$
\sigma \mapsto \begin{cases}
\sigma + 1, & \text{if } \sigma \in [-1, 0], \\
1 - \sigma, & \text{if } \sigma \in [0, 1].
\end{cases}
$$

Now we have to find the right boundary layer size and expansion scale. We now compute the first non trivial boundary layer profile. We take $k = 2l$ in (3.4b) (at this point it does not matter if $2l$ is lower or greater than $s - l$ because all the $R^k$ are zero for $k < l$) and boundary condition (3.4g) to find that $R^l$ satisfies:

$$
\begin{cases}
- \mu c_l \partial_x^2 R^l(\pm 1, \cdot) + R^l(\pm 1, \cdot) = 0, \\
\partial_z R^l(-1, 0) = 1, \\
\partial_z R^l(1, 0) = 1.
\end{cases}
$$

whose solution is $R^l(\pm 1, z) = \sqrt{\mu c_l} \exp(-z/\sqrt{\mu c_l})$.

Now let us examine the equations (3.4a) obeyed by $\tilde{Q}^k$. The first equation to have $\tilde{Q}^0$ in a nontrivial manner is for $k = 2l + p$ because the first nontrivial $\tilde{c}_k$ is for $k = 2l$. In this equation there are two possible “external forces”. First are the boundary conditions (3.4e). The first non zero boundary condition for a $\tilde{Q}^k$ are for $k = l$. The second “external force” is the term of the form $\partial_x^{k-s}$. The first $\tilde{Q}^k$ to have a non trivial term of this form is when $k = s$ that is for the equation introducing $\tilde{Q}^{s-2l}$.

It is easy to see by induction that for $1 \leq k \leq \min(s - 2l, l) - 1$ the $\tilde{Q}^k$ are zero because they have no “external force” and they actually follow a linear equation (also (3.4h) gives the constraint $\int_{-1}^{1} \tilde{Q}^k(\sigma) d\sigma = 0$ as long as $1 \leq k \leq 2l - 1$ which is of course satisfied by a trivial $\tilde{Q}^k$).

If we had $l < s - 2l$ then we would have $2l + l \leq s - 1$ and $\tilde{Q}^j$ would satisfy the following problem:

$$
\begin{cases}
- \mu c_l \partial_x^2 \tilde{Q} = 0, \\
\tilde{Q}(\pm 1) = R^l(\pm 1, 0), \\
\int_{-1}^{1} \tilde{Q}(\sigma) d\sigma = 0.
\end{cases}
$$
Since we have $R_l(1, 0) = R_l(-1, 0) = \sqrt{\mu c_l}$ we clearly have $Q_l(\sigma) = \sqrt{\mu c_l}$, but this leads to a contradiction between $c_l \neq 0$ and $\int_{-1}^1 Q_l(\sigma) d\sigma = 0$. This means we must have $s - 2l \leq l$ and thus the first term to have an external force is $Q^{s-2l}$, whose equation is given by (3.4a) with $k = s$. This leads to $Q^{s-2l}$ to satisfy the problem
\[
\begin{cases}
-\mu c_l \partial_\sigma^2 Q^{s-2l} + \partial_\sigma Q^0 = 0, \\
Q^{s-2l}(\pm 1) = R_l(\pm 1, 0), & \text{if } s - 2l = l, \\
Q^{s-2l}(\pm 1) = 0, & \text{if } s - 2l < l,
\end{cases}
\]
\[
\int_{-1}^1 Q^{s-2l}(\sigma) d\sigma = 0.
\]

We now prove that $s - 2l$ cannot be $l$. If it were we could decompose $Q^{s-2l}$ into a sum $A + B$ by linearity where we have
\[
\begin{cases}
-\mu c_l \partial_\sigma^2 A = 0, \\
A(\pm 1) = R_l(\pm 1, 0),
\end{cases}
\]
and
\[
\begin{cases}
-\mu c_l \partial_\sigma^2 B = -\partial_\sigma Q^0, \\
B(\pm 1) = 0.
\end{cases}
\]

Since $R_l(1, 0) = R_l(-1, 0) = \sqrt{\mu c_l}$ we once again have $A = \sqrt{\mu c_l}$. Now since we have $\partial_\sigma Q^0$ odd we have that $B$ is the sum of the odd primitive of $1/(\mu c_l)Q^0$ and of an affine function and considering the boundary condition $B$ is in fact odd. This put together leads to a contradiction since we should have
\[
0 = \int_{-1}^1 Q^{s-2l}(\sigma) d\sigma = \int_{-1}^1 A(\sigma) d\sigma + \int_{-1}^1 B(\sigma) d\sigma = 2\sqrt{\mu c_l},
\]
with $c_l \neq 0$. Thus $s - 2l < l$. From $2l + 1 \leq s \leq 3l - 1$ we get $l \geq 2$. And we have that the problem satisfied by $Q^{s-2l}$ is well posed. We actually need $Q^{s-2l}$. One can prove that it is the function
\[
\sigma \mapsto \begin{cases}
\frac{1}{c_l} \sigma (\sigma + 1), & \text{if } \sigma \in [-1, 0], \\
\frac{1}{c_l} \sigma (1 - \sigma), & \text{if } \sigma \in [-1, 0].
\end{cases}
\]

But $c_l$ is still not defined! We thus push the study to the following non zero term. From the “external force” point of view, the boundary condition still drives $Q^l$ first. For the derivative it is $\partial_\sigma Q^{s-2l}$ which appears first in the equation (3.4a) with $k = 2s - 2l$ which gives the equation of $Q^{2s-4l}$.

What we prove now is that both “external forces” need to be active simultaneously to have a well posed problem. This leads to $l = 2s - 4l$ which is equivalent to $2s = 5l$. 
If $l < 2s - 4l$ then the problem satisfied by $Q_l$ is now

$$\begin{cases}
- \mu_c \partial^2_q Q_l = \mu_{c_{l-4}} \partial^2_q Q_l^{s-2l}, \\
Q_l'(\pm 1) = R_l'(\pm 1, 0), \\
\int_{-1}^{1} Q_l(\sigma) d\sigma = 0.
\end{cases}$$

Since $Q_l^{s-2l}$ is odd this "external force" does not contribute to the integral constraint. Moreover we can still lift the boundary condition with the constant function $\sqrt{\mu_c}$. This leads to a contradiction between the integral vanishing and $c_l$ being non zero.

On the other hand if $2s - 4l < l$, then the problem solved by $Q_{2s-4l}$ is

$$\begin{cases}
- \mu_c \partial^2_q Q_{2s-4l} = \mu_{c_{s-l}} \partial^2_q Q_{s-2l} - \partial_q Q_{s-2l}, \\
Q_{2s-4l}'(\pm 1) = 0, \\
\int_{-1}^{1} Q_{2s-4l}(\sigma) d\sigma = 0.
\end{cases}$$

Once again, $\partial^2_q Q_{s-2l}$ being odd, $\int_{-1}^{1} Q_{2s-4l}(\sigma) d\sigma$ by linearity is really only $\int_{-1}^{1} C(\sigma) d\sigma$ where $C$ solves:

$$\begin{cases}
- \mu_c \partial^2_q C = - \partial_q Q_{s-2l}, \\
C(\pm 1) = 0.
\end{cases}$$

If we multiply the equation for $C$ by $(\sigma^2 - 1)/2$ and integrate twice by parts we find :

$$\mu_c \int_{-1}^{1} C(\sigma) d\sigma = - \int_{-1}^{1} Q_{2s-4l}(\sigma) \sigma d\sigma.$$

Now using the expression of $Q_{s-2l}$ one finds $\int_{-1}^{1} C(\sigma) d\sigma = 1/(3c_l^2)$ and this cannot be 0. We then necessarily have $2s - 4l = l$ that is $2s = 5l$. It is left to the reader to see that $s = 5$ and $l = 2$ lead to the minimum of non trivial profiles. One also finds that $c_l$ satisfies the following equation,

$$2\sqrt{\mu_c} - \frac{1}{3c_l^2} = 0$$

which leads to $c_l = 1/(3\sqrt{2})^{2/5}$. \[\square\]


4.1. Reformulation as a Two Parameter Problem. We shall now give a rigorous proof that the behavior of (2.1a)–(2.1h) is the one described by Theorem 2.1. For this, we first rewrite the system in new variables. We then explain the argument of the proof using some technical assumptions. We finally check that these assumptions are true (unique solvability of the fundamental problem, analyticity and derivatives of the implicit function).

Reformulation Through New Parameters. A crucial step in the analysis is to rewrite the system with new variables: instead of having $y$ and $\phi$ we set

$$a = \frac{y}{\phi} \quad \text{and} \quad b = \sqrt{\phi}.$$
Let us remark that this kind of change of variable was also necessary in our previous paper [12] in order to study the behavior near the singularity \( (y, \phi) = (0, 0) \). However, \( y \) and \( \phi \) have clear physical meaning (the first is a shear rate, the second is the fluidity).

We then define from the solution \( p \) the functions
\[
q = p_{\mid[-1,1]},
\]
\[
r_d(\theta) = p(1+b\theta), \quad \text{for } \theta > 0,
\]
\[
r_g(\theta) = p(-1-b\theta), \quad \text{for } \theta > 0.
\]

In these variables Eq. 1.1 can be written:

\[
-\partial_x^2 q + a\partial_y q = \frac{1}{\mu} \delta_0, \quad \text{in } ]-1,1[, \quad (4.1a)
\]
\[
-\partial_x^2 r_g - ab \partial_y r_g + r_g = 0, \quad \text{in } ]0,\infty[, \quad (4.1b)
\]
\[
-\partial_x^2 r_d + ab \partial_y r_d + r_d = 0, \quad \text{in } ]0,\infty[, \quad (4.1c)
\]
\[
r(\pm \infty) = 0, \quad (4.1d)
\]
\[
q \geq 0, \quad (4.1e)
\]
\[
r \geq 0, \quad (4.1f)
\]
\[
r_g(0) = q(-1), \quad (4.1g)
\]
\[
r_d(0) = q(1), \quad (4.1h)
\]
\[
-\partial_y r_g(0) = b\partial_y q(-1), \quad (4.1i)
\]
\[
\partial_y r_d(0) = b\partial_y q(1), \quad (4.1j)
\]
\[
b \int_0^{+\infty} r_g(\theta) d\theta + b \int_0^{+\infty} r_d(\theta) d\theta + \int_{-1}^{1} q(\sigma) d\sigma = 1. \quad (4.1k)
\]

A major advantage of this formulation is that the reformulated problem no longer is of a singularly perturbed nature. This is the deeper reason why in the formal expansion of the previous section there was never any true two-parameter series; either all the \( R^k \) or all the \( \bar{R}^k \) were zero. In several dimensions, the singularly perturbed nature of the problem cannot be removed by a mere rescaling of the independent variable, and this will complicate the analysis. In particular, analyticity of the solution with respect to \( a \) and \( b \) as shown below cannot be expected.

**Proof of Theorem 2.1.** In terms of the two parameter expansion, the three cases encountered in the previous section are as follows:

1. \( \mu > 1/2 \): \( a \to 0, b \to b_0 > 0 \).
2. \( \mu < 1/2 \): \( b \to 0, a \to a_0 > 0 \).
3. \( \mu = 1/2 \): Both \( a \) and \( b \) tend to 0 and \( b \) is of the same order as \( a^2 \).

The structure of our argument essentially proceeds as follows: First we show that if either \( a \) or \( b \) is close to 0, we can solve the system \((4.1a)-(4.1j)\) uniquely for \( q, r_d \) and \( r_g \) as functions of \( \mu, a \) and \( b \). The remaining equation \((4.1k)\) is then of the form \( F(\mu, a, b) = 0 \). The function \( F \) depends analytically on its arguments.

**Case \( \mu > 1/2 \).** We can show that \( \partial F/\partial b(0,b_0,\mu) \neq 0 \), hence we solve the equation \( F = 0 \) for \( b \): \( b = g(\mu, a) \). In terms of \( y \) this reads \( \sqrt{\phi(y)} = g(\mu, y/\phi(y)) \). We can now use the implicit function theorem again to solve for \( \phi \) as an analytical function of \( y \).

**Case \( \mu < 1/2 \).** We can show that \( \partial F/\partial a(a_0,0,\mu) \neq 0 \), and hence solve for \( a \): \( a = g(\mu, b) \). In terms of the original variables, this reads \( y = \phi(y)g(\mu, \sqrt{\phi(y)}) \). We can
now apply another implicit function argument to solve for \( \sqrt{q(y)} \) as an analytical function of \( \sqrt{y} \); the leading order term is \( \sqrt{q(y)} \sim y/\sqrt{\mu} = \sqrt{y}/a_0 \).

*Case \( \mu = 1/2 \).* For \( \mu \) near \( 1/2 \), we can solve \( F \) uniquely for \( \mu \): \( \mu - 1/2 = g(a, b) \). This confirms that \( \mu = 1/2 \) is the only case in which \( a = b = 0 \). We shall show that, to leading order \( g(a, b) = c_1b - c_2a^2 \) with positive constants \( c_1 \) and \( c_2 \). Thus, at \( \mu = 1/2 \), we have a balance of \( b \) and \( a^2 \). If we fix \( \mu \) at \( 1/2 \), we can solve for \( b \) in terms of \( a \):

\[
 b = \gamma_2a^2 + \gamma_3a^3 + ..., \quad \gamma_2 = c_2/c_1.
\]

Now we substitute \( b = \beta y^{2/5} \), leading to \( a = \beta^{-2}y^{1/5} \). We find the following new equation for \( \beta \):

\[
 \beta = \gamma_2/\beta^4 + \gamma_3y^{1/5}/\beta^6 + ...
\]

Another implicit function argument shows that \( \beta \) is an analytical function of \( y^{1/5} \), with leading term \( \beta \sim (\gamma_2)^{1/5} \).

4.2. The Fundamental Problems. We are now exhibiting the three fundamental solutions we will perturb; that is, we will uniquely solve (4.1a)–(4.1j) for specific sets of the parameters \((\mu, a, b)\). Of course we will find the three fundamental problems of §3 rewritten in the new variables (without satisfying the integral constraint yet).

Set \( a = 0 \) in the previous equations and let us find \( \mu \) and \( b \) that satisfy the equations. We can solve the ODEs and use the various constraints to set the integration constants. This leads to

\[
 q^{\mu,0,b}(\sigma) = \begin{cases} 
 \frac{\sigma + 1}{2\mu^2} & \text{if } \sigma \in ]0,1[, \\
 \frac{1}{2\mu}(1 - \sigma) + \frac{b}{2\mu} & \text{if } \sigma \in ]-1,0[,
\end{cases}
\]

\[
 r^{\mu,0,b}_g(\theta) = \frac{b}{2\mu} \exp(-\theta),
\]

\[
 r^{\mu,0,b}_d(\theta) = \frac{b}{2\mu} \exp(-\theta). \tag{4.2}
\]

On the other hand set \( b = 0 \). Then the equations of \( q \) on one hand and \( r_g, r_d \) on the other hand decouple. We can set \( r_g = r_d = 0 \) by linearity and this makes the problem set on the interval \([-1,1]\) only, with homogeneous Dirichlet boundary conditions. This leads to

\[
 q^{\mu,a,0}(\sigma) = \begin{cases} 
 1 - \exp(-a) & \text{if } \sigma \in ]0,1[,
\end{cases}
\]

\[
 r^{\mu,a,0}_g = 0,
\]

\[
 r^{\mu,a,0}_d = 0. \tag{4.3}
\]

Finally, taking the limit \( b \to 0 \) in the first case or \( a \to 0 \) in the second one leads to the same solution for \( a = b = 0 \)

\[
 q^{\mu,0,0}(\sigma) = \begin{cases} 
 \frac{\sigma + 1}{2\mu} & \text{if } \sigma \in ]-1,0[,
\end{cases}
\]

\[
 \frac{1}{2\mu}(1 - \sigma) & \text{if } \sigma \in ]0,1[.
\]
4.3. Unique Solvability of the ODE system. We define the weighted Sobolev space

$$H^s(0, \infty) = \{ u \mid \exp(\varepsilon \theta) u(\theta) \in H^s(0, \infty) \}.$$  

We fix a small $\varepsilon > 0$, and we seek solutions $(q, r_d, r_g)$ in $H^1(-1, 1) \times (H^1(0, \infty))^2$. We can get rid of the weight $\exp(\varepsilon \theta)$ by setting $r_{g,d} = \exp(-\varepsilon \theta) s_{g,d}$. Moreover, we isolate the singularity of $q$ at the origin by setting $\tilde{q}(\sigma) = q(\sigma) - (1 - |\sigma|)/(2\mu)$. This leads to the new system

$$- \partial_\sigma^2 \tilde{q} + a \partial_\sigma \tilde{q} = \frac{a}{2\mu} \sign \sigma \quad \text{in } [0, \infty[, \quad \text{(4.4a)}$$

$$- \partial_\sigma^2 s_g - (ab - 2\varepsilon) \partial_\sigma s_g + (1 - \varepsilon^2 + ab\varepsilon) s_g = 0 \quad \text{in } [0, \infty[, \quad \text{(4.4b)}$$

$$- \partial_\sigma^2 s_d + (ab - 2\varepsilon) \partial_\sigma s_d + (1 - \varepsilon^2 - ab\varepsilon) s_d = 0 \quad \text{in } [0, \infty[, \quad \text{(4.4c)}$$

$$s_g(0) = \tilde{q}(-1), \quad s_d(0) = \tilde{q}(1), \quad \text{(4.4d)}$$

We now look for solutions in the function space

$$X = \{(\tilde{q}, s_d, s_g) \in H^2(-1, 1) \times (H^2(0, \infty))^2\}.$$  

Define $Y$ as

$$Y = L^2(-1, 1) \times (L^2(0, \infty))^2 \times (\mathbb{R})^4.$$  

We define the operator $L_{a,b}$ from $X$ to $Y$ by

$$L_{a,b}(\tilde{q}, s_d, s_g) = (-\tilde{q}'' + a \tilde{q}', s_g'' - (ab - 2\varepsilon) s_g' + (1 - \varepsilon^2 + ab\varepsilon) s_g, s_d'' + (ab - 2\varepsilon) s_d' + (1 - \varepsilon^2 - ab\varepsilon) s_d, s_g(0) - \tilde{q}(-1), s_d(0) - \tilde{q}(1), -s_g'(0) + \varepsilon s_g(0) - b\tilde{q}'(-1), s_d'(0) - \varepsilon s_d(0) - b\tilde{q}'(1)).$$

Let us note $E = \mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R}_+$. We claim the following Proposition.

**Proposition 4.1.**

- If $\varepsilon$ is sufficiently small and $(a, b) \in E$ then $L_{a,b}$ is an invertible operator from $X$ onto $Y$.
- For $(a_0, b_0) \in E$ and $\varepsilon$ chosen as above, then there exist a neighborhood of $(a_0, b_0)$ such that $L_{a,b}$ is analytical and invertible.
- The mapping $(\mu, a, b) \mapsto (q^{\mu,a,b}, r_\mu, r_g^{\mu,a,b}, r_d^{\mu,a,b})$ is analytic.

**Proof.** For the proof of the first point, it clearly suffices to consider $\varepsilon = 0$ and either $a = 0$ or $b = 0$; the rest follows from perturbation theory (see for instance Kato[11]). If $\varepsilon = b = 0$, the problems for $\tilde{q}$, $s_g$ and $s_d$ decouple, and the decoupled problems are easy to analyze. Let $\varepsilon = a = 0$, $b > 0$, and assume $L_{a,b}(\tilde{q}, s_d, s_g) = 0$. It is easily checked that then

$$\int_{-1}^{1} (\tilde{q}')^2 d\sigma + \frac{1}{b} \int_{0}^{\infty} (s_g')^2 + s_g^2 + (s_d')^2 + s_d^2 d\theta = 0.$$
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which immediately implies \( \tilde{q} = 0, s_d = s_g = 0 \).

The second point is proved by perturbation theory, and the third point uses the fact that \( L_{a,b}^{-1} \), as a function of \( a \) and \( b \), is analytic, so is

\[
(\mu, a, b) \mapsto \left( \frac{a}{2\mu} \text{ sign } \sigma, 0, 0, 0, \frac{b}{2\mu}, -\frac{b}{2\mu} \right),
\]

and finally so are the various change of variables. \( \square \)

4.4. Derivatives of \( F \). We now set

\[
F(\mu, a, b) = \int_{-1}^{1} q(\sigma) \, d\sigma + b \int_{0}^{\infty} g(\theta) \, d\theta + b \int_{0}^{\infty} d(\theta) \, d\theta,
\]

where \( q, r_d \) and \( r_g \) have been determined as solutions of the ODE system as discussed in the previous subsection.

If \( a = 0 \), we compute, using (4.2),

\[
F(\mu, 0, b) = \frac{b + b^2}{\mu} + \frac{1}{2\mu}.
\]

A nonnegative solution \( b_0 \) of the equation \( F(\mu, 0, b_0) = 1 \) exists if and only if \( \mu \geq 1/2 \). Moreover, we find

\[
\frac{\partial F}{\partial b}(\mu, 0, b_0) = \frac{1 + 2b_0}{\mu} > 0,
\]

\[
\frac{\partial F}{\partial \mu}(\mu, 0, b_0) = -\frac{F(\mu, 0, b_0)}{\mu} = -\frac{1}{\mu} < 0.
\]

If \( b = 0 \), we calculate, using (4.3),

\[
F(\mu, a, 0) = \frac{\tanh(a/2)}{a\mu}.
\]  

(4.5)

As \( a \) varies, this function decreases monotonically from a limit of \( 1/(2\mu) \) at \( a = 0 \) to zero as \( a \to \infty \). Hence the equation \( F(\mu, a_0, 0) = 1 \) has a solution \( a_0 \) if and only if \( \mu \leq 1/2 \). The derivative \( \partial F/\partial a(\mu, a, 0) \) is negative for \( a > 0 \), but \( \partial F/\partial a(\mu, 0, 0) = 0 \). Moreover, we find

\[
\frac{\partial^2 F}{\partial a^2}(\mu, 0, 0) = -\frac{1}{12\mu} < 0.
\]  

(4.6)

This verifies all the sign properties of derivatives of \( F \) claimed above.

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