Homogenization via unfolding in periodic elasticity with contact on closed and open cracks

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Abstract. We consider the elasticity problem in a heterogeneous domain with an $\varepsilon$-periodic micro-structure, $\varepsilon \ll 1$, including multiple micro-contacts between the structural components. These components can be a simply connected matrix domain with open cracks or inclusions completely surrounded by cracks, which do not touch the outer boundary. The contacts are described by the Signorini and Tresca-friction contact conditions. The Signorini condition is described mathematically by a closed convex cone, while the friction condition is a nonlinear convex functional over the interface jump of the solution on the oscillating interface.

The difficulties appear when the inclusions are completely surrounded by cracks and can have rigid displacements. In this case, in order to obtain preliminary estimates for the solution in the $\varepsilon$-domain, the Korn inequality should be modified, first in the fixed context and then for the $\varepsilon$-dependent periodic case. Additionally, for all states of the contact (inclusions can freely move, or are locked/stuck to the interface with the matrix, or the frictional traction is achieved on the inclusion-matrix interface and the inclusions can slide in the tangential to the interface direction) we obtain estimates for the solution in the $\varepsilon$-domain, uniform with respect to $\varepsilon$.

An asymptotic analysis (as $\varepsilon \to 0$) for the nonlinear functionals over the growing interface is carried out, based on the application of the periodic unfolding method for sequences of jumps of the solution on the oscillating interface. This allows to obtain the homogenized limit as well as a corrector result.

Keywords: ???

1. Introduction

Contact problems for inner oscillating interface were considered in [14,17,18] and in the books [16] and [13]. In particular, [14] and [16] (Chapter 3, Section 5) deal with frictionless Signorini problems on oscillating interfaces, while in [13] (Chapter 10), contact problems with friction are considered. In [16] small cracks are imposed and the homogenization is done in a formal way. In [14] one considers the case

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of particles (which can rotate) diluted in the matrix material. The problem is handled via the two-scale homogenization.\(^1\)

The papers [17,18] provide tools for weak convergence of some nonlinear Robin-type conditions on the inner oscillating interface under appropriate assumptions on the nonlinearity. The physically interesting scaling of frictional functional on the inner oscillating interface \(\lim_{\varepsilon \to 0} \int_{S_\varepsilon} \Psi^{-1}(\varepsilon^{-1} x, u^\varepsilon(x)) \, ds\) is considered also, but under some strong restrictions on the growth and the mean value of the function \(\Psi\).

In [12], the correct scaling is considered and the linear Robin-type interface conditions are handled by the two-scale convergence on the oscillating interface.

In this paper, we consider a problem which combines unilateral contacts (which gives a condition on the jump of the normal components of the strain) with a Tresca-type friction condition (which is on the jump of the tangential strain – cf. (5.7) for their explicit form). We make extensive use of the periodic unfolding technique [4,8], which has two advantages: first, it transforms sequences of functions defined on varying domains and interfaces in sequences defined on fixed domains and interfaces; second, it significantly simplifies the proofs.

We consider an example in which the stiffness of the matrix and of inclusions surrounded by open cracks are of the same order. An interesting result is that if the initial gap between the inclusions and the matrix is of the order of \(\varepsilon\), the rigid part of the displacements of the particles strongly converges to the homogenized displacements.

The non-penetration and friction contact conditions are given by periodic nonlinear functions of a weak convergent unfolded argument. Several possibilities are known to prove their weak or two-scale convergence. In [11] and [18], the function \(\Psi = \Psi(y, u^\varepsilon)\) in the nonlinear Robin-type condition on the oscillating interface is linearized via approximations by Taylor series and reduced on this way to the linear case [12]. We use the convexity of these nonlinear functions and prove their convergence by lower semi-continuity with respect to weak convergences.

One can apply the method in the two simpler physically relevant cases:

- perfect normal contact and tangential friction;
- frictionless unilateral non-penetration.

But, since the proof is essentially the same, we give it for a combination of both condition, even though the mechanical relevance of such a problem is not clear. The full problem of non-penetration together with a friction condition only where there is actual contact is not variational therefore much more complex and beyond the scope of this contribution. It is actually a dynamic problem, and is usually time-discretized, and it is this situation we consider here (see [9] and [10] for details). One key ingredient in the proofs is a couple of unilateral Korn inequalities for Lipschitz domains which seem new (see Proposition 4.5).

The paper is organized as follows. Section 2 gives the geometric setting for the \(\varepsilon\)-periodic problem, including the unit cell. Section 3 presents the standard forms of the Korn inequality which will be used in the paper. In Section 4, we give a few inequalities related to the unfolding operator on the interface surfaces, then establish a uniform Korn inequality for the perforated matrix domain. Then, two unilateral Korn inequalities are proved with their applications to the oscillating inclusions in the domain. Section 5 concerns the problem for fixed \(\varepsilon\)-periodicity. Existence of the solution is proved and its uniqueness is analyzed. Finally, in Section 6, under more precise hypotheses, the homogenization of the problem is proved and the corrector result is established.

\(^1\)However, in [14], there is a flaw in the main estimate which implicitly assumes no jump in the tangential strain; hardly frictionless!
1.1. Notations

- The normal component of a vector field \( v \) on the boundary of a domain is denoted \( v_\nu \), while the tangential component \( v - v_\nu \nu \) is denoted \( v_\tau \) (where \( \nu \) is the outward unit normal to the boundary);
- the strain tensor of a vector field \( v \) is denoted by \( e(v) \); its values are symmetric \( 3 \times 3 \) real matrices, the set of which is denoted \( M^3_\text{sym}(\mathbb{R}) \);
- the kernel of \( e \) in a connected domain is the finite dimensional space of rigid motions denoted \( \mathcal{R} \);
- \( C \) indicates a generic constant which does not depend upon the scaling factor;

2. Geometric set up and first hypotheses

The problem is set in the natural space \( \mathbb{R}^3 \), although the results can be obtained in any dimension \( N \geq 2 \).

In the following, \( Y \subset \mathbb{R}^3 \) is the unit cell (or a set having the paving property with respect to the group of periods), and \( Y^* \) is the “cracked” \( Y \), i.e. \( Y \) without all the cracks within \( Y \) (see Fig. 1).

The set \( Y^* \) has one or several single connected components which intersect the boundary \( \partial Y \), it is assumed that their union with the corresponding part of neighboring cells is still connected. We denote them by \( Y^0 \) and call this set the “matrix”. We have in mind either a cracked composite structure or a meshlike fiber set such as textiles.

The other connected components of \( Y^* \), denoted \( Y^1, \ldots, Y^m \), are finitely many, they correspond to the inclusions completely surrounded by “closed” cracks: \( S^j = \partial Y^j \) inside \( Y \). The other cracks, whose union is denoted \( S^0 \), are “open cracks” which are included in the boundary of \( Y^0 \) inside \( Y \). They are open in the sense that \( Y^0 \) lies on both sides of the connected components of \( S^0 \). They can go up to the boundary of \( Y \) and even in such a way that, when extended by periodicity, such a crack produces a connected crack across several neighboring cells. Note that

\[
S^0 = (\partial Y^0 \cap Y) \setminus \bigcup_{j=1}^m S^j.
\]

We make the assumption that the set \( Y^0 \) satisfies the hypotheses of the paper on the unfolding for holes [3] (concerning the set which is denoted in [3] by \( Y^* \)). The results from [3] will be our essential tools in the present paper. As in [3], we require that \( Y^0 \) satisfy the Poincaré–Wirtinger inequality (for the space \( H^1 \)), and that the union of \( Y^0 \) and its (3) direct neighbors be connected.

Fig. 1. The unit cell \( Y \). (Colors are visible in the online version of the article; http://dx.doi.org/10.3233/ASY-2012-1141.)
Recall that in the periodic setting, (almost) every point \( z \in \mathbb{R}^3 \), can be written as

\[
z = \lfloor z \rfloor_Y + \{ z \}_Y,
\]

where, as in the one-dimensional case, \( \lfloor z \rfloor_Y \) for \( z \in \mathbb{R}^3 \), denotes the unique integer combination of periods such that \( \{ z \}_Y = z - \lfloor z \rfloor_Y \) belongs to \( Y \) (see Fig. 2).

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^3 \) with Lipschitz boundary, and \( \Gamma_D \) be a non-empty open subset of the boundary \( \partial \Omega \) (\( \Gamma_D \) is the set where a Dirichlet condition will be prescribed).

Let \( \tilde{\Omega}_\varepsilon \) denote the set of the points \( x \in \Omega \) whose \( \varepsilon \)-cell, \( \varepsilon \lfloor \frac{x}{\varepsilon} \rfloor_Y + \varepsilon Y \), does not intersect the Neumann boundary \( \Gamma_N = \partial \Omega \setminus \Gamma_D \). Denote also by \( \Lambda_\varepsilon = \Omega \setminus \tilde{\Omega}_\varepsilon \), the subset of \( \Omega \) containing the parts from cells intersecting the boundary \( \Gamma_N \).

For \( j = 1, \ldots, m \), introduce the set

\[
\Omega^j_\varepsilon = \left\{ x \mid x \in \tilde{\Omega}_\varepsilon \text{ such that } \varepsilon \left\{ \frac{x}{\varepsilon} \right\}_Y \in Y^j \right\}.
\]
The boundary $\partial \Omega^j_\varepsilon$ is the set of “closed cracks” associated with $S^j_\varepsilon$,

$$\partial \Omega^j_\varepsilon \equiv S^j_\varepsilon = \left\{ x \mid x \in \hat{\Omega}_\varepsilon \text{ such that } \varepsilon \{ \frac{x}{\varepsilon} \} \in S^j_\varepsilon \right\}.$$  

For $j = 0$, set

$$S^0_\varepsilon = \left\{ x \mid x \in \hat{\Omega}_\varepsilon \text{ such that } \varepsilon \{ \frac{x}{\varepsilon} \} \in S^0_\varepsilon \right\}$$

and

$$\Omega^0_\varepsilon = \Omega \setminus \bigcup_{j=1, \ldots, m} \overline{\Omega^j_\varepsilon \cup S^0_\varepsilon}.$$  

The union of all the cracks is denoted $S^*_\varepsilon$,

$$S^*_\varepsilon = \bigcup_{j=0,1,\ldots,m} S^j_\varepsilon.$$  

Finally, set

$$\Omega^*_\varepsilon = \Omega \setminus S^*_\varepsilon.$$  

Note that from these definitions, it is clear that there are no cracks in the layer $\Lambda_\varepsilon$.

For a function $v$ defined on $\Omega^*_\varepsilon$, for simplicity, we denote its restriction to $\Omega^j_\varepsilon$ by $v^j$:

$$v^j = v|_{\Omega^j_\varepsilon} \quad \text{for } j = 0, \ldots, m.$$  

We use the same notation $\mathcal{T}_\varepsilon$ for the unfolding operator applied to functions defined on $\Omega$ or subsets of $\Omega$ (such as the $\Omega^j_\varepsilon$’s and the $S^j_\varepsilon$’s) with all its properties from [4]. The properties of its restrictions to functions defined on $\Omega^0_\varepsilon$ or on the $S^j_\varepsilon$ follow from [3], with a slight variation on the open cracks $S^0_\varepsilon$.

In the following, for any bounded set $\mathcal{O}$ and $\varphi \in L^1(\mathcal{O})$, $\mathcal{M}_\mathcal{O}(\varphi)$ denotes the mean value of $\varphi$ over $\mathcal{O}$, i.e.,

$$\mathcal{M}_\mathcal{O}(\varphi) = \frac{1}{|\mathcal{O}|} \int_{\mathcal{O}} \varphi \, dy.$$  

3. Korn inequalities

As mentioned in the Introduction, we will need some Korn inequalities in order to obtain a priori estimates for the solutions of the problem to be considered (see (5.5) and (5.6) below). Recall that the kernel of the strain operator $\varepsilon$ in an open set is formed by the vector fields which are a rigid motion (in each connected component of the set). We denote by $\mathcal{R}$ the finite vector space of rigid motions:

$$\mathcal{R} = \{ x \mapsto v_{a,b}(x) = a \land x + b; a \text{ and } b \in \mathbb{R}^3 \}.$$
Definition 3.1. A bounded connected domain $O$ is a Korn domain whenever it satisfies the second Korn inequality, i.e., if there exists a constant $C$ such that

$$\forall v \in H^1(O), \quad \|v\|_{H^1(O)} \leq C\left(\|v\|_{L^2(O)} + \|e(v)\|_{L^2(O)}\right).$$

(3.1)

It is known (see [15]) that this holds for every connected bounded Lipschitz domain, but it is true for more general domains.

Analogously, the following Korn inequality (which, because of its similarity with the Poincaré–Wirtinger inequality, we call the Korn-W inequality) holds for connected bounded Lipschitz domains, but does hold in more general connected domains:

Definition 3.2. The bounded domain $O$ satisfies the Korn-W inequality if there exists a constant $C$ such that for every $v \in H^1(O)$ there is a rigid motion $r_v \in \mathbb{R}$ with

$$\|v - r_v\|_{H^1(O)} \leq C\|e(v)\|_{L^2(O)}.$$

One could expect that the Korn-W and the Poincaré–Wirtinger inequalities are related. This seems to be an open question.

Definition 3.3. For a bounded domain $O$ which satisfies the Korn-W inequality, we denote by $W^1(O)$ the orthogonal subspace of $\mathbb{R}$ in $H^1(O)$. Therefore $H^1(O)$ is the orthogonal sum of $W^1(O)$ and $\mathbb{R}$.

Remark 3.4. If the Korn-W inequality holds, it follows that

$$\|\text{proj}_{W^1(O)} v\|_{H^1(O)} \leq C\|e(v)\|_{L^2(O)}.$$

In particular, for $v \in W(O)$,

$$\|v\|_{H^1(O)} \leq C\|e(v)\|_{L^2(O)}.$$

As a consequence, $P = \text{Id} - \text{proj}_{W^1(O)}$ is the orthogonal projection from $H^1(O)$ to $\mathbb{R}$ and $\|v\|^2 = \|e(v)\|^2_{L^2(O)} + \|P(v)\|^2_{L^2(O)}$ defines an equivalent norm on $H^1(O)$. Of course, the operator $P$ depends upon $O$, but, for simplicity, we will not indicate this dependence (as it is always obvious for which domain it applies and it needlessly complicates the notations).

Consider a bounded Lipschitz domain $O$, hence satisfying both the Korn-W and the Poincaré–Wirtinger inequalities together with a Dirichlet part $\Gamma_D$ (with positive measure) of its boundary $\partial O$.

Then the following result holds:

Proposition 3.5. Let $O$ and $\Gamma_D$ be as above. There is a constant $C$ such that

$$u \in H^1(O) \text{ with } u = 0 \text{ on } \Gamma_D \quad \Rightarrow \quad \|u\|_{H^1(O)} \leq C\|e(v)\|_{L^2(O)}.$$  

(3.2)
Proof. By (3.2), there is a constant $C$ with \[ \|v - r_v\|_{H^1(O)} \leq C \|e(v)\|_{L^2(O)}. \] By the trace theorem, \[ \|r_v\|_{L^2(\Gamma_D)} \leq C \|e(v)\|_{L^2(O)}. \] Since $\mathcal{R}$ is finite dimensional, this implies \[ \|r_v\|_{L^2(\partial\Omega)} \leq C \|e(v)\|_{L^2(O)}. \] whence the result.

Regarding the set $Y^0$, we make the assumption that it also satisfies the Korn-W inequality. This holds whenever $Y^0$ is connected and Lipschitz, even with Lipschitz open cracks that intersect $\partial Y$ (since it is then a finite union of overlapping Lipschitz domains).

Consequently, by the same type of argument as that in the proof of Proposition 3.5, if we consider the subspace $V(\omega)$ of $H^1(Y^0)$ consisting of the functions vanishing on the given open subset $\omega$ of $Y^0$, the following holds:

\[ \|v\|_{H^1(Y^0)} \leq C \|e(v)\|_{L^2(Y^0)}. \] (3.3)

4. Some inequalities related to unfolding and the geometric domain

4.1. Some extra formulas from unfolding

Recall some formulas related to the unfolding operator (see for more details [3]) defined for any function $\phi$ Lebesgue-measurable on $\Omega^j$ by

\[ T_\varepsilon(\phi)(x, y) = \begin{cases} \phi\left(\varepsilon\frac{x}{\varepsilon} + \varepsilon y\right) & \text{a.e. for } (x, y) \in \tilde{\Omega} \times Y^j, \\ 0 & \text{otherwise}. \end{cases} \]

For $\varphi \in H^1(\Omega^j)$, recall that $\nabla_y(T_\varepsilon(\varphi)) = \varepsilon T_\varepsilon(\nabla \varphi)$ on $\Omega \times Y^j$. In a similar way, $e_y(T_\varepsilon(\varphi)) = \varepsilon T_\varepsilon(e(\varphi))$.

For $\varphi \in L^p(S^j)$, $T_\varepsilon(\varphi)$ is similarly defined for $y$ in $S^j$. Furthermore,

\[ \int_{S^j} \varphi \, d\sigma(x) = \frac{1}{\varepsilon|Y|} \int_{\Omega \times S^j} T_\varepsilon(\varphi)(x, y) \, dx \, d\sigma(y) \] (4.1)

and

\[ \|\varphi\|_{L^p(S^j)} = \frac{1}{(\varepsilon|Y|)^{1/p}} \|T_\varepsilon(\varphi)\|_{L^p(\Omega \times S^j)}. \] (4.2)

We will need a reverse Hölder inequality for functions such as images via unfolding which are piecewise constant on $\varepsilon$-cells.
Let $\psi$ be a function defined on $\hat{\Omega}_e \times Y^j$ and piece-wise constant with respect to $x$ on subsets all of measure $\varepsilon^3 |Y|$. Then, for $1 < p \leq \infty$, $1/p + 1/p' = 1$,

$$
\|\psi\|_{L^p(\hat{\Omega}_e \times Y^j)} \leq |\Omega|^{1/p'} \|\psi\|_{L^p(\hat{\Omega}_e \times L^1(Y^j))},
$$

$$
\|\psi\|_{L^p(\hat{\Omega}_e \times L^1(Y^j))} \leq |Y|^{-1/p'} \varepsilon^{-3/p'} \|\psi\|_{L^p(\hat{\Omega}_e \times Y^j)}.
$$

Similarly,

$$
\|\psi\|_{L^1(\hat{\Omega}_e \times S^j)} \leq |\Omega|^{1/p'} \|\psi\|_{L^p(\hat{\Omega}_e \times L^1(S^j))},
$$

$$
\|\psi\|_{L^p(\hat{\Omega}_e \times L^1(S^j))} \leq |Y|^{-1/p'} \varepsilon^{-3/p'} \|\psi\|_{L^p(\hat{\Omega}_e \times S^j)}.
$$

Note that the reverse inequalities are sharp (in case that $\psi$ is supported only in one cell).

**4.2. A Korn inequality for the perforated matrix domain $\Omega^0_e$**

Let $\eta$ be a fixed positive number, sufficiently small in order that the set

$$
Y^0_{\eta} = \{y \in Y^0 \mid \text{dist}(y, Y \cap \partial Y^0) > \eta\},
$$

be connected. Clearly, $Y^0_{\eta}$ is Lipschitz. Therefore

- there is an extension operator $P$ from $H^1(Y^0_{\eta})$ to $H^1(Y)$ which is the identity on constant functions and a constant $C$ such that

$$
\|P(v)\|_{L^2(Y)} \leq C\|v\|_{L^2(Y^0_{\eta})} \quad \text{and} \quad \|\nabla P(v)\|_{L^2(Y)} \leq C\|\nabla v\|_{L^2(Y^0_{\eta})}
$$

(see Cioranescu-Saint Jean Paulin extension from [7]),

- $Y^0_{\eta}$ is a Korn-domain. So there is an equivalent Hilbert norm on $H^1(Y^0_{\eta})$ such that the projection on the subspace of rigid motions $R$ in the sense of this norm, is the same as the projection in the $L^2$ norm (choose as equivalent norm $\|v\|^2 = \|v\|^2_{L^2(Y^0_{\eta})} + \|e(v)\|^2_{L^2(Y^0_{\eta})}$). We denote the projection of $v$ on $R$ as $r_v$. By the previous argument (all the norms being equivalent on $R$),

$$
\|r_v\|_{R} \leq C\|r_v\|_{L^2(Y^0_{\eta})} \leq C\|v\|_{L^2(Y^0_{\eta})},
$$

- by the Korn-W inequality there is a constant $C$ such that

$$
\|v - r_v\|_{H^1(Y^0_{\eta})} \leq C\|e(v)\|_{L^2(Y^0_{\eta})}.
$$

Now, we define another extension operator $Q$ from $P$. To do so, for $v \in H^1(Y^0_{\eta})$, set $Q(v) = P(v - r_v) + r_v$. It is easily seen that $Q$ is the identity on constant functions and that

$$
\|Q(v)\|_{L^2(Y)} \leq C\|v\|_{L^2(Y^0_{\eta})}, \quad \|Q(v)\|_{H^1(Y)} \leq C\|v\|_{H^1(Y^0_{\eta})},
$$

$$
\|e(Q(v))\|_{L^2(Y)} \leq C\|e(v)\|_{L^2(Y^0_{\eta})}.
$$
Hence, by applying the second inequality of (4.5) to \( v - M_{\eta} v \) (keeping in mind that \( M_{\eta} v \) belongs to \( \mathcal{R} \)) gives
\[
\| \nabla Q(v) \|_{L^2(Y^0)} \leq C \| v - M_{\eta} v \|_{L^2(Y^0)} + \| \nabla v \|_{L^2(Y^0)},
\]
from which, by the Poincaré–Wirtinger inequality (which holds in \( Y^0 \)),
\[
\| \nabla Q(v) \|_{L^2(Y^0)} \leq C \| \nabla v \|_{L^2(Y^0)}.
\]

Since \( Q(v) - v \) vanishes on \( Y^0 \), by (3.3)
\[
\| Q(v) - v \|_{H^1(Y^0)} \leq C \| e(Q(v) - v) \|_{L^2(Y^0)} \leq C \| e(v) \|_{L^2(Y^0)}.
\]

**Proposition 4.1.** With the choice of \( \eta \) indicated above, there exists an extension operator \( Q \) from \( H^1(Y^0) \) to \( H^1(Y^0) \) such that (4.5), (4.6) and (4.7) hold. As a consequence, there exists a constant \( C \) independent of \( \varepsilon \) such that
\[
\forall u \in H^1(\Omega^0) \text{ with } u|\Gamma_D = 0, \quad \| u \|_{H^1(\Omega^0)} \leq C \| e(u) \|_{L^2(\Omega^0)}.
\] (4.8)

**Proof.** The properties of \( Q \) are already proven.
Let \( \Omega^0_{\varepsilon, \eta} \) denote the subset of \( (\Omega^0_{\varepsilon, \eta})^c \) corresponding to \( Y^0 \). By rescaling (4.5), (4.6) and (4.7), then adding on all the small periodic cells of \( \Omega^0_{\varepsilon, \eta} \), one defines an “extension” operator \( Q_\varepsilon \) from \( H^1(\Omega^0_{\varepsilon, \eta}) \) to \( H^1(\Omega) \) such that for \( u \in H^1(\Omega^0_{\varepsilon, \eta}) \)
\[
\| Q_\varepsilon(u) \|_{L^2(\Omega)} \leq C \| u \|_{L^2(\Omega^0_{\varepsilon, \eta})},
\]
\[
\| Q_\varepsilon(u) - u \|_{L^2(\Omega^0_{\varepsilon, \eta})} \leq \varepsilon C \| e(u) \|_{L^2(\Omega^0_{\varepsilon, \eta})},
\]
\[
\| \nabla (Q_\varepsilon(u) - u) \|_{L^2(\Omega^0_{\varepsilon, \eta})} \leq C \| e(u) \|_{L^2(\Omega^0_{\varepsilon, \eta})},
\]
\[
\| e(Q_\varepsilon(u)) \|_{L^2(\Omega^0_{\varepsilon, \eta})} \leq C \| e(u) \|_{L^2(\Omega^0_{\varepsilon, \eta})}.
\]
Furthermore, since \( u \) vanishes on \( \Gamma_D \) so does \( Q_\varepsilon(u) \). By the Korn inequality (3.2) in \( \Omega \) it follows that
\[
\| Q_\varepsilon(u) \|_{H^1(\Omega)} \leq C \| e(u) \|_{L^2(\Omega^0_{\varepsilon, \eta})},
\]
from which (4.8) follows. \( \square \)

**4.3. Two unilateral Korn inequalities for the inclusions \( \Omega^0_{\varepsilon, \eta} \)**

Regarding the inclusions \( Y^0 \), we need to distinguish two classes according to a geometric property (of their shapes).

**Definition 4.2.** A locked domain is a bounded domain with Lipschitz boundary for which the only rigid motion which is tangent on its boundary is zero.
Remark 4.3. Via the exponential map, one can see that a locked domain is a bounded Lipschitz domain whose group of isometries is discrete.

In the next inequalities, it is of interest that only the positive part of $v_\nu$ (the normal component of $v$, see Notations in Section 1) appears. Obviously, applying them to $-v$ would involve $v_\nu^-$ alone.

We start by a simple remark concerning the rigid motions in a bounded domain with a Lipschitz boundary.

Lemma 4.4. Let $O$ be a bounded connected and Lipschitz domain. Then,

$$ r \mapsto \|r\| \equiv \|r_r\|_{L^1(\partial O)} + \|(r_\nu)^+\|_{L^1(\partial O)} \text{ is a norm on } \mathcal{R}. $$

If furthermore, $O$ is a locked domain, then

$$ r \mapsto \|r\|_t \equiv \|(r_\nu)^+\|_{L^1(\partial O)} \text{ is a norm on } \mathcal{R}. $$

Proof. Every $r \in \mathcal{R}$ is divergence-free. Hence $\int_{\partial O} r_\nu \, d\sigma = 0$, so that

$$ \|(r_\nu)^+\|_{L^1(\partial O)} = 2\|(r_\nu)^+\|_{L^1(\partial O)}. $$

Therefore $\| \cdot \|$ is a norm provided it vanishes only for $r = 0$. If $\|r\| = 0$, then $r|_{\partial O}$ is identically zero. It is well-known that such a rigid motion has to be 0. This follows from the equi-projectivity of $r$ itself (for each point $x$ of $O$ the projection of $r(x + te_i)$ on a basis vector $e_i$ is independent of $t$, and for some $t$, $x + te_i$ belongs to $\partial O$).

Similarly, if $\|r\| = 0$, then $r|_{\partial O}$ is tangent to $\partial O$. If $O$ is a locked domain, this implies $r = 0$. \hfill \Box

The previous lemma yields two new Korn-type inequalities which are essential in order to get uniform estimates for our problem. These inequalities seem new.

Proposition 4.5 (Unilateral Korn inequalities). If $O$ is a bounded connected and Lipschitz domain, there exists a constant $C_1$ such that

$$ \forall v \in H^1(O), \quad \|v\|_{H^1(O)} \leq C_1 \left( \|v_\nu\|_{L^1(\partial O)} + \|(v_\nu)^+\|_{L^1(\partial O)} + \|v_\tau\|_{L^1(\partial O)} \right). \quad (4.9) $$

If $O$ is a locked domain, then there exists a constant $C_2$ such that

$$ \forall v \in H^1(O), \quad \|v\|_{H^1(O)} \leq C_2 \left( \|v_\nu\|_{L^1(\partial O)} + \|(v_\nu)^+\|_{L^1(\partial O)} \right). \quad (4.10) $$

Proof. We give the proof for the second inequality, the first is obtained in a similar way.

The proof is done by contradiction. If inequality (4.10) is false, there exists a sequence $v_n$ in $H^1(O)$ such that

$$ \|v_n\|_{H^1(O)} \equiv 1, \quad \|e(v_n)\|_{L^2(O)} \to 0, \quad \|(v_\nu)^+\|_{L^1(\partial O)} \to 0. $$
By the compactness of the embedding of $H^1(O)$ into $L^2(O)$, there is a $v$ in $H^1(O)$ such that

\[ v_n \rightarrow v \quad \text{in} \quad H^1(O), \quad v_n \rightarrow v \quad \text{in} \quad L^2(O), \quad e(v_n) \rightarrow 0 \quad \text{in} \quad L^2(O), \]

together with $(v_n)^+ \rightarrow 0$ in $L^1(O)$. Consequently, $e(v) = 0$ so that $v$ is a rigid motion, and by the continuity of the trace operator, $(v_n)^+ \rightarrow 0$. Since $O$ is a locked domain, this implies that $v = 0$ and $\|v_n\|_{L^2(O)} \rightarrow 0$. But, recalling that $O$ is a Korn domain, by (3.1),

\[ \|v_n\|_{H^1(O)} \leq C(\|v_n\|_{L^2(O)} + \|e(v_n)\|_{L^2(O)}) \rightarrow 0, \]

which contradicts the fact that $\|v_n\|_{H^1(O)} \equiv 1$. □

We now turn back to functions defined on the domain $\Omega^j_e$. From now on, the number $\delta_j$ will be defined for $j = 1, \ldots, m$ as follows:

\[ \delta_j = \begin{cases} 0 & \text{if } Y^j\text{ is a locked domain,} \\ 1 & \text{otherwise.} \end{cases} \]

By scaling, we get the following result:

**Proposition 4.6.** For every $j = 1, \ldots, m$, there exists a constant $C$ such that for every $u$ in $H^1(\Omega^j_e)$,

\[ \|u\|_{L^2(\Omega^j_e)} + \varepsilon\|\nabla u\|_{L^2(\Omega^j_e)} \leq C\varepsilon\|e(u)\|_{L^2(\Omega^j_e)} + C(\|T_e(u_n)\|_{L^2(\Omega^j_e \times |\partial Y^j_e|)} + \delta_j\|T_e(u)\|_{L^2(\Omega^j_e \times |\partial Y^j_e|)}). \quad (4.11) \]

**Proof.** We consider the case $\delta_j = 1$, the proof being similar for $\delta_j = 0$.

Apply (4.9) or (4.10) to $v = T_e(u)$ to obtain

\[ \|T_e(u_x)\|_{L^2(\Omega^j_e \times |\partial Y^j_e|)} \leq C\varepsilon\|T_e(u)\|_{L^2(\Omega^j_e \times |\partial Y^j_e|)}^2 + C(\|T_e(u_n)\|_{L^2(\Omega^j_e \times |\partial Y^j_e|)} + \delta_j\|T_e(u)\|_{L^2(\Omega^j_e \times |\partial Y^j_e|)}^2). \]

Now using the facts that $e(T_e(u)) = eT_e(e(u))$ and $Y^j$ is a Korn domain, we integrate over $\Omega$ to get

\[ \|T_e(u_x)\|_{L^2(\Omega \times |\partial Y^j_e|)} \leq C\varepsilon\|T_e(u)\|_{L^2(\Omega \times |\partial Y^j_e|)}^2 + C(\|T_e(u_n)\|_{L^2(\Omega \times |\partial Y^j_e|)} + \delta_j\|T_e(u)\|_{L^2(\Omega \times |\partial Y^j_e|)}^2). \quad (4.12) \]

But from (4.1) and (4.2), the following equalities hold:

\[ \|T_e(u_x)\|_{L^2(\Omega \times |\partial Y^j_e|)} = \sqrt{|Y^j|}\|u_x\|_{L^2(\Omega^j_e)} \cdot \sqrt{|\nabla_y T_e(u)\|_{L^2(\Omega^j_e \times |\partial Y^j_e|)} = \varepsilon\sqrt{|Y^j|}\|\nabla u\|_{L^2(\Omega^j_e)} \cdot \sqrt{|\nabla_y T_e(u)\|_{L^2(\Omega^j_e \times |\partial Y^j_e|)} = \varepsilon\sqrt{|Y^j|}\|e(u)\|_{L^2(\Omega^j_e)}, \]

which, used in (4.12), give the result. □
5. The contact problem for fixed $\varepsilon$

For every $\varepsilon$ we assume that the tensor field $a^\varepsilon = (a^\varepsilon_{\alpha\beta\gamma\delta})$ is given with the usual properties of symmetry

$$a^\varepsilon_{\alpha\beta\gamma\delta} = a^\varepsilon_{\beta\alpha\gamma\delta} = a^\varepsilon_{\alpha\beta\delta\gamma} = a^\varepsilon_{\gamma\delta\alpha\beta},$$

and satisfying

$$\sup \max_{\varepsilon} \|a^\varepsilon_{\alpha\beta\gamma\delta}\|_{L^\infty(\Omega)} < \infty, \quad \bar{\alpha}_{\alpha\beta\gamma\delta} \leq a^\varepsilon_{\alpha\beta\gamma\delta} \bar{\alpha}_{\alpha\beta\gamma\delta}, \quad (5.1)$$

with $\bar{\alpha}$ independent of $\varepsilon$.

Introduce the following symmetric bilinear form:

$$a^\varepsilon(e(u), e(v)) = \sum_{j=0}^m \sum_{\alpha\beta\gamma\delta=1}^3 a^\varepsilon_{\alpha\beta\gamma\delta}(x)e(u)_{\gamma\delta}(x)e(v)_{\alpha\beta}(x) \, dx.$$ 

Let $K^\varepsilon$ be the convex set defined, for non-negative $g^\varepsilon$ on $S^\varepsilon_l$, by

$$K^\varepsilon = \{ v = (v^0, \ldots, v^m) | v \in H^1(\Omega^0, \Gamma_D) \times \cdots \times H^1(\Omega^m, \nu^0), v^j - v^j_0 \leq g^\varepsilon_j \text{ on } S^\varepsilon_l \text{ and } [v^j]_{S^0_l} \leq g^0_j \} \quad (5.2)$$

The vector fields $v$ are the admissible deformation fields with respect to the reference configuration $\Omega^\varepsilon$. We will denote by $[v]_{S^\varepsilon_l}$ the jump of the vector field across the surface $S^\varepsilon_l$. For $j = 1, \ldots, m$, one has $[v^j]_{S^\varepsilon_l} = v^j - v^j_0$. For simplicity, we may use the notations $[v^j]_{S^\varepsilon_l}$ and $[v^j]_{S^\varepsilon_l}$ in place of $([v]_{S^\varepsilon_l})$, and $([v]_{S^\varepsilon_l})$, respectively. By standard trace theorems, these jumps belong to $H^{1/2}(S^\varepsilon_l)$.

The tensor field

$$\sigma^\varepsilon_{\alpha\beta}(v) = \sum_{\gamma=1}^3 a^\varepsilon_{\alpha\beta\gamma\delta} \tau_{\gamma\delta}(v)$$

is the stress tensor associated to the deformation $v$ (not to be confused with the surface measures $d\sigma$).

The functions $g^\varepsilon_j$ and the $g^\varepsilon_j$’s are the original gaps (in the reference configuration), and the corresponding inequalities in the definition of $K^\varepsilon$ represent the non-penetration conditions. In case there is contact in the reference configuration, these functions are just 0.

Consider also the family of convex functions $\Psi^j_\varepsilon$, $0 \leq j \leq m$, where $\Psi^0_j$ is non-negative and lower semi-continuous on $H^{1/2}(S^\varepsilon_l)$, while for $j = 1, \ldots, m$, $\Psi^j_\varepsilon$ is continuous on $H^{1/2}(S^\varepsilon_l)$ and satisfies

$$\Psi^j_\varepsilon(w) \geq M^j_\varepsilon \|w\|_{L^1(S^\varepsilon_l)} \quad \text{for non-negative real numbers } M^j_\varepsilon. \quad (5.3)$$

In case of Tresca friction, the functions $\Psi^j_\varepsilon$ are actually explicitly given by

$$\Psi^j_\varepsilon(w) = \int_{S^\varepsilon_l} G^j_\varepsilon(x) |w| \, d\sigma(x), \quad (5.4)$$

with $\overline{\left[\Psi^j_\varepsilon\right]}_{S^\varepsilon_l}$.
with $G^j_\epsilon$ bounded below by $M^j_\epsilon$ for $j = 1, \ldots, m$.

**Problem $P_\epsilon$.** Given $f_\epsilon = (f^0_\epsilon, \ldots, f^m_\epsilon)$ in $L^2(\Omega_\epsilon)$ find a minimizer over $K^\epsilon$ of the functional

$$
E_\epsilon(v) = \frac{1}{2} \mathbf{a}^\epsilon(e(v), e(v)) + \sum_{j=0}^m \Psi^j_\epsilon([v_\tau]_{S^j_\epsilon}) - \int_{\Omega^*_\epsilon} f_\epsilon v \, dx.
$$

(5.5)

From the properties of convexity of the $\Psi^j_\epsilon$, $j = 0, \ldots, m$, the solutions of $P_\epsilon$ are the same as that of the following problem:

**Problem $P'_\epsilon$.** Find $u_\epsilon \in K^\epsilon$ such that for every $v \in K^\epsilon$,

$$
\mathbf{a}^\epsilon(e(u_\epsilon), e(v - u_\epsilon)) + \sum_{j=0}^m \left( \Psi^j_\epsilon([v_\tau]_{S^j_\epsilon}) - \Psi^j_\epsilon([u_\epsilon]_{S^j_\epsilon}) \right) \geq \int_{\Omega^*_\epsilon} f_\epsilon(v - u_\epsilon) \, dx.
$$

(5.6)

The “strong formulation” of this problem is given here, with the notation $\sigma^\epsilon(u_\epsilon)$ for the stress tensor $\sigma^\epsilon$:

- Find $u_\epsilon \in K^\epsilon$ such that
  - $\text{div} \, \sigma^\epsilon = f_\epsilon$ in $\Omega^*_\epsilon$,
  - $[(u_\epsilon)_\nu]_{S^j_\epsilon} - g^j_\epsilon \leq 0$,
  - $\sigma^\epsilon(\nu)_\nu \leq 0$,
  - $\sigma^\epsilon(\nu)_\nu([u_\epsilon]_{S^j_\epsilon}) - g^j_\epsilon = 0$,
  - $\sigma^\epsilon(\nu)_\tau \in \partial \Psi^j_\epsilon([u_\epsilon]_{S^j_\epsilon})$ on $S^j_\epsilon$ for $j = 0, \ldots, m$,

where $\partial \Psi^j_\epsilon$ denotes the subdifferential of $\Psi^j_\epsilon$ (here taken in the sense of the $L^2(S^j_\epsilon)$ duality).

The corresponding explicit Tresca conditions on the interface $S^j_\epsilon$ with the function $\Psi^j_\epsilon$ given in (5.4) are as follows:

$$
\begin{align*}
|[(u_\epsilon)_\tau]|_{S^j_\epsilon} + \lambda_j^\epsilon |\sigma^\epsilon(\nu)_\tau| = 0 & \quad \text{a.e. on } S^j_\epsilon, \\
\lambda_j^\epsilon \geq 0, & \quad G^j_\epsilon - |\sigma^\epsilon(\nu)_\tau| \geq 0,
\end{align*}
$$

(5.7)

where $\sigma^\epsilon(\nu)$ is the stress vector on the surface $S^j$ (whose normal is $\nu$), and $\sigma^\epsilon(\nu)_\tau$ is its tangential component (see, for example [13] Section 10.3 for a detailed derivation).

The first step in the proof of existence of a solution consists in obtaining a bound for minimizing sequences. We denote by $u$ the general term of such a sequence. We use the generic notation $C$ for constants which can be expressed independently of the data.

We start with the usual estimate obtained for the variational inequality, by taking into account that $\nu = 0$ belongs to $K^\epsilon$,

$$
\alpha \sum_{j=0}^m \|e(u^j)\|_{L^2(\Omega^*_\epsilon)}^2 + \sum_{j=0}^m \Psi^j_\epsilon([u_\tau]_{S^j_\epsilon}) \leq \int_{\Omega^*_\epsilon} f^0_\epsilon u^0 \, dx + \sum_{j=1}^m \int_{\Omega^*_\epsilon} f^j_\epsilon u^j \, dx + C.
$$

(5.8)
We now use (4.8) to control the first term on the right-hand side to get
\[
\int_{\Omega_\varepsilon} f_\varepsilon u_\varepsilon^0 \, dx \leq C \| f_\varepsilon^0 \|_{L^2(\Omega_\varepsilon^0)} \| e(u_\varepsilon^0) \|_{L^2(\Omega_\varepsilon^0)}.
\] (5.9)

The other terms of the right-hand side for \(1 \leq j \leq m\), are simply bounded as follows,
\[
\int_{\Omega_\varepsilon} f_j^j u_j^j \, dx \leq \| f_j^j \|_{L^2(\Omega_\varepsilon^0)} \| u_j^j \|_{L^2(\Omega_\varepsilon^0)},
\]
which requires to control \(\| u_j^j \|_{L^2(\Omega_\varepsilon^0)}\).

**Proposition 5.1.** There exists a constant \(C\) such that for all \(u \in H^1(\Omega_\varepsilon^0), j = 1, \ldots, m\),
\[
\| u_j^j \|_{L^2(\Omega_\varepsilon^0)} + \varepsilon \| \nabla u_j^j \|_{L^2(\Omega_\varepsilon^0)} \leq C(\| e(u_\varepsilon^0) \|_{L^2(\Omega_\varepsilon^0)} + \varepsilon \| e(u^0) \|_{L^2(\Omega_\varepsilon^0)})
\] + \(C(\| T_\varepsilon(g_j^j) \|_{L^2(\Omega; L^1(\varepsilon \partial Y^{j}))} + \delta_j \varepsilon^{-1/2} \| [u_j^j] \|_{L^1(S_j^0)}).\) (5.10)

**Proof.** By the definition (5.2) of \(K^\varepsilon\), the jump of the normal component of \(u\) across \(S_j^0\) is bounded above by \(g_j^j\). Applying the unfolding operator, it follows that one can control \(T_\varepsilon(u_j^j)^+_{S_j^0}\) (the positive part of the normal component of the trace of \(T_\varepsilon(u_j^j)\) on the inside of \(\partial Y^j\)), by \(T_\varepsilon(u_\varepsilon^0)\) (its trace on the outside of \(\partial Y^j\)), and by \(T_\varepsilon(g_j^j)\). By the trace theorem, the outer trace is controlled by \(\| T_\varepsilon(u_j^j) \|_{L^2(\Omega; H^1(\varepsilon \partial Y^j))}\). More precisely,
\[
\| T_\varepsilon(u_j^j) \|_{L^2(\Omega; H^1(\varepsilon \partial Y^j))} \leq C \| T_\varepsilon(u_j^j)^+_{S_j^0} \|_{L^2(\Omega; H^1(\varepsilon \partial Y^j))} \leq C \| u_j^j \|_{H^1(\Omega_\varepsilon^0)} \leq C \| e(u_\varepsilon^0) \|_{L^2(\Omega_\varepsilon^0)}.
\]

Then, for the positive part of the normal component, using (4.3), one gets
\[
\| T_\varepsilon(u_j^j)^+_{S_j^0} \|_{L^2(\Omega; L^1(\varepsilon \partial Y^j))} \leq \| T_\varepsilon(g_j^j) \|_{L^2(\Omega; L^1(\varepsilon \partial Y^j))} + \varepsilon \| e(u_\varepsilon^0) \|_{L^2(\Omega_\varepsilon^0)}.
\]

Similarly, concerning the tangential components, one has
\[
\| T_\varepsilon(u_j^j)^+_{S_j^0} \|_{L^2(\Omega; L^1(\varepsilon \partial Y^j))} \leq \| T_\varepsilon(u_j^j)^+_{S_j^0} \|_{L^2(\Omega; L^1(\varepsilon \partial Y^j))} + \varepsilon \| e(u_\varepsilon^0) \|_{L^2(\Omega_\varepsilon^0)}.
\]

By (4.4) and (4.2) one observes that
\[
\| T_\varepsilon(u_j^j)^+_{S_j^0} \|_{L^2(\Omega; L^1(\varepsilon \partial Y^j))} \leq C \frac{1}{\varepsilon^3 |Y^j|} \| T_\varepsilon(u_j^j)^+_{S_j^0} \|_{L^2(\Omega; L^1(\varepsilon \partial Y^j))} = C \varepsilon^{-1} \| [u_j^j] \|_{L^2(\Omega_\varepsilon^0)}.
\]

Combining all these inequalities with (4.11), we get the result. \(\Box\)

**Remark 5.2.** Note that if one uses the \(L^2\)-norms, then
\[
\| T_\varepsilon(g_j^j) \|_{L^2(\Omega; L^1(\varepsilon \partial Y^j))} \leq C \| T_\varepsilon(g_j^j) \|_{L^2(\Omega; L^1(\varepsilon \partial Y^j))} = \sqrt{\varepsilon} C \| g_j^j \|_{L^2(S_j^0)}.
\]
Remark 5.3. An analogous estimate can be found in [12], but the estimate of Proposition 5.1 above is sharper (with $\varepsilon$ instead of $\varepsilon^{-1}$).

From (5.10) it follows that

$$
\int_{\Omega^j} f^j_2 u^j \, dx \leq C \| f^j_2 \|_{L^2(\Omega^j)} (\| e(u^0) \|_{L^2(\Omega^j)} + \varepsilon \| e(u^j) \|_{L^2(\Omega^j)} + \| T_\varepsilon(g^j_0) \|_{L^2(\Omega^j)} + \varepsilon^{-1/2} \| [u_\varepsilon]_{S^j_1} \|_{L^2(S^j_1)}) \tag{5.11}
$$

One can improve this last inequality, provided the restriction $f^j_2$ to the non-locked domains (case $\delta_j = 1$) satisfies an extra condition.

Lemma 5.4. Suppose that $f^j_2$, $j = 1, \ldots, m$, is locally orthogonal to the rigid motions, that is

$$
\int_{Y^j} T_\varepsilon(f^j_2) r \, dy = 0 \quad \text{for every } r \in \mathcal{R} \text{ and for a.e. } x \in \Omega. \tag{5.12}
$$

Then

$$
\int_{\Omega^j} f^j_2 u^j \, dx \leq C \varepsilon \| f^j_2 \|_{L^2(\Omega^j)} \| e(u^j) \|_{L^2(\Omega^j)}. \tag{5.13}
$$

Proof. By the Korn-W inequality in $Y^j$, there exists a measurable $r^j_2(x) \in \mathcal{M}_Y(T_\varepsilon(u^j))$ in $\mathcal{R}$ such that

$$
\| T_\varepsilon(u^j) - r^j_2(x) \|_{L^2(\Omega \times Y^j)} \leq C \| e(T_\varepsilon(u^j)) \|_{L^2(\Omega \times Y^j)} = C \varepsilon \| e(u^j) \|_{L^2(\Omega^j)}. \tag{5.14}
$$

We write successively

$$
\int_{\Omega^j} f^j_2 u^j \, dx = \frac{1}{|Y|} \int_{\Omega \times Y^j} T_\varepsilon(f^j_2) T_\varepsilon(u^j) \, dx \, dy = \frac{1}{|Y|} \int_{\Omega \times Y^j} T_\varepsilon(f^j_2) (T_\varepsilon(u^j) - r^j_2(x)) \, dx \, dy,
$$

from which (5.13) follows in view of (5.14). \qed

Proposition 5.5. Suppose that for non-locked domains (case $\delta_j = 1$)

(i) either $\frac{1}{\varepsilon^{1/2} M^j_1} \| f^j_2 \|_{L^2(\Omega^j)}$ is small enough (compared to $C_1$ from (4.9)),

(ii) or $f^j_2$ satisfies condition (5.12),

(iii) or $f^j_2 = f^{j,1}_2 + f^{j,2}_2$, with $f^{j,1}_2$ satisfying condition (i) and $f^{j,2}_2$ condition (ii),

where $M^j_1$ are defined by (5.3) (and assumed to be strictly positive for cases (i) and (iii)).

Then there exists a constant $C$ independent of $\varepsilon$ such that for $u$ in a minimizing sequence,

$$
\| e(u^0) \|_{L^2(\Omega^j)} + \frac{m}{j=1} \| e(u^j) \|_{L^2(\Omega^j)} + \frac{m}{j=0} \| \psi^j_2 ([u_\varepsilon]_{S^j_1}) \|_{L^2(\Omega \times \partial Y^j)} \leq C \left( \| f^j_2 \|_{L^2(\Omega^j)} + \frac{m}{j=1} \| T_\varepsilon(g^j_0) \|_{L^2(\Omega \times \partial Y^j)} \right). \tag{5.15}
$$
Proof. For \( \delta_j = 0 \), use (5.11) to get an upper bound for \( \int_{\Omega_j} f^j \, u^j \, dx \). Similarly, for the case \( \delta_j = 1 \) with \( f^j \) orthogonal to solid motions, use (5.13). For the case \( \delta_j = 1 \) with \( f^j = f^{j,1} + f^{j,2} \), we use the same estimate for \( \int_{\Omega_j} f^{j,2} \, u^j \, dx \), while \( \int_{\Omega_j} f^{j,1} \, u^j \, dx \) is estimated by (5.11) with the following inequality:

\[
C \frac{1}{\varepsilon} f^j \| f^j \|_{L^2(\Omega_j)} \| [u_r]_{S_j} \|_{L^1(\Omega_j)} \leq \rho \Phi^j \| [u_r]_{S_j} \|
\]

for some \( \rho \in (0, 1) \). Consequently, we obtain

\[
\int_{\Omega_j} f^j \, u^j \, dx \leq \rho \Phi^j (u) + C \| e(u^0) \|_{L^2(\Omega_j)} \sum_{j=0}^m \| f^j \|_{L^2(\Omega_j)}
\]

\[
+ C \sum_{j=1}^m \| f^j \|_{L^2(\Omega_j)} \| e(u^j) \|_{L^2(\Omega_0)} + \| T_e(g_j^j) \|_{L^2(\Omega; L^1(\partial Y_j))}
\]

Combining (5.8) and (5.9), (5.11), (5.13) and (5.16) finally yields

\[
\sum_{j=0}^m \| e(u^j) \|_{L^2(\Omega_j)} + \sum_{j=0}^m \Phi_j^j ([u_r]_{S_j})
\]

\[
\leq C \| e(u^0) \|_{L^2(\Omega_j)} \sum_{j=0}^m \| f^j \|_{L^2(\Omega_j)}
\]

\[
+ C \sum_{j=1}^m \| f^j \|_{L^2(\Omega_j)} \| e(u^j) \|_{L^2(\Omega_0)} + \| T_e(g_j^j) \|_{L^2(\Omega; L^1(\partial Y_j))}
\]

Estimate (5.15) is obtained simply by multiple applications of the classical inequality \( ab \leq \ell a^2 + \frac{1}{4\ell} b^2 \) (for any \( \ell > 0 \)).

Corollary 5.6. Set

\[
D(\varepsilon) = \| f \|_{L^2(\Omega; \mathbb{R})}^2 + \sum_{j=1}^m \| T_e(g_j^j) \|_{L^2(\Omega; L^1(\partial Y_j))}^2
\]

Then under the hypotheses of Proposition 5.5, there exists a fixed constant \( C \) such that for \( u = (u^0, \ldots, u^m) \) element of a minimizing sequence

\[
\sum_{j=0}^m \| e(u^j) \|_{L^2(\Omega_j)} + \sum_{j=0}^m \Phi_j^j ([u_r]_{S_j}) \leq CD(\varepsilon),
\]

\[
\| r^j \|_{L^2(\Omega; \mathbb{R})} + \| T_e(u^j) \|_{L^2(\Omega; H^1(\partial Y_j))} \leq CD(\varepsilon)(1 + \varepsilon) \quad \text{for } \delta_j = 0,
\]

\[
\| r^j \|_{L^2(\Omega; \mathbb{R})} + \| T_e(u^j) \|_{L^2(\Omega; H^1(\partial Y_j))^1 + L^2(\Omega; \mathbb{R})} \leq CD(\varepsilon) \left( 1 + \frac{\varepsilon}{M^j_0} \right) \quad \text{for } \delta_j = 1.
\]
Furthermore, the following estimate holds:

\[ \| u^j \|_{H^1(\Omega^j)} \leq CD(\varepsilon) \left[ 1 + \delta_j \left( \varepsilon^{-3/2} + \varepsilon \left( 1 + \frac{1}{M_\varepsilon^2} \right) \right) \right]. \]  

(5.19)

**Proof.** To obtain the first estimate of (5.18), apply (5.15) together with (4.8).

For \( \delta_j = 0 \), consider (5.10) rewritten in terms of \( T_\varepsilon(u^j) \).

\[ \| T_\varepsilon(u^j) \|_{L^2(\Omega; H^1(Y^j))} \leq C \| e(u^0) \|_{L^2(\Omega^j)} + \varepsilon \| e(u^j) \|_{L^2(\Omega^j)} + \| T_\varepsilon(g^j) \|_{L^2(\Omega; L^2(\partial Y^j))}. \]  

(5.20)

This implies the second estimate of (5.18) for \( T_\varepsilon(u^j) \).

To prove the remaining estimates of (5.18), as in the beginning of the proof of Lemma 5.4, use the Korn-W inequality (in \( Y^j \)) to derive the existence of piecewise constant functions \( r^j \) with values in \( \mathcal{R} \) satisfying (5.14). Then,

\[ \| T_\varepsilon(u^j) - r^j \|_{L^2(\Omega; H^1(Y^j))} \leq C \| e(u^0) \|_{L^2(\Omega^j)} \]

\[ = C\varepsilon \| e(u^j) \|_{L^2(\Omega^j)} \leq C\varepsilon D(\varepsilon). \]  

(5.21)

This, together with (5.20) proves the second estimate for \( r^j \) in \( L^2(\Omega; \mathcal{R}) \).

For \( \delta_j = 1 \), (5.21) implies

\[ \| T_\varepsilon(u^j) - r^j \|_{L^2(\Omega; \mathcal{R})} \leq C\varepsilon D(\varepsilon). \]

On the other hand, from the definition of \( \mathcal{K}^j \) we have the estimate

\[ \| (T_\varepsilon(w^j) - T_\varepsilon(u^0))_\nu \|_{L^2(\Omega; L^2(\partial S^j))} \leq \varepsilon \| T_\varepsilon(g^j) \|_{L^2(\Omega; L^1(S^j))}. \]

Also, from (5.3),

\[ \| (T_\varepsilon(w^j) - T_\varepsilon(u^0))_\tau \|_{L^2(\Omega; S^j)} \leq \frac{\varepsilon}{M_\varepsilon^2} \Phi^j(\varepsilon). \]

From these three last estimates, it follows that

\[ \| (r^j_\nu)_{\partial \Omega \times S^j} \|_{L^2(\Omega \times S^j)} \leq \varepsilon \| T_\varepsilon(g^j) \|_{L^2(\Omega; L^1(S^j))} + C\varepsilon D(\varepsilon) + C\| T_\varepsilon(u^0) \|_{L^2(\Omega \times S^j)} \]

and

\[ \| (r^j_\tau)_{\partial \Omega \times S^j} \|_{L^2(\Omega \times S^j)} \leq \varepsilon \left( 1 + \frac{\varepsilon}{M_\varepsilon^2} \Phi^j(u_\tau_{\partial \Omega \times S^j}) \right) + \| T_\varepsilon(u^0) \|_{L^2(\Omega \times S^j)}. \]

Since by (4.8) and the trace theorem,

\[ \| T_\varepsilon(u^0) \|_{L^2(\Omega; H^1(\Omega^j))} \leq C \| e(u^0) \|_{L^2(\Omega^j)}, \]
Lemma 4.4 implies the estimates of (5.18) for \( \|r^j_\varepsilon\|_{L^1(\hat{\Omega}_\varepsilon; \mathbb{R})} \) (recall that \( \delta_j = 1 \)).

Reverting to \( u^j_\varepsilon \), and making use of (5.21), this in turn implies the corresponding estimate of (5.18) for \( \|T^\varepsilon(u^j_\varepsilon)\|_{L^2(\Omega; H^1(Y_j)) + L^1(\Omega; \mathbb{R})} \).

Finally, we note that the function \( r^j_\varepsilon \) is in the finite dimensional space of \( \varepsilon \)-piecewise constant functions on \( \hat{\Omega}_\varepsilon \) (i.e. constant on each \( \varepsilon [\frac{x}{\varepsilon}]_Y + \varepsilon Y \)). Therefore by (4.3),

\[
\|r^j_\varepsilon\|_{L^1(\hat{\Omega}_\varepsilon; \mathbb{R})} \leq \varepsilon^{-3/2} \|r^j_\varepsilon\|_{L^1(\Omega; \mathbb{R})}.
\]

This, together with (5.21) again, proves (5.19). \( \Box \)

By standard minimizing arguments, we obtain the following result:

**Corollary 5.7.** For each \( \varepsilon > 0 \), there is a solution \( u^\varepsilon = (u^0_\varepsilon, \ldots, u^m_\varepsilon) \) in \( K^\varepsilon \) for the Problem \( P^\varepsilon \).

Uniqueness holds only for \( u^0_\varepsilon \) and for \( u^j_\varepsilon - \text{proj}_R u^j_\varepsilon \), \( j = 1, \ldots, m \) (equivalently for \( u^0_\varepsilon \) and for \( f(u^j_\varepsilon) \)).

**Proof.** Any minimizing sequence is bounded in \( K^\varepsilon \), and by weak lower semi-continuity, every weak limit point of such a sequence is a solution. Using formulation (5.6) for two solutions \( u^\varepsilon \) and \( \tilde{u}^\varepsilon \), one easily gets that \( a^\varepsilon(u^\varepsilon - \tilde{u}^\varepsilon, u^\varepsilon - \tilde{u}^\varepsilon) = 0 \). It follows that \( e(u^\varepsilon - \tilde{u}^\varepsilon) = 0 \), which by (4.8), implies \( u^0_\varepsilon - \tilde{u}^0_\varepsilon = 0 \), and \( u^j_\varepsilon - \tilde{u}^j_\varepsilon \in \mathcal{R} \) in each connected component of \( \Omega^j_\varepsilon \). \( \Box \)

**Remark 5.8.** Uniqueness for the \( j \)-th components (for \( j = 1, \ldots, m \)) cannot be obtained because of the lack of global strict convexity of the functional (5.5) on \( K^\varepsilon \).

### 6. Homogenization

The aim of this section is to pass to the limit as \( \varepsilon \to 0 \) in problem (5.6). We will obtain a limit “homogenized” problem which is given in Theorem 6.10. For the proof we use the unfolding method and the results from Sections 4 and 5 above.

We start by formulating the hypotheses we make in order to prove the homogenization result.

#### 6.1. Boundedness hypotheses on the data

In this subsection we give hypotheses on \( f_\varepsilon, g^j_\varepsilon, \Psi^j_0 \) and \( \Psi^j_\varepsilon \) (\( j = 1, \ldots, m \)) which allow to obtain uniform estimates with respect to \( \varepsilon \) for the solutions \( u^\varepsilon \).

First of all, we suppose that \( D(\varepsilon) \) (defined by (5.17)) is bounded uniformly with respect to \( \varepsilon \). We also suppose that for all \( j \) with \( \delta_j = 1 \) (non-locked domains), \( f_\varepsilon \) satisfies hypotheses of Proposition 5.5.

**6.1.1. Hypotheses on \( g^j_\varepsilon \) and \( \Psi^j_\varepsilon \) (\( j = 0, \ldots, m \))**

To pass to the limit in the homogenization process, we need structural assumptions concerning the contact data which are more precise than those of Section 5 (see (5.17)), namely

1. Concerning the function \( g^j_\varepsilon \), there is a \( g^j \) in \( L^1(S^j) \) with

\[
g^j_\varepsilon(x) = \varepsilon g^j \left( \left\{ \frac{x}{\varepsilon} \right\} \right) 1_{\hat{\Omega}_\varepsilon}(x) \quad \text{for} \ x \in S^j,
\]
so that
\[ T_\varepsilon (g^j_\varepsilon (x, y)) = \varepsilon g^j (y) 1_{\Omega_\varepsilon} (x) \quad \text{for } (x, y) \in \Omega \times S^j. \]

2. We assume that \( \Psi_j^0 \) is a convex continuous function of its argument. More precisely, there is a positive normal convex integrand \( \psi_0 \) on \( \Omega \times \mathbb{H}^1/2(S^0) \) which vanishes on 0, such that
\[
\Psi_j^0 (w) \doteq \int_\Omega \psi_0 \left( x, \frac{1}{\varepsilon} T_\varepsilon (w) \right) \, dx.
\]

3. For \( j = 1, \ldots, m \), we assume that \( \Psi_j^0 \) is of the form
\[
\Psi_j^0 (w) = \Theta_j \left( \frac{1}{\varepsilon} w_\tau \right),
\]
where
\[
\Theta_j (v) = \int_{\Omega \times S^j} G_j (x, y) |v| \, dx \, d\sigma (y),
\]
for continuous and positive \( G_j \)'s. Then,
\[
M_j = \min G_j.
\]
It is further assumed that for all \( j = 1, \ldots, m \) such that \( \delta_j = 1 \), \( M_j \) is strictly positive.

Note that \( \Theta_j \) has a natural extension to the space of bounded measures on \( \Omega \times S^j \) which is continuous with respect to the strict topology on that space, namely
\[
\int_{\Omega \times S^j} G_j \, d(\rho (|v|)), \quad \text{where } d(\rho (|v|)) \text{ is the total variation of the measure } \rho.
\]

We will still denote this extension by \( \Theta_j \).

This choice of \( \Psi_j^0 \) corresponds to a Tresca friction coefficient \( \varepsilon^{-1} G_j (x, \{ x/\varepsilon \} Y) \).

6.1.2. Convergence hypotheses on \( f_\varepsilon \)

From the boundedness hypotheses of proposition 5.5 it follows that, up to a subsequence (for \( \varepsilon \)), one can assume some weak convergences. We make the convergence hypotheses on \( f_\varepsilon \) explicit here.

There exists some \( f^0 \in L^2 (\Omega \times Y^0) \) with
\[
f_\varepsilon 1_{A_\varepsilon} \to 0 \quad \text{strongly in } L^2 (\Omega),
\]
\[
T_\varepsilon (f_\varepsilon^0) \rightharpoonup f^0 \quad \text{weakly in } L^2 (\Omega \times Y^0),
\]
\[
(6.1)
\]

\[ \text{This hypothesis means that } x \mapsto \psi^0 (x, \rho) \text{ is measurable for every } \rho \in H^{1/2} (S^0) \text{ while for a.e. } x \in \Omega, \text{ the map } \rho \mapsto \psi^0 (x, \rho) \text{ is positive and lower semi-continuous convex on } H^{1/2} (S^0). \]

\[ \text{This is the so-called “étroite” topology which combines the weak-\ast topology and the convergence of the total variation norm.} \]
and for all \( j = 1, \ldots, m \), with \( \delta_j = 0 \) (locked domains), there exist \( f^j \in L^2(\Omega \times Y^j) \) with

\[
\mathcal{T}_\varepsilon(f^j) \rightarrow f^j \quad \text{strongly in } L^2(\Omega \times Y^j).
\]

(6.2)

In the sequel \( F \) denotes the function

\[
F = \mathcal{M}_Y \left( f^0 1_{Y^0} + \sum_{\delta_j=0} f^j 1_{Y^j} \right),
\]

(6.3)

which is the weak limit in \( L^2(\Omega) \) of \( f_\varepsilon \).

6.2. Convergence of the solutions of problem (5.6)

We start by proving that the uniform hypotheses of Sections 6.1, 6.1.1 and 6.1.2, in combination with the properties of the unfolding operator, allow to obtain uniform estimates with respect to \( \varepsilon \) of all the solutions \( u_\varepsilon \) of (5.6).

**Proposition 6.1.** Under the hypotheses of Sections 6.1 and 6.1.1, the following quantities are bounded uniformly in \( \varepsilon \):

\[
\|u_\varepsilon^0\|_{H^1(\Omega^0)}, \quad \left\| \left( \mathcal{T}_\varepsilon(u_\varepsilon) - \mathcal{T}_\varepsilon(u_\varepsilon^0) \right) \right\|_{L^2(\Omega; L^2((S^j)))},
\]

\[
\frac{1}{\varepsilon} \left\| \mathcal{T}_\varepsilon(u_\varepsilon) \right\|_{L^2(\Omega \times Y^j)}, \quad \frac{1}{\varepsilon} \left\| \mathcal{T}_\varepsilon(u_\varepsilon) - u_\varepsilon \right\|_{L^2(\Omega; H^1(Y^j))},
\]

as well as for \( \delta_j = 0, \)

\[
\|r_\varepsilon^0\|_{L^2(\Omega; R)}, \quad \left\| \mathcal{T}_\varepsilon(u_\varepsilon) \right\|_{L^2(\Omega; H^1(Y^j))},
\]

and for \( \delta_j = 1, \)

\[
\frac{1}{\varepsilon} \left\| \mathcal{T}_\varepsilon(u_\varepsilon) \right\|_{L^2(\Omega \times Y^j)}, \quad \left\| r_\varepsilon^0 \right\|_{L^1(\Omega; R)} \quad \text{and} \quad \left\| \mathcal{T}_\varepsilon(u_\varepsilon) \right\|_{L^2(\Omega; L^2(Y^j))}.
\]

**Proof.** The result follows from the properties of the unfolding operator together with estimates (5.18) of Corollary 5.6 which are uniform bounds as soon as \( D(\varepsilon) \) is bounded uniformly with respect to \( \varepsilon \). □

As usual, in order to obtain a limit problem, one uses weak (or weak-* compactness to extract convergent subsequences (still using the notation \( \varepsilon \) which will now belong to an appropriate subsequence going to 0). Eventually, for the components of the solution of the limit problem which are unique, this will imply the convergence of the corresponding complete sequences.

At this point, one can apply Theorem 2.13 of [3] to the sequence \( \{u_\varepsilon^0\} \). We also need a variant of this theorem, which is adapted to the case of vector-valued functions, using the symmetric gradient instead of the full gradient.

For simplicity, the notation \( \mathcal{W}_1^{1, \text{per}}(Y^0) \) indicates the subspace of \( Y \)-periodic elements \( w \) of \( H^1(Y^0) \) with \( \mathcal{M}_{Y^0}(w) = 0 \) (this notation is only used for \( j = 0 \) since the other sets \( Y^j \) are all compact in \( Y \) so periodicity is meaningless).
Proposition 6.2 (Convergences for \( j = 0 \)). Up to a subsequence, there exist

\[ u^0 \in H^1(\Omega; \Gamma_D), \quad \tilde{u}^0 \in L^2(\Omega; W^{1}_{\text{per}}(Y^0)) \]

such that

\[
\begin{align*}
T_\varepsilon(u^0_\varepsilon) & \to u^0 \quad \text{strongly in } L^2_{\text{loc}}(\Omega; H^1(Y^0)), \\
T_\varepsilon(u^0_\varepsilon) & \rightharpoonup u^0 \quad \text{weakly in } L^2(\Omega; H^1(Y^0)), \\
\frac{1}{\varepsilon}(T_\varepsilon(u^0_\varepsilon) - \mathcal{M}_Y u(T_\varepsilon(u^0_\varepsilon))) & \rightharpoonup \nabla u^0 \cdot y_M + \tilde{u}^0 \quad \text{weakly in } L^2(\Omega; H^1(Y^0)), \\
T_\varepsilon(\nabla u^0_\varepsilon) & \rightharpoonup \nabla u^0 + \nabla y \tilde{u}^0 \quad \text{weakly in } L^2(\Omega \times Y^0),
\end{align*}
\]

(6.4)

where \( y_M = y - \mathcal{M}_Y(y) \). Furthermore,

(i) \( T_\varepsilon(e(u^0_\varepsilon)) \to e(u^0) + e_y(\tilde{u}^0) \) weakly in \( L^2(\Omega \times Y^0) \),

(ii) \( \frac{1}{\varepsilon}[T_\varepsilon(u^0_\varepsilon)]_{S^0} \to [\tilde{u}^0]_{S^0} \) weakly in \( L^2(\Omega; H^{1/2}(S^0)) \),

and consequently

\[ [\tilde{u}^0]_{S^0} \leq g^0 \quad \text{on } S^0. \]

(6.6)

Proof. Applying Theorem 3.12 of [3] to the sequence \( \{ u^0_\varepsilon \} \) gives convergences (6.4). Convergence (6.5)(i) follows by taking the symmetric part of the last convergence of (6.4). The jump operator \([\cdot]_{S^0}\) is continuous from \( H^1(Y^0) \) to \( H^{1/2}(S^0) \). Applying it to the third convergence of (6.4) implies (6.5)(ii) and (6.6).

We turn to the convergences for \( j = 1, \ldots, m \). The estimates of Proposition 6.1 provide weak/weak-* convergences for sequences involving \( u^j_\varepsilon \) and \( r^j_\varepsilon \).

Proposition 6.3 (Convergences for \( j = 1, \ldots, m \)). Up to a subsequence, there exists a function \( \tilde{u}^j \in L^2(\Omega; W^{1}(Y^j)) \) such that

\[
\frac{1}{\varepsilon}(T_\varepsilon(u^j_\varepsilon) - r^j_\varepsilon) \rightharpoonup \tilde{u}^j \quad \text{weakly in } L^2(\Omega; W^{1}(Y^j)),
\]

(6.7)

\[
T_\varepsilon(e(u^j_\varepsilon)) \rightharpoonup e_y(\tilde{u}^j) \quad \text{weakly in } L^2(\Omega \times Y^j).
\]

Moreover,

\[ r^j_\varepsilon \rightharpoonup u^0 \quad \text{weakly in } L^2(\Omega; \mathcal{R}) \quad \text{and \ strongly in } L^1(\Omega; \mathcal{R}), \]

(6.8)

and

\[
\begin{align*}
T_\varepsilon(u^j_\varepsilon) & \rightharpoonup u^0 \quad \text{weakly in } L^2(\Omega; H^{1}(Y^j)), \\
T_\varepsilon(u^j_\varepsilon) & \rightharpoonup u^0 \quad \text{strongly in } L^2(\Omega; H^{1}(Y^j)) + L^1(\Omega; \mathcal{R}).
\end{align*}
\]

(6.9)
Proof. The first convergence in (6.7) is just the application of the usual weak compactness criterion in the space $L^2(\Omega; H^1(Y^j))$, the second convergence is obtained by applying the strain operator $e_y$.

From the estimates in Proposition 6.1 it follows that

$$
\| (T_\epsilon(u^j) - T_\epsilon(u_\epsilon^j)) \|_{L^2(\Omega; L^1(S^j))} \text{ and } \| T_\epsilon(u^j) - r_\epsilon^j \|_{L^2(\Omega; H^1(S^j))},
$$

both converge to 0, as well as $\| T_\epsilon(u_\epsilon^j) - T_\epsilon(u_\epsilon^0) \|_{L^1(\Omega; S^j)}$ for $\delta_j = 1$. Consequently,

$$
\| (r_\epsilon^j - T_\epsilon(u_\epsilon^0)) - r_\epsilon^j \|_{L^1(\Omega; S^j)} \to 0.
$$

Recall that $T_\epsilon(u_\epsilon^0)$ converges weakly to $u^0$ in $L^2(\Omega; H^1(Y^0))$. Therefore,

$$
\| (r_\epsilon^j - u^0)^+ \|_{L^2(\Omega; L^1(S^j))} \to 0 \quad \text{and for } \delta_j = 1, \quad \| (r_\epsilon^j - u^0) \|_{L^1(\Omega; S^j)} \to 0.
$$

In view of Lemma 4.4, this implies that $\| r_\epsilon^j - u^0 \|_{L^1(\Omega; S^j)} \to 0$, whence (6.8).

Finally, (6.9) is a consequence of the convergence $\| T_\epsilon(u_\epsilon^j) - r_\epsilon^j \|_{L^2(\Omega; W^{1,1}(Y^j))} \to 0$. \hfill $\square$

The next step is to obtain a weak-* convergence for the jumps across $S^j$ for $j \geq 1$. For this purpose, set

$$
r_\epsilon^0 = \mathcal{M}_Y(u_\epsilon^0),
$$

which is independent of $y$ and converges to $u^0$ in $L^2(\Omega)$.

Proposition 6.4. Under the hypotheses of Sections 6.1, there exists a measure $\beta^j$ in $\mathcal{M}^1(\Omega; \mathbb{R})$ such that, up to a subsequence,

$$
\frac{1}{\epsilon} (r_\epsilon^j - r_\epsilon^0) \rightharpoonup \beta^j \quad \text{weakly-* in } \mathcal{M}^1(\Omega \times S^j),
$$

$$
\frac{1}{\epsilon} (T_\epsilon(u^j) - T_\epsilon(u_\epsilon^0)) \rightharpoonup (\tilde{u}^j + \beta^j - \nabla u^0 \cdot y_M - \tilde{w}^0)|_{S^j} \quad \text{weakly-* in } L^2(\Omega; H^{1/2}(S^j)) + \mathcal{M}^1(\Omega; \mathbb{R}),
$$

and so

$$
(\tilde{u}^j + \beta^j - \nabla u^0 \cdot y_M - \tilde{w}^0)_{|_{S^j}} \leq g^j. \quad (6.10)
$$

Proof. From the definitions of $\tilde{u}^0, \tilde{w}^j, r_\epsilon^0$ and $r_\epsilon^j$, the following convergences are straightforward:

$$
\frac{1}{\epsilon} (T_\epsilon(u_\epsilon^j) - r_\epsilon^0) \rightharpoonup \nabla u^0 \cdot y_M + \tilde{u}^0 \quad \text{weakly in } L^2(\Omega; W^1(Y^0)),
$$

$$
\frac{1}{\epsilon} (T_\epsilon(u_\epsilon^j) - r_\epsilon^j) \rightharpoonup \tilde{u}^j \quad \text{weakly in } L^2(\Omega; W^1(Y^j)). \quad (6.12)
$$
On the other hand, by estimates (5.2), \( \frac{1}{\varepsilon} \| T_\varepsilon (u^0) - T_\varepsilon (u^0) \|_{L^1(\Omega \times S^j)} \) is bounded and by the definition of \( K^\varepsilon \), \( \frac{1}{\varepsilon} (T_\varepsilon (u^0) - T_\varepsilon (u^0)) \leq g^j \). Therefore \( \frac{1}{\varepsilon} \| (r^j - r^0) \|_{L^1(\Omega \times S^j)} \) is also bounded.

For \( \delta_j = 1 \), we also have that \( \frac{1}{\varepsilon} \| T_\varepsilon (u^0) - T_\varepsilon (u^0) \|_{L^1(\Omega \times S^j)} \) is bounded. Consequently, \( \frac{1}{\varepsilon} \| (r^j - r^0) \|_{L^1(\Omega \times S^j)} \) is bounded too.

Using Lemma 4.4, we conclude that \( \frac{1}{\varepsilon} (r^j - r^0) \) converges weakly * in \( M^1(\Omega \times S^j) \) to some \( \beta_j \). Going back to \( \frac{1}{\varepsilon} (T_\varepsilon (u^0) - T_\varepsilon (u^0)) \), this, together with (6.12), imply (6.10)–(6.11). \( \square \)

6.3. The corrector problem

We assume that there exists a tensor \( (a^0) \) such that
\[
T_\varepsilon (a^0) \rightarrow a^0 \quad \text{a.e. in } \Omega \times Y^*.
\] (6.13)

By hypothesis (5.1), \( (a^0) \) is bounded in \( L^\infty (\Omega \times Y^*) \) and \( \overline{\sigma} \) coercive.

In order to state the main result, we introduce the so-called corrector problem: for every symmetric tensor \( U \) and for a.e. \( x \in \Omega \), find \( \chi = (\chi^0(x, \cdot), \ldots, \chi^m(x, \cdot)) \) in the closed convex set \( \overline{K} \) (which is the set of local admissible deformations), defined as
\[
\begin{align*}
\overline{K} & = \left\{ v = (v_0, \ldots, v_m) \mid v_0 \in W^1_{\text{per}} (Y^0), v_j \in H^1 (Y^j), j \in \{1, \ldots, m\}, \\
& \text{ such that } \\
& \frac{1}{|Y^j|} \int_{Y^j} a^0 (U + e_y (\chi)) (e_y (W) - e_y (\chi)) \, dy + \psi^0 (x, [(W^0)_{\tau}]_{S^j}) - \psi^0 (x, [(\chi^0)_{\tau}]_{S^j}) \\
& + \sum_{j=1}^m \int_{S^j} G^j (x, y) [(W^j (y) - W^0 (y))_{|S^j \tau}] \, d\sigma (y) \\
& - \sum_{j=1}^m \int_{S^j} G^j (x, y) [(\chi^j - \chi^0)_{|S^j \tau}] \, d\sigma (y) \geq 0 
\right\}.
\end{align*}
\] (6.14)

for all \( W \in \overline{K} \).

By an argument similar to that used in the proof of the existence of a solution for (5.6), one shows that the corrector problem (6.14) has at least one solution \( \chi \). For similar reasons, there is not necessarily uniqueness of the solution, but the nonlinear map \( U \mapsto (e_y (\chi^0), \ldots, e_y (\chi^m)) \) is itself single valued. This map has properties which are stated below.

**Proposition 6.5.** Let \( \Xi \) be the map
\[
(x, y, U) \mapsto \Xi = (e_y (\chi^0), \ldots, e_y (\chi^m)),
\]
where \( \chi = (\chi^0(x, \cdot), \ldots, \chi^m(x, \cdot)) \) in \( \overline{K} \) is any solution of (6.14).
The map $\Xi$ is a Caratheodory map from $\Omega^* \times M_2^S(\mathbb{R})$ with values in $w$-$L^2(Y^*)$ (i.e. $L^2(Y^*)$ endowed of its weak topology).

**Proof.** If, for fixed $x \in \Omega$, one considers a sequence $U_n$ in $M_2^S(\mathbb{R})$ converging to some $U$ and the associated $\chi_n$, it is easy to show from the problem $(CP)_n$ that the sequence $\chi_n$ converges weakly in $\hat{K}$ to the solution of problem $(CP)$ (all the terms converge except for the quadratic one where one uses the weak lower semi-continuity). This implies that the map $U \mapsto \Xi$ is continuous from $M_2^S(\mathbb{R})$ to $w$-$L^2(Y^*)$.

A similar argument shows that if the map $x \mapsto a^0$ is continuous, so is the map $\varphi \mapsto \Xi(x, \varphi)$, when using the weak topology of $L^2(Y^*)$. With $a^0$ measurable, by Lusin’s theorem, this implies that the map $x \mapsto \Xi(x, \varphi)$ is weakly hence strongly measurable for every $\varphi$. \(\square\)

**Definition 6.6.** The map $\sigma^{\text{hom}}$ is defined by

$$\sigma^{\text{hom}}(x, \varphi) \doteq \frac{1}{|Y|} \int_{Y^*} a^0(x, y)(\varphi + \Xi(x, y, \varphi)) \, dy.$$  

As a direct consequence of Proposition 6.5, the map $\sigma^{\text{hom}}(x, \varphi)$ is a Caratheodory function on $\Omega \times M_2^S(\mathbb{R})$ with values in $M_2^S(\mathbb{R})$.

Going back to the corrector problem (6.14), one can see that it is equivalent to minimizing over $\hat{K}$ the functional

$$\frac{1}{2|Y|} \int_{Y^*} a^0(\varphi + e_y(W))(\varphi + e_y(W)) \, dy + \psi^0(\varphi, [W^0], [\sigma^0]) + \sum_{j=1}^m \int_{S_j} G^j(x, y) d \left( [(W^j - W^0), S_j] \right).$$

The corresponding minimum is denoted as $\sigma^{\text{hom}}(x, \varphi)$:

**Definition 6.7.** For every symmetric tensor $\varphi$ and for a.e. $x \in \Omega$, set

$$\sigma^{\text{hom}}(x, \varphi) \doteq \frac{1}{2|Y|} \int_{Y^*} a^0(\varphi + e_y(x))(\varphi + e_y(x)) \, dy + \psi^0(\varphi, [(\chi^0)], [\sigma^0]) + \sum_{j=1}^m \int_{S_j} G^j(x, y) d \left( [(\chi^0 - \chi^0), S_j] \right).$$

One can check by a direct computation that $\sigma^{\text{hom}}$ is convex with respect to $\varphi$.

It turns out that $\sigma^{\text{hom}}$ and $\sigma^{\text{hom}}$ are very much connected. This is the subject of the next proposition.

**Proposition 6.8.** The function $\sigma^{\text{hom}}$ is a strictly convex Caratheodory function defined on $\Omega \times M_2^S(\mathbb{R})$. It is Gâteaux-differentiable with respect to its second argument and this derivative is the tensor $\sigma^{\text{hom}}(x, \varphi)$.

In other words, for a.e. $x \in \Omega$, $\varphi$ and $\varphi$ in $M_2^S(\mathbb{R})$,

$$\sigma^{\text{hom}}(\varphi) + \langle \sigma^{\text{hom}}(\varphi), \varphi - \varphi \rangle \leq \sigma^{\text{hom}}(\varphi).$$

In particular, the map $\varphi \mapsto \sigma^{\text{hom}}(x, \varphi)$ is (maximal) monotone. It is actually strictly monotone.
Proof. From (6.14) written for two tensors $U$ and $\hat{U}$ and denoting for short $\Xi$ for $\Xi(U)$ and $\hat{\Xi}$ for $\Xi(\hat{U})$, it follows that
\[
\frac{1}{|Y|} \int_{Y^*} a^0(U + \Xi - (\hat{U} + \hat{\Xi}))(\hat{\Xi} - \Xi) \, dy \geq 0.
\]
Combining this with the coercivity relation
\[
\frac{1}{|Y|} \int_{Y^*} a^0(U + \Xi - (\hat{U} + \hat{\Xi}))(U + \Xi - (\hat{U} + \hat{\Xi})) \, dy \geq \frac{\alpha}{|Y|} \int_{Y^*} (U + \Xi - (\hat{U} + \hat{\Xi}))^2 \, dy,
\]
one gets
\[
\frac{1}{|Y|} \int_{Y^*} a^0(U + \Xi - (\hat{U} + \hat{\Xi}))(U - \hat{U}) \, dy \geq \frac{\alpha}{|Y|} \int_{Y^*} (U + \Xi - (\hat{U} + \hat{\Xi}))^2 \, dy.
\]
This implies that for a.e. $x \in \Omega$
\[
\langle \sigma^{\text{hom}}(U) - \sigma^{\text{hom}}(\hat{U}), U - \hat{U} \rangle \geq \frac{\alpha}{|Y|} \int_{Y^*} (U + \Xi - (\hat{U} + \hat{\Xi}))^2 \, dy,
\]
which shows that the map $U \mapsto \sigma^{\text{hom}}(x, U)$ is monotone.

It is actually strictly monotone. Indeed, if the left-hand side of (6.16) vanishes, this implies that the right-hand side also vanishes, implying that $\Xi - \hat{\Xi} \equiv \hat{U} - U$, which is independent of $y$. In particular, $e_y(\chi^0 - \hat{\chi}^0)$ is constant in $Y^0$ which is connected. Therefore, $\chi^0 - \hat{\chi}^0$ is an affine function of the variable $y$ in $Y^0$. But by definition, it also is $Y$-periodic and with mean value 0 in $Y^0$. This implies that $\chi^0 - \hat{\chi}^0$ is identically zero. Therefore, $\hat{U} - U = 0$, which is the condition for strict monotonicity of $\sigma^{\text{hom}}$.

It remains to show that $\sigma^{\text{hom}}$ is the subdifferential of $E^{\text{hom}}$. By definition of the minimizer $\hat{\chi}$ associated with $\hat{U}$, one sees that (we drop the variable $x$ for clarity)
\[
E^{\text{hom}}(\hat{U}) \leq E^{\text{hom}}(\hat{U}) + \frac{1}{2|Y|} \int_{Y^*} (a^0(U + \hat{\Xi})(\hat{U} + \hat{\Xi}) - a^0(\hat{U} + \hat{\Xi})(\hat{U} + \hat{\Xi})) \, dy,
\]
from which follows
\[
E^{\text{hom}}(\hat{U}) + \frac{1}{|Y|} \int_{Y^*} a^0(U + \hat{\Xi})(\hat{U} - U) \, dy \leq E^{\text{hom}}(\hat{U}) \, dy.
\]
By the definition of $\sigma^{\text{hom}}(U)$, this implies
\[
E^{\text{hom}}(\hat{U}) + \langle \sigma^{\text{hom}}(U), \hat{U} - U \rangle \leq E^{\text{hom}}(\hat{U}) + \frac{1}{|Y|} \int_{Y^*} a^0(\Xi - \hat{\Xi})(\hat{U} - U) \, dy.
\]
In this last inequality, replacing $\hat{U}$ by $U_t = (1 - t)U + \hat{U}$ (for $t \in (0, 1)$), using the convexity inequality for $E^{\text{hom}}$ and then dividing by $t$ gives
\[
E^{\text{hom}}(U) + \langle \sigma^{\text{hom}}(U), \hat{U} - U \rangle \leq E^{\text{hom}}(\hat{U}) + \frac{1}{|Y|} \int_{Y^*} a^0(\Xi - \Xi(U_t))(\hat{U} - U) \, dy.
\]
By the weak convergence of $\Xi(\mathcal{U}_\varepsilon) - \Xi$ to zero in $L^2(Y^*)$, one finally gets (6.15). □

**Proposition 6.9.** $\mathcal{E}^{\text{hom}}$ is coercive.

**Proof.** Let show that $\mathcal{E}^{\text{hom}}(\mathcal{U})$ is bounded below by a multiple of $|\mathcal{U}|^2$. Recall (see (5.1)) that the tensor $(a^0)$ is $\Theta$ coercive. Furthermore, the family of functions $\Psi_j^0$, $0 \leq j \leq m$ is non-negative. Hence,

$$
\mathcal{E}^{\text{hom}}(\mathcal{U}) \geq \frac{\alpha}{2|Y|} \min_{\chi \in W^1_0(Y)} \int_Y |\mathcal{U} + e_y(\chi)|^2 \, dy.
$$

This minimum is a non-negative quadratic form of $\mathcal{U}$ on the finite dimensional space $\mathcal{M}_2^2(\mathbb{R})$ and defines an equivalent norm on it. To see that it is enough to show that it vanishes only for $\mathcal{U} = 0$. But the proof is similar to that of the strict monotonicity of $\mathcal{E}^{\text{hom}}$. Consequently, there is strictly positive constant $\gamma$ such that $\mathcal{E}^{\text{hom}}(\mathcal{U})$ is bounded below by $\frac{\gamma}{2} |\mathcal{U}|^2$. □

6.4. The main homogenization result

For $j = 1, \ldots, m$, set

$$
U^j \doteq \bar{u}^j + \beta^j - \nabla u^0 \cdot y_M \quad \text{and} \quad \hat{u} = (\bar{u}^0, U^1, \ldots, U^m),
$$

with $u^0$ defined in Proposition 6.2, convergences (6.4). Note that by definition,

$$
e_y(U^j) = e_y(\hat{u}^j) - e(u^0).
$$

By its definition, for almost every $x \in \Omega$, the function $\hat{u}(x, \cdot)$ belongs to the following closed convex set of admissible unfolded deformations

$$
\tilde{\mathcal{K}} \doteq \{(w_0, \ldots, w_m) | w_0 \in L^2(\Omega; W^1_0(Y)), [w_0]_{\nu |_{S^0}} \leq g^0 \in L^2(\Omega; H^{1/2}(S^0)),
$$

for $j \in \{1, \ldots, m\}$, $w_j \in L^2(\Omega; W^1(Y^j)) + \mathcal{M}(\Omega; \mathfrak{R}_{\mathcal{K}})$, $w_j - w_0|_{\nu = 0} \leq g^j \in L^2(\Omega; H^{1/2}(S^j)) + \mathcal{M}(\Omega),

$$
\}.
$$

**Theorem 6.10.** Assume that $f_\varepsilon$, $g^j$, $\Psi_j^0$ and $\Psi_j^0$ ($j = 1, \ldots, m$) satisfy the hypotheses from Sections 6.1.1 and 6.1.2. Suppose furthermore that the assumption (6.13) on the tensor field $\mathcal{T}_\varepsilon(\sigma^\varepsilon)$ together with the boundedness and coercivity of $(a^\varepsilon)$ holds.

Let $u_\varepsilon \in \tilde{\mathcal{K}}$ be a solution of Problem $\mathcal{P}_\varepsilon$ and $u^0 \in H^1(\Omega; \Gamma_D)$ be the limit function given by Proposition 6.2.

Then $u^0$ is the unique minimizer over $H^1(\Omega; \Gamma_D)$ of the functional

$$
\int_\Omega \left(\mathcal{E}^{\text{hom}}(x, e(v)) - F(v)\right) \, dx.
$$

The limit problem can equivalently be written as

$$
\begin{cases}
\ u^0 \in H^1(\Omega; \Gamma_D), \\
- \text{div} \sigma^{\text{hom}}(x, e(u^0)) = F & \text{in } \Omega.
\end{cases}
$$
**Proof.** Problem (6.19) has a unique solutions because of the strict convexity of $\varepsilon^{\text{hom}}$ or the strict monotonicity of $\sigma^{\text{hom}}$. It can easily be seen that problem (6.20) is actually of generalized Leray–Lions type, so that it has one solution.

It remains to show that the limits furnished by the convergences of Propositions 6.2–6.4 satisfy the homogenized limit problem. In the previous subsection, we have already established that $\hat{u} = (\hat{u}^0, U^1, \ldots, U^m)$ (defined in (6.17)) belongs to the set $\hat{K}$ defined by (6.18). The main point now is to pass to the limit in (5.6). To do so, we will use the following test function:

$$v_\varepsilon(x) = \Phi(x) + \varepsilon w \left( x, \frac{x}{\varepsilon} \right),$$

where $\Phi$ is in $C^1(\overline{\Omega})$ and vanishes near $\Gamma_D$, and $w$ belongs to $\mathcal{D}(\Omega; \hat{K})$ (this choice makes sense since $\hat{K}$ contains 0).

We have the obvious convergence

$$v_\varepsilon \rightarrow \Phi \quad \text{strongly in } L^2(\Omega).$$

We rewrite (5.6) in the form

$$a^\varepsilon(e(u_\varepsilon), e(v_\varepsilon)) + \sum_{j=0}^m \Psi_\varepsilon^j([e(v_\varepsilon)]^j_{S^2_j}) - \int_{\Omega^\varepsilon} f_\varepsilon v_\varepsilon \, dx \geq a^\varepsilon(e(u_\varepsilon), e(u_\varepsilon)) + \sum_{j=0}^m \Psi_\varepsilon^j([e(u_\varepsilon)]^j_{S^2_j}) - \int_{\Omega^\varepsilon} f_\varepsilon u_\varepsilon \, dx.$$  

(6.21)

We study the terms appearing in this inequality. Note that

$$e(v_\varepsilon) = e(\Phi(x)) + \varepsilon e_x \left( w \left( x, \frac{x}{\varepsilon} \right) \right) + e_y \left( w \left( x, \frac{x}{\varepsilon} \right) \right),$$

so that

$$\mathcal{T}_\varepsilon(e(v_\varepsilon)) \rightarrow e(\Phi(x)) + e_y(w(x,y)) \quad \text{uniformly and in } L^2(\Omega \times \mathbb{R}^2).$$

By unfolding, and recalling Hypotheses 6.1 and 6.1.1, for $\varepsilon$ small enough

$$a^\varepsilon(e(u_\varepsilon), e(v_\varepsilon)) = \frac{1}{|Y|} \sum_{j=0}^m \int_{\Omega \times Y^j} \mathcal{T}_\varepsilon(a^\varepsilon) \mathcal{T}_\varepsilon(e(u^j_\varepsilon)) \mathcal{T}_\varepsilon(e(v^j_\varepsilon)) \, dx \, dy$$

$$+ \int_{A_\varepsilon} a^\varepsilon(e(u_\varepsilon)e(v_\varepsilon)) \, dx,$$

$$\Psi_\varepsilon^j([e(v_\varepsilon)]^j_{S^2_j}) = \Theta^j \left( (w^j(x,y) - w^0(x,y))_{|S^2_j} \right)$$

(6.22)

which is independent of $\varepsilon$, 

$$\Psi_\varepsilon^j([e(u_\varepsilon)]^j_{S^2_j}) = \Theta^j \left( \frac{1}{\varepsilon} (\mathcal{T}_\varepsilon(u^j_\varepsilon) - \mathcal{T}_\varepsilon(u^0_\varepsilon))_{|S^2_j} \right).$$
By the lower semi-continuity with respect to weak (or weak-$*$) convergences,

\[
\liminf \mathbf{a}^\varepsilon(e(u_\varepsilon), e(u_\varepsilon)) \geq \frac{1}{|Y|} \sum_{j=0}^m \int_{\Omega \times Y^j} T_\varepsilon(a^\varepsilon) T_\varepsilon(e(u^\varepsilon_j)) \, dx \, dy + \pi \int_{\Lambda_\varepsilon} |e(u_\varepsilon)|^2 \, dx.
\]

By the lower semi-continuity with respect to weak (or weak-$*$) convergences,

\[
\liminf \mathbf{a}^\varepsilon(e(u_\varepsilon), e(u_\varepsilon)) \geq \frac{1}{|Y|} \int_{\Omega \times Y} a_0^0(e(u^0) + e_y(\tilde{u})) \, dx \, dy,
\]

\[
\liminf \Psi^0_\varepsilon([u_\varepsilon]_{S^0}) \geq \Psi^0(\tilde{u}^0)_{|S^0},
\]

\[
\liminf \Psi^0_\varepsilon([u_\varepsilon]_{S^0}) \geq \Psi^0((U^j - \tilde{u}^0)_{|S^0}).
\]

(6.23)

Let us now consider the right-hand side terms in (6.21). Recalling hypotheses (6.1) and (6.2) on $f_\varepsilon$, we first get the convergence

\[
\int_{\Omega^0_\varepsilon} f_\varepsilon^0 u^0_\varepsilon \, dx \to \frac{1}{|Y|} \int_{\Omega \times Y^0} f^0 u^0 \, dx \, dy = \frac{|Y^0|}{|Y|} \int_{\Omega} \mathcal{M}_{Y^0}(f^0) u^0 \, dx.
\]

For $\delta_j = 0$ (locked domains), by Proposition 6.3

\[
\int_{\Omega^j_\varepsilon} f_\varepsilon^j u^j_\varepsilon \, dx \to \frac{1}{|Y|} \int_{\Omega \times Y^j} f^j u^j \, dx \, dy = \frac{|Y^j|}{|Y|} \int_{\Omega} \mathcal{M}_{Y^j}(f^j) u^0 \, dx.
\]

By (5.10), if $\delta_j = 1$ and $\frac{1}{e^{N/2-1}} \|f^j\|_{L^2(\Omega^j)} \to 0$, then $\int_{\Omega^j_\varepsilon} f_\varepsilon^j u^j_\varepsilon \, dx \to 0$.

For $\delta_j = 1$ and if condition (5.12) is satisfied, then the same convergence holds true due to (5.13), since $\varepsilon \|f^j\|_{L^2(\Omega^j)} \to 0$.

Similarly, one gets

\[
\int_{\Omega^0_\varepsilon} f_\varepsilon^0 v_\varepsilon \, dx \to \frac{|Y^0|}{|Y|} \int_{\Omega} \mathcal{M}_{Y^0}(f^0) \Phi \, dx,
\]

for $\delta_j = 1$, $\int_{\Omega^j_\varepsilon} f_\varepsilon^j v_\varepsilon \, dx \to 0$,

\[
\int_{\Omega^j_\varepsilon} f_\varepsilon^j v_\varepsilon \, dx \to \frac{|Y^j|}{|Y|} \int_{\Omega} \mathcal{M}_{Y^j}(f^j) \Phi \, dx.
\]
Recall the notation $\mathcal{E}_\epsilon$ for the energy related to Problem $\mathcal{P}_\epsilon$, (5.6), i.e.

$$\mathcal{E}_\epsilon(u_\epsilon) = a^\ast(e(u_\epsilon), e(u_\epsilon)) + \sum_{j=0}^{m} \mathcal{F}_\epsilon^j \left( [u_\epsilon]_{Y^j} \right).$$

Taking the inf-limit in (5.6) and using the established convergences yield

$$\frac{1}{|Y|} \int_{\Omega \times Y^*} a^0(e(u^0) + e_y(\tilde{u})) \left( e(\Phi)(x) + e_y(w) \right) dx \, dy$$

$$+ \Phi^0([w^0]_{Y^*} |_{\Theta^0}) + \sum_{j=1}^{m} \Theta^j((w^j(y) - w^0(y))_{Y^j}) - \int_{\Omega} F_{\Phi} \, dx$$

$$\geq \limsup \mathcal{E}_\epsilon - \int_{\Omega} F w^0 \, dx \geq \liminf \mathcal{E}_\epsilon - \int_{\Omega} F w^0 \, dx$$

$$\geq \frac{1}{|Y|} \int_{\Omega \times Y^*} a^0(e(u^0) + e_y(\tilde{u})) \left( e(u^0) + e_y(\tilde{u}) \right) dx \, dy$$

$$+ \Phi^0([\tilde{u}^0]_{Y^*} |_{\Theta^0}) + \sum_{j=1}^{m} \Theta^j((U^j - \tilde{u}^0)_{Y^j}) - \int_{\Omega} F_{\Phi} \, dx.$$

Here we used the formula

$$F = \frac{|Y^0|}{|Y|} \mathcal{M}_{Y^0}(f^0) + \sum_{\delta_j=0}^{m} \frac{|Y^j|}{|Y|} \mathcal{M}_{Y^j}(f^j),$$

which is equivalent to (6.3).

By a density argument,\(^4\) (6.24) holds for $\phi \in H^1(\Omega, \Gamma_D)$ and for $w$ in the set $\tilde{\mathcal{C}}$. Recall that $\tilde{u}$ belongs to $\tilde{\mathcal{C}}$. Thus, with this choice for $w$ and replacing $\Phi$ by $u^0 + \Phi$ (which is still in $H^1(\Omega, \Gamma_D)$ since $u^0$ has no jumps), yields

$$\frac{1}{|Y|} \int_{\Omega \times Y^*} a^0(e(u^0) + e_y(\tilde{u})) \left( e(\Phi)(x) + e(u^0) + e_y(\tilde{u}) \right) dx \, dy$$

$$+ \Phi^0([\tilde{u}^0]_{Y^*} |_{\Theta^0}) + \sum_{j=1}^{m} \Theta^j((U^j - \tilde{u}^0)_{Y^j}) - \int_{\Omega} F_{\Phi} \, dx$$

$$\geq \limsup \mathcal{E}_\epsilon - \int_{\Omega} F w^0 \, dx \geq \liminf \mathcal{E}_\epsilon - \int_{\Omega} F w^0 \, dx$$

$$\geq \frac{1}{|Y|} \int_{\Omega \times Y^*} a^0(e(u^0) + e_y(\tilde{u})) \left( e(u^0) + e_y(\tilde{u}) \right) dx \, dy.$$

\(^4\)Using regularization by convolution in $x$, this applies also to the various convex functions, making use of the continuity with respect to the strict topology for the $\Theta^j$'s.
\[ + \psi^0([\hat{u}^0],_{\Omega^0}) + \sum_{j=1}^{m} \Theta^j((U^j - \hat{u}^0)_{\tau|S^j}) - \int_{\Omega} Fu^0 \, dx \doteq \mathcal{I}. \] (6.25)

Subtracting \( \mathcal{I} \) from each term in (6.25) gives

\[
\frac{1}{|Y|} \int_{\Omega \times Y^*} a^0 (e(u^0) + e_y(\hat{u})) (e(\Phi)(x)) \, dx \, dy - \int_{\Omega} F \Phi \, dx \geq \limsup E_{\epsilon} - \mathcal{I} \geq \liminf E_{\epsilon} - \mathcal{I} \geq 0. \] (6.26)

Since \( \Phi \) can be replaced by \( \pm \Phi \) in (6.26), this gives the identity

\[
\frac{1}{|Y|} \int_{\Omega \times Y^*} a^0 (e(u^0) + e_y(\hat{u})) e(\Phi)(x) \, dx \, dy = \int_{\Omega} F \Phi \, dx, \] (6.27)

while, by choosing \( \Phi = 0 \) in (6.26), one gets

\[
\lim E_{\epsilon} = \mathcal{I} \left( \text{as well as } \mathcal{I} \int_{\Lambda_{\epsilon}} |e(u_{\epsilon})|^2 \, dx \rightarrow 0 \text{ upon closer inspection}! \right). \] (6.28)

Equation (6.27) is the limit problem for \( u^0 \), provided we express each \( e_y(\hat{u}) \) in terms of \( u^0 \), while (6.28) is the convergence of the energy.

To get the expression of \( e_y(\hat{u}) \), consider (6.24) with \( \Phi = u^0 \) and substract the last term from the first one to get

\[
\frac{1}{|Y|} \int_{\Omega \times Y^*} a^0 (e(u^0) + e_y(\hat{u})) (e_y(w) - e_y(\hat{u})) \, dx \, dy + \psi^0 ([u^0]_{\tau|S^0}) - \psi^0 ([\hat{u}^0]_{\tau|S^0}) + \Theta^j (w^j(y) - w^0(y)_{\tau|S^j}) - \Theta^j ((U^j - \hat{u}^0)_{\tau|S^j}) \geq 0. \] (6.29)

Clearly, (6.25) is equivalent to the pair (6.27) (6.29). Then in order to satisfy (6.29) it is enough to solve the following problem for a.e. \( x \) in \( \Omega \):

\[
\frac{1}{|Y|} \int_{\Omega \times Y^*} a^0 (e(u^0) + e_y(\hat{u})) (e_y(w) - e_y(\hat{u})) \, dx \, dy + \psi^0 ([u^0]_{\tau|S^0}) - \psi^0 ([\hat{u}^0]_{\tau|S^0}) + \sum_{j=1}^{m} \int_{S^j} G^j(x, y) (w^j(y) - w^0(y)_{\tau|S^j}) \, d\sigma - \sum_{j=1}^{m} \int_{S^j} G^j(x, y) ((U^j - \hat{u}^0)_{\tau|S^j}) \, d\sigma \geq 0. \]
This is the same as the (6.14) where \( \mathcal{U} \) is replaced by \( e(u^0) \) so that \( e_y(\tilde{u}) \) is nothing else than \( \Xi(x, y; e(u^0)(x)) \). Plugging this into (6.27), gives the variational formulation for the homogenized problem (6.20), i.e.

\[
\begin{align*}
\text{Find } u^0 & \in H^1(\Omega; \Gamma_D) \quad \forall v \in H^1(\Omega; \Gamma_D) \\
\iint_\Omega \sigma^\text{hom}(x, e(u^0)) e(v) \, dx = & \iint_\Omega F v \, dx.
\end{align*}
\]

The proof of the main theorem is now complete. \( \square \)

### 6.5. Convergence of the energy and correctors

**Proposition 6.11.** Under the hypotheses of the preceding sections, one has the convergences

\[
\lim \mathcal{A}^\varepsilon(e(u_\varepsilon), e(u_\varepsilon)) = \frac{1}{|Y|} \iint_{\Omega \times Y} \mathcal{A}^0(e(u^0) + e_y(\tilde{u}^0)(e(u^0) + e_y(\tilde{u}^0)) \, dx \, dy,
\]

\[
\lim \Psi^0_\varepsilon([[(u_\varepsilon)]_{\Gamma_0}]) = \Psi^0([\tilde{u}^0]_{\Gamma_0}),
\]

\[
\lim \Psi^1_\varepsilon([[(u_\varepsilon)]_{\Gamma_1}]) = \Psi^1((U^j - \tilde{u}^0)_{\Gamma_1}),
\]

where

\[
\Psi^1((U^j - \tilde{u}^0)_{\Gamma_1}) = \int_{\Omega \times \Gamma_1} G_j(x, y) \, dy \in \Omega \times \Gamma_1.
\]

**Proof.** This is just a restatement of (6.28) by taking into account (6.23). \( \square \)

**Corollary 6.12.** The following strong convergences hold:

\[
T_\varepsilon(e(u_\varepsilon)) |_{\Omega \times Y} \rightarrow e(u^0) + e_y(\tilde{u}^0) \quad \text{strongly in } L^2(\Omega \times Y^0),
\]

\[
T_\varepsilon(e(u_\varepsilon)) |_{\Omega \times Y^j} \rightarrow e_y(\tilde{u}^0) \quad \text{strongly in } L^2(\Omega \times Y^j) \quad \text{for } j = 1, \ldots, m,
\]

as well as

\[
\int_{A_\varepsilon} |e(u_\varepsilon)|^2 \, dx \rightarrow 0.
\]

**Proof.** By (6.22),

\[
\mathcal{A}^\varepsilon(e(u_\varepsilon), e(u_\varepsilon)) = \frac{1}{|Y|} \sum_{j=0}^m \iint_{\Omega \times Y^j} \mathcal{T}_\varepsilon(a^\varepsilon) \mathcal{T}_\varepsilon(e(u^0)) \, dy + \int_{A_\varepsilon} a^\varepsilon e(u_\varepsilon) e(u_\varepsilon) \, dx.
\]

The result follows by passing to the limit by using Lemma 4.9 of [3], Assumption (6.13) and the convergences established in Proposition 6.2. \( \square \)

Classically in the unfolding method (see [4] and [3]) the convergences of Corollary 6.12 imply the following corrector result:
Corollary 6.13. Under the hypotheses of the preceding sections, as $\varepsilon \to 0$,

$$
\| e(u_\varepsilon) - e(u_0) - U_\varepsilon(e_y(\tilde{u}_0)) \|_{L^2(\Omega_0^\varepsilon)} \to 0,
$$

$$
\| e(u_\varepsilon) - U_\varepsilon(e_y(\tilde{u}_j)) \|_{L^2(\Omega_j^\varepsilon)} \to 0 \quad \text{for} \quad j = 1, \ldots, m,
$$

where $U_\varepsilon$ is the right inverse of $T_\varepsilon$.

One can also obtain correctors for $u_\varepsilon$ itself making use of $Q_1$ interpolates. We refer to [4] and [3] for such correctors.

References