Existence of mild solution for evolution equation with Hilfer fractional derivative

Haibo Gu\textsuperscript{a,b,}*\textsuperscript{,}Juan J. Trujillo\textsuperscript{c}

\textsuperscript{a} School of Mathematics and Computational Science, Xiangtan University, Xiangtan, Hunan 411105, China
\textsuperscript{b} School of Mathematics Sciences, Xinjiang Normal University, Urumqi, Xinjiang 830054, China
\textsuperscript{c} Departamento de Análisis Matemático, Universidad de La Laguna, 38271 La Laguna, Tenerife, Spain

\section{1. Introduction}

Nowadays, fractional calculus receives increasing attention in the scientific community, with a growing number of applications in physics, electrochemistry, biophysics, viscoelasticity, biomedicine, control theory, signal processing, etc\textsuperscript{(see [22,29] and the references therein). Fractional differential equations also have been proved to be useful tools in the modeling of many phenomena in various fields of science and engineering. There has been a significant development in fractional differential equations in recent years, see the monographs of Kilbas et al. [16], Miller and Ross [21], Podlubny [23], Lakshmikantham et al. [17], Zhou [32], the papers [1,4–7,9,11,18,19,27,28] and the references therein.}

A strong motivation for investigating fractional evolution equations comes from physics. Fractional diffusion equations are abstract partial differential equations that involve fractional derivative in space and time. For example, El-Sayed [10] discussed fractional order diffusion-wave equation. Eidelman and Kochubei [11] investigated the Cauchy problem for fractional diffusion equation. As stated in [11], fractional diffusion equations describe anomalous diffusion on fractals. Physical objects of fractional dimension, like some amorphous semiconductors or strongly porous materials. This class of equations can provide a nice instrument for the description of memory and hereditary properties of various materials and processes.

Some recent papers investigated the problem of the existence of mild solution for abstract differential equations with fractional derivative [2,8,15,25]. Since the mild solution definition in integer order abstract differential equations obtained by variation of constant formulas can not be generalized directly to fractional order abstract differential equations, Zhou and Jiao [30] gave a suit concept on mild solutions by applying laplace transform and probability density functions for evolution equation with Caputo fractional derivative. Using the same method, Zhou et al. [31] gave a suit concept on mild solutions for evolution equation with Riemann–Liouville fractional derivative. By using sectorial operator, Su et al. [25] gave a definition of mild solution for fractional differential equation with order $1 < \alpha < 2$ and investigated it's existence. Agarwal et al. [2] studied the existence and dimension of the set for mild solutions of semilinear fractional differential equations inclusions. Wang [26] researched the abstract fractional Cauchy problem with almost sectorial operators. On the other hand, Hilfer [14]...
proposed a generalized Riemann–Liouville fractional derivative, for short, Hilfer fractional derivative, which includes Riemann–Liouville fractional derivative and Caputo fractional derivative. This operator appeared in the theoretical simulation of dielectric relaxation in glass forming materials. In [12], Furati et al. considered an initial value problem for a class of nonlinear fractional differential equations involving Hilfer fractional derivative. In [24], the solution of a fractional diffusion equation with a Hilfer time fractional derivative was obtained in terms of Mittag–Leffler functions and Fox’s $H$-function. To the best of our knowledge, there has no results about the evolution equations with Hilfer fractional derivative.

Inspired by the above discussion, in this paper, we will investigate a class of evolution equation with Hilfer fractional derivative. By Laplace transform and density function, we firstly give the mild solution definition. Then we obtain some sufficient conditions ensuring the existence of mild solution by using noncompact measure method and Ascoli–Arzela Theorem. Because Hilfer fractional derivative is more general than Riemann–Liouville fractional derivative. So, the results we obtained are also more general than known results.

In this paper, we consider the following fractional order evolution equation:

$$\begin{aligned}
D_{0+}^{\alpha,\beta} x(t) &= Ax(t) + f(t,x(t)), \quad t \in J = (0, b], \\
I_{0+}^{1-\alpha-(1-\mu)} x(0) &= x_0,
\end{aligned}$$

where $D_{0+}^{\alpha,\beta}$ is the Hilfer fractional derivative which will be given in next section, $0 < \alpha \leq 1$, $0 < \beta < 1$, the state $x(\cdot)$ takes value in a Banach space $X$ with norm $\| \cdot \|$. $A$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators (i.e. $C_0$ semigroup) $\{ Q(t) \}_{t \geq 0}$ in Banach space $X$. $f : J \times X \to X$ is given functions satisfying some assumptions, $x_0 \in X$.

The rest of this paper is organized as follows. In Section 2, some notations and preparation are given. A suitable concept on mild solution for our problem is introduced. In Section 3, some sufficient conditions are obtained to ensure the existence of mild solution. Some conclusions are given in Section 4.

2. Preliminaries

In this section, we will firstly introduce fractional integral and derivative, some notations and then give the definition of a mild solution of system (1.1). Finally, we will give some assumptions and lemmas which are useful in next section.

Throughout this paper, $\mathbb{R}$ represents the set of real numbers, and $\mathbb{R}^+ = [0, \infty)$. Let $J = [0, b]$ and $J' = (0, b)$, by $C(J, X)$ and $C(J', X)$ we denote the spaces of all continuous functions from $J$ to $X$ and $J'$ to $X$, respectively.

Define

$$Y = \{ x \in C(J', X) : \lim_{t \to 0^+} t^{1-\alpha}(1-\beta)x(t) \text{ exists and infinite} \},$$

with the norm $\| \cdot \|_Y$ defined by

$$\| x \|_Y = \sup_{t \in J'} t^{1-\alpha}(1-\beta)|x(t)|.$$

Obviously, $Y$ is a Banach space. We also note that:

(i) When $\alpha = 1$, then $Y = C(J, X)$ and $\| \cdot \|_Y = \| \cdot \|$

(ii) Let $x(t) = t^{\alpha-1}(1-\beta)y(t)$ for $t \in J', x \in Y$ if and only if $y \in C(J, X)$, and $\| x \|_Y = \| y \|.$

Let $B_r(J) = \{ y \in C(J, X) : \| y \| \leq r \}$ and $B_r(J') = \{ x \in Y : \| x \|_Y \leq r \}$, then $B_r$ and $B_r'$ are two bounded closed and convex subsets of $C(J, X)$ and $Y$, respectively.

**Definition 2.1** (see [23]). The fractional integral of order $p$ with the lower limit $a$ for a function $f : [a, \infty) \to \mathbb{R}$ is defined as

$$I_a^p f(t) = \frac{1}{\Gamma(p)} \int_a^t \frac{f(s)}{(t-s)^{1-p}} ds, \quad t > a, \quad p > 0,$$

provided the right side is point-wise defined on $[a, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

**Definition 2.2** (see [23]). The Riemann–Liouville derivative of order $p > 0$ for a function $f : [a, \infty) \to \mathbb{R}$ is defined as

$$D_a^{p} f(t) = \frac{1}{\Gamma(n-p)} \frac{d^n}{dr^n} \int_a^t \frac{f(s)}{(t-s)^{n-1-p}} ds, \quad t > a, \quad n - 1 < p < n.$$

**Definition 2.3** (see [23]). The Caputo derivative of order $p > 0$ for a function $f : [a, \infty) \to \mathbb{R}$ is defined as

$$^C D_a^{p} f(t) = D_a^{p} [ f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(a) ], \quad t > a, \quad n - 1 < p < n.$$
Remark 2.4

(i) If \( f \in C^0[a, \infty) \), then

\[
\mathcal{D}_a^p f(t) = \frac{1}{\Gamma(n-p)} \int_a^t f^{(n)}(s)(t-s)^{p-1-n} \, ds, \quad t > a, \; n - 1 < p < n.
\]

(ii) The Caputo derivative of a constant is equal to zero.

(iii) If \( f \) is an abstract function with values in \( X \), then integrals which appear in Definitions 2.1 and 2.2 are taken in Bochner’s sense.

**Definition 2.5 (Hilfer fractional derivative, see [14])**. The generalized Riemann–Liouville fractional derivative of order \( 0 \leq \nu \leq 1 \) and every nonempty subset \( B \subseteq X \)

\[
\mathcal{D}_{a}^\nu f(t) = I_{a}^{1-\nu} \frac{d}{dt} I_{a}^{\nu} f(t),
\]

for functions such that the expression on the right hand side exists.

Remark 2.6

(i) When \( \nu = 0, 0 < \mu < 1 \) and \( a = 0 \), the Hilfer fractional derivative corresponds to the classical Riemann–Liouville fractional derivative:

\[
\mathcal{D}_{a}^{\mu, \nu} f(t) = \frac{d}{dt} I_{a}^{\nu} f(t) = D_{a}^{\mu} f(t).
\]

(ii) When \( \nu = 1, 0 < \mu < 1 \) and \( a = 0 \), the Hilfer fractional derivative corresponds to the classical Caputo fractional derivative:

\[
\mathcal{D}_{a}^{\mu, \nu} f(t) = I_{a}^{1-\nu} \frac{d}{dt} I_{a}^{\nu} f(t) = D_{a}^{\mu} f(t).
\]

Next, we introduce the Hausdorff noncompact measure \( \alpha(\cdot) \) defined on each bounded subset \( \Omega \) of Banach space \( X \) by

\[
\alpha(\Omega) = \inf \{ \varepsilon > 0 : \Omega \text{ has a finite-net in } X \}.
\]

Some basic properties of \( \alpha(\cdot) \) are given in the following Lemmas.

**Lemma 2.7** (see [3]). The noncompact measure \( \alpha(\cdot) \) satisfies:

(i) if for all bounded subsets \( B_1, B_2 \) of \( X \), \( B_1 \subseteq B_2 \) implies \( \alpha(B_1) \subseteq \alpha(B_2) \);

(ii) if \( \alpha(\{x\} \cup B) = \alpha(B) \) for every \( x \in X \) and every nonempty subset \( B \subseteq X \);

(iii) \( \alpha(B) = 0 \) if and only if \( B \) is relatively compact in \( X \).

(iv) \( \alpha(B_1 + B_2) \leq \alpha(B_1) + \alpha(B_2) \), where \( B_1 + B_2 = \{x + y : x \in B_1, y \in B_2\} \);

(v) \( \alpha(B_1 \cup B_2) \leq \max\{\alpha(B_1), \alpha(B_2)\} \);

(vi) \( \alpha(\lambda B) \leq |\lambda| \alpha(B) \) for any \( \lambda \in \mathbb{R} \).

For any \( W \subset C(J, X) \), we define

\[
\int_0^t W(s) \, ds = \left\{ \int_0^t u(s) \, ds : u \in W \right\}, \text{ for } t \in J,
\]

where \( W(s) = \{u(s) \in X : u \in W\} \).

**Lemma 2.8** (see [13]). If \( W \subset C(J, X) \) is bounded and equicontinuous, then \( t \rightarrow \alpha(W(t)) \) is continuous on \( J \), and

\[
\alpha(W) = \max_{t \in J} \alpha(W(t)), \quad \alpha\left( \int_0^t W(s) \, ds \right) \leq \int_0^t \alpha(W(s)) \, ds, \text{ for } t \in J.
\]

**Lemma 2.9** (see [20]). Let \( (u_n)_{n=1}^\infty \) be a sequence of Bochner integrable functions from \( J \) into \( X \) with \( |u_n(t)| \leq m(t) \) for almost all \( t \in J \) and every \( n \geq 1 \), where \( m \in L(J, \mathbb{R}^+) \), then the function \( \psi(t) = \alpha((u_n(t))_{n=1}^\infty) \) belongs to \( L(J, \mathbb{R}^+) \) and satisfies

\[
\alpha\left( \left\{ \int_0^t u_n(s) \, ds : n \geq 1 \right\} \right) \leq 2 \int_0^t \psi(s) \, ds.
\]
Lemma 2.10 (see [28]). Suppose $\beta > 0$, $a(t)$ is a nonnegative function locally integrable on $0 \leq t < T$ (some $T \leq +\infty$) and $g(t)$ is a nonnegative, nondecreasing continuous function defined on $0 \leq t < T$, $g(t) \leq M$(constant), and suppose $u(t)$ is nonnegative and locally integrable on $0 \leq t < T$ with

$$u(t) \leq a(t) + g(t) \int_0^t (t-s)^{\beta-1} u(s) dt,$$

on this interval. Then

$$u(t) \leq a(t) + \int_0^t \left[ \sum_{n=1}^{\infty} \frac{(g(t) \Gamma(\beta))^n}{n!} (t-s)^{\beta-1} a(s) \right] ds, 0 \leq t < T.$$

Especially, when $a(t) = 0$, then $u(t) = 0$ for all $0 \leq t < T$.

Throughout this paper, we introduce the following hypotheses:

(H1) $Q(t)$ is continuous in the uniform operator topology for $t > 0$, and $(Q(t))_{t>0}$ is uniformly bounded, i.e., there exists $M > 1$ such that $\sup_{t \in [0, +\infty)} |Q(t)| < M$;

(H2) for each $t \in J$, the function $f(t, \cdot) : X \to X$ is continuous and for each $x \in X$, the function $f(\cdot, x) : J \to X$ is strongly measurable;

(H3) there exists a function $m \in L^1(J, \mathbb{R}^+)$ such that

$$\lim_{t \to 0^+} (1-x(t)) m(t) = 0,$$

and

$$|f(t, x)| \leq m(t) \quad \text{for all } x \in X \text{ and almost all } t \in J;$$

It is obvious that if (H3) holds, there exists a constant $r > 0$ such that

$$M\left( \frac{|x_0|}{\Gamma(\nu(1-\mu) + \mu)} + \frac{\sup_{t \in J} f(t, x(t))}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} m(s) ds \right) \leq r.$$

(H4) there exists a constant $l > 0$ such that for any bounded $D \subseteq X$,

$$\lambda(f(t, D)) \leq l^{1-\nu(1-\mu)} \lambda(D), \quad \text{for a.e. } t \in [0, b].$$

Lemma 2.11. The Cauchy problem (1.1) is equivalent to the integral equation

$$x(t) = \frac{x_0}{\Gamma(\nu(1-\mu) + \mu)} t^{(\nu-1)(1-\mu)} + \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} [Ax + f(s, x(s))] ds, t \in J'. \quad (2.1)$$

Proof. We can refer to [12], here omit it.

The Wright function $M_{\mu}(q)$ is defined by

$$M_{\mu}(q) = \sum_{n=1}^{\infty} \frac{(-q)^{n-1}}{(n-1)! \Gamma(1-\mu n)}, \quad 0 < \mu < 1, \quad q \in \mathbb{C},$$

which satisfies the following equality.

$$\int_0^\infty \theta^\mu M_{\mu}(\theta) d\theta = \frac{\Gamma(1+\delta)}{\Gamma(1+\delta \mu)}, \quad \text{for } \theta > 0. \quad \square$$

Lemma 2.12. If integral Eq. (2.1) holds, then we have

$$x(t) = S_{v, \mu}(t)x_0 + \int_0^t K_{\mu}(t-s)f(s, x(s)) ds, \quad t \in J'. \quad (2.2)$$

where

$$K_{\mu}(t) = t^{\mu-1} P_{\mu}(t), \quad P_{\mu}(t) = \int_0^\infty \mu \theta M_{\mu}(\theta) Q(t^\mu \theta) d\theta \quad \text{and} \quad S_{v, \mu}(t) = I_0^{(1-\nu)\mu} K_{\mu}(t).$$
Proof. Let \( \lambda > 0 \). Applying the Laplace transform
\[
\mathcal{L}(\lambda) = \int_0^\infty e^{-\lambda s} x(s) ds \quad \text{and} \quad \omega(\lambda) = \int_0^\infty e^{-\lambda s} f(s, x(s)) ds
\]
to (2.1), we have
\[
\mathcal{L}(\lambda) = \lambda^{(1-\gamma)(1-\beta)-1} x_0 + \frac{1}{\lambda^\gamma} \mathcal{L}^\gamma(\lambda) + \frac{1}{\lambda^\alpha} \omega(\lambda) = \lambda^{\gamma(\mu-1)} (\lambda^\alpha I - A)^{-1} x_0 + (\lambda^\gamma I - A)^{-1} \omega(\lambda)
\]
provided that the integrals in (2.3) exist, where I is the identity operator defined on \( X \).

Let
\[
\psi_\mu(\theta) = \frac{\mu}{\theta^\mu} M_\mu(\theta^{-\mu}),
\]
whose Laplace transform is given by
\[
\int_0^\infty e^{-\lambda \psi_\mu(\theta)} d\theta = e^{-\mu}, \quad \text{where} \quad \mu \in (0, 1).
\]

Using (2.4), we have
\[
\int_0^\infty e^{-\lambda s} Q(s) x_0 ds = \int_0^\infty \int_0^\infty \mu \psi_\mu(0) e^{-\lambda t} Q(t^{\mu}) t^{\mu-1} x_0 dt d\mu = \int_0^\infty \int_0^\infty e^{-\lambda t} \left[ \int_0^\infty \psi_\mu(0) Q(t^{\mu}) t^{\mu-1} x_0 d\mu \right] dt
\]
\[
= \int_0^\infty e^{-\lambda t} t^{\mu-1} P_\mu(t) x_0 dt.
\]
\[
\int_0^\infty e^{-\lambda s} Q(s) f(s) \omega(\lambda) ds = \int_0^\infty \int_0^\infty \mu \psi_\mu(0) e^{-\lambda t} Q(t^{\mu}) e^{-\lambda s} f(s, x(s)) ds dt
\]
\[
= \int_0^\infty \int_0^\infty \int_0^\infty \psi_\mu(0) e^{-\lambda t} Q(t^{\mu}) e^{-\lambda s} f(s, x(s)) ds dt d\mu dt
\]
\[
= \int_0^\infty \int_0^\infty \int_0^\infty e^{-\lambda t} \left[ \int_0^t \psi_\mu(0) Q(t^{\mu}) (t^{\mu} - s)^{\mu-1} f(s, x(s)) ds \right] dt d\mu dt
\]
\[
= \int_0^\infty \int_0^\infty e^{-\lambda t} \left[ \int_0^t (t-s)^{\mu-1} P_\mu(t-s)f(s, x(s)) ds \right] dt.
\]
Since the Laplace inverse transform of \( \lambda^{\gamma(\mu-1)} \) is
\[
\mathcal{L}^{-1}(\lambda^{\gamma(\mu-1)}) = \begin{cases} \frac{t^{\mu-1}}{\Gamma(\mu)}, & 0 < \mu \leq 1, \\ \delta(t), & \mu = 0, \end{cases}
\]
where \( \delta(t) \) is the Delta function.

Thus, by (2.3), (2.5), and (2.6), for \( t \in \mathcal{J} \) we obtain
\[
x(t) = \left( \mathcal{L}^{-1}(\lambda^{\gamma(\mu-1)}) \ast K_\mu(t) \right) x_0 + \int_0^t K_\mu(t-s)f(s, x(s)) ds = \left( \mathcal{L}^{-1}(\lambda^{\gamma(\mu-1)}) \ast K_\mu(t) \right) x_0 + \int_0^t K_\mu(t-s)f(s, x(s)) ds
\]
\[
= S_{\gamma, \mu}(t) x_0 + \int_0^t K_\mu(t-s)f(s, x(s)) ds.
\]
This completes the proof. \( \square \)

Due to Lemma 2.12, we give the following definition of the mild solution of (1.1).

**Definition 2.13.** By the mild solution of the Cauchy problem (1.1), we mean that the function \( x \in \mathcal{C}(\mathcal{J}, X) \) which satisfies
\[
x(t) = S_{\gamma, \mu}(t) x_0 + \int_0^t K_\mu(t-s)f(s, x(s)) ds, \quad t \in \mathcal{J}.
\]
Remark 2.14

(i) By (2.7), we are easy to know that

\[ D_{\alpha}^{(1+\mu)} S_{\alpha}(t) = K_{\alpha}(t), \quad t \in J. \]

(ii) When \( \nu = 0 \), the fractional Eq. (1.1) degenerated to the classical Riemann–Liouville fractional equation which has been studied by Zhou et al. in [31]. In there,

\[ S_{\alpha}(t) = K_{\alpha}(t) = t^{\mu-1} P_{\alpha}(t), \quad t \in J. \]

(iii) When \( \nu = 1 \), the fractional Eq. (1.1) degenerated to the classical Caputo fractional equation which had been studied by Zhou and Jiao in [30]. In there,

\[ S_{1,\alpha}(t) = S_{\alpha}(t), \quad t \in J, \]

where \( S_{\alpha}(t) \) is defined in [30].

Proposition 2.15. Under assumption (H1), \( P_{\alpha}(t) \) is continuous in the uniform operator topology for \( t > 0 \).

Proof. For any \( t > 0 \), \( h > 0 \) and \( x \in X \), we have

\[ |P_{\alpha}(t+h)x - P_{\alpha}(t)x| = \left| \int_0^\infty \mu \partial M_{\alpha}(\theta) [Q((t+h)^{\mu}\theta) - Q(t^{\mu}\theta)] \partial x \theta \right|. \]

Since

\[ \left| \int_0^\infty \mu \partial M_{\alpha}(\theta) [Q((t+h)^{\mu}\theta) - Q(t^{\mu}\theta)] \partial x \theta \right| \leq 2M \int_0^\infty \mu \partial M_{\alpha}(\theta) |\partial x| \theta = \frac{2M}{\Gamma(\mu)} |x|, \]

then by Lebesgue dominated convergence Theorem, we have

\[ |P_{\alpha}(t+h)x - P_{\alpha}(t)x| \rightarrow 0 \quad \text{independently of} \ t \ \text{and} \ x, \ \text{as} \ h \rightarrow 0. \]

Therefore, \( P_{\alpha}(t) \) is continuous in the uniform operator topology for \( t > 0 \). This completes the proof. \( \square \)

Proposition 2.16. Under assumption (H1), for any fixed \( t > 0 \), \( \{K_{\alpha}(t)\} \) and \( \{S_{\alpha}(t)\} \) are linear operators, and for any \( x \in X \)

\[ |K_{\alpha}(t)x| \leq \frac{M \mu^{\mu-1}}{\Gamma(\mu)} |x| \quad \text{and} \quad |S_{\alpha}(t)x| \leq \frac{M \mu^{(\nu-1)(\mu-1)}}{\Gamma(\nu(1-\mu) + \mu)} |x|. \]

Proof. From the equality

\[ \int_0^\infty \theta^\nu M_{\alpha}(\theta) d\theta = \frac{\Gamma(1+\nu)}{\Gamma(1+\mu\theta)}, \]

we know that

\[ |P_{\alpha}(t)x| = \left| \int_0^\infty \mu \partial M_{\alpha}(\theta) Q(t^{\mu}\theta) \partial x \theta \right| \leq \frac{M}{\Gamma(\mu)} |x|, \quad \text{for} \ t \in J \ \text{and} \ x \in X, \]

then we have

\[ |K_{\alpha}(t)x| \leq \frac{M \mu^{\mu-1}}{\Gamma(\mu)} |x|, \quad \text{for} \ t \in J^* \ \text{and} \ x \in X. \]

For \( t \in J^* \) and \( x \in X \),

\[ |S_{\alpha}(t)x| = \left| \int_0^{(1-\nu) K_{\alpha}(t)x} = \frac{1}{\Gamma(\nu(1-\mu))} \int_0^1 (t-s)^{(\nu-1)(\mu-1)} K_{\alpha}(s) x ds \right| \]

\[ = \frac{1}{\Gamma(\nu(1-\mu))} \int_0^1 (t-s)^{(\nu-1)(\mu-1)} s^{\mu-1} P_{\alpha}(s) x ds \leq \frac{M}{\Gamma(\nu(1-\mu)) \Gamma(\mu)} \int_0^1 (1-s)^{(\nu-1)(\mu-1)} s^{\mu-1} |x|. \]

This completes the proof. \( \square \)
Proposition 2.17. Under assumption (H1), \( \{K_\mu(t)\}_{t>0} \) and \( \{S_\mu(t)\}_{t>0} \) are strongly continuous, which means that, for any \( x \in X \) and \( 0 < t' < t'' \leq b \), we have

\[
|K_\mu(t')x - K_\mu(t'')x| \to 0 \quad \text{and} \quad |S_\mu(t')x - S_\mu(t'')x| \to 0, \quad \text{as} \quad t'' \to t'.
\]

Proof. By Proposition 2.15, we know that \( \{P_\mu(t)\}_{t>0} \) is strongly continuous, then we easily obtain \( \{K_\mu(t)\}_{t>0} \) is also strongly continuous.

For any \( x \in X \) and \( 0 < t_1 < t_2 \leq b \), we have

\[
|S_\mu(t_2)x - S_\mu(t_1)x| = \frac{1}{\Gamma(v(1-\mu))} \left| \int_0^{t_2} (t_2 - s)^{v(1-\mu)-1}K_\mu(s)ds - \int_0^{t_1} (t_1 - s)^{v(1-\mu)-1}K_\mu(s)ds \right|
\]

\[
= \frac{1}{\Gamma(v(1-\mu))} \left| \int_0^{t_2} (t_2 - s)^{v(1-\mu)-1}P_\mu(s)ds - \int_0^{t_1} (t_1 - s)^{v(1-\mu)-1}P_\mu(s)ds \right|
\]

\[
\leq \frac{1}{\Gamma(v(1-\mu))} \left| \int_0^{t_1} (t_2 - s)^{v(1-\mu)-1}P_\mu(s)ds \right|
\]

\[
+ \frac{1}{\Gamma(v(1-\mu))} \left| \int_0^{t_1} (t_2 - s)^{v(1-\mu)-1}P_\mu(s)ds \right|
\]

\[
\leq \frac{M_\mu}{\Gamma(v(1-\mu))} \left| t_2 - t_1 \right|^{v(1-\mu)}
\]

\[
+ \frac{M}{\Gamma(v(1-\mu))} \left| \int_0^{t_1} \left( (t_2 - s)^{v(1-\mu)-1} - (t_1 - s)^{v(1-\mu)-1} \right) P_\mu(s)ds \right|
\]

(2.9)

Since

\[
\left| \int_0^{t_1} \left( (t_2 - s)^{v(1-\mu)-1} - (t_1 - s)^{v(1-\mu)-1} \right) P_\mu(s)ds \right| \leq 2 \int_0^{t_1} (t_1 - s)^{v(1-\mu)-1}ds \text{ exists},
\]

then by Lebesgue dominated convergence Theorem, we have

\[
\left| \int_0^{t_1} \left( (t_2 - s)^{v(1-\mu)-1} - (t_1 - s)^{v(1-\mu)-1} \right) P_\mu(s)ds \right| \to 0 \quad \text{as} \quad t_2 \to t_1.
\]

Consequently, we have

\[
|S_\mu(t_2)x - S_\mu(t_1)x| \to 0 \quad \text{as} \quad t_2 \to t_1,
\]

i.e., \( \{S_\mu(t)\}_{t>0} \) is strongly continuous. This completes the proof. \( \Box \)

For any \( x \in Y \), Define an operator \( T \) as follows

\[
(Tx)(t) = (T_1x)(t) + (T_2x)(t),
\]

where

\[
(T_1x)(t) = S_\mu(t)x_0 \quad \text{and} \quad (T_2x)(t) = \int_0^t K_\mu(t-s)f(s,x(s))ds \quad \text{for all} \quad t \in J.
\]

By (2.8) and (H3), we have

\[
\lim_{t \to 0^+} t^{\nu(1-\mu)} S_\mu(t)x_0 = \frac{1}{\Gamma(v(1-\mu))} \left| \int_0^1 (1-s)^{v(1-\mu)-1} x_0 ds \right| = \frac{x_0}{\Gamma(v(1-\mu) + 1)}
\]

(2.10)

and

\[
\left| t^{\nu(1-\mu)} \int_0^t K_\mu(t-s)f(s,x(s))ds \right| \leq \frac{M t^{(\nu-1)(1-\mu)}}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1}m(s)ds \to 0.
\]

as \( t \to 0^+ \).

Thus, we can define operator \( \mathcal{F} \) as follows. For any \( y \in C(J,X) \), let \( x(t) = t^{(\nu-1)(1-\mu)}y(t) \),

\[
(\mathcal{F}y)(t) = (\mathcal{F}_1y)(t) + (\mathcal{F}_2y)(t),
\]
\[
(\mathcal{F}_1 y)(t) = \begin{cases} 
 t^{(1-v)/(1-\mu)}(T_1 x)(t), & \text{for } t \in (0, b], \\
 \frac{x_0}{\Gamma(1-\mu + s)}, & \text{for } t = 0, 
\end{cases}
\]
\[
(\mathcal{F}_2 y)(t) = \begin{cases} 
 t^{(1-v)/(1-\mu)}(T_2 x)(t), & \text{for } t \in (0, b], \\
 0, & \text{for } t = 0. 
\end{cases}
\]

Obviously, \(x\) is a mild solution of (1.1) in \(Y\) if and only if the operator equation \(y = \mathcal{F} y\) has a solution \(y \in C(I, X)\).

### 3. Main results

In this section, by using the measure of noncompactness and Ascoli–Arzelà Theorem, we will obtain some sufficient conditions ensuring the existence of the mild solution.

**Theorem 3.1.** Assume that (H1)–(H3) hold, then \(\{\mathcal{F} y : y \in \mathcal{B}(J)\}\) is equicontinuous.

**Proof.**

**Step I.** \(\{\mathcal{F}_1 y : y \in \mathcal{B}(J)\}\) is equicontinuous.

For any \(y \in \mathcal{B}(J)\), let \(x(t) = t^{\mu-1} y(t), t \in (0, b]\). For \(0 \leq t_1 < t_2 \leq b\), we have

\[
|(\mathcal{F}_1 y)(t_2) - (\mathcal{F}_1 y)(t_1)| \leq \left| \int_0^{t_2} t^{(1-v)/(1-\mu)}(T_1 x)(t_2) - t^{(1-v)/(1-\mu)}(T_1 x)(t_1) \, ds \right|,
\]

and

\[
\|\mathcal{F}_1 y\|_{\mathcal{C}(I, X)} \leq M \mathcal{N}(\mu).
\]

**Step II.** \(\{\mathcal{F}_2 y : y \in \mathcal{B}(J)\}\) is equicontinuous.

For any \(y \in \mathcal{B}(J)\), let \(x(t) = t^{\mu-1} y(t), t \in (0, b]\). For \(t_1 = 0, 0 < t_2 \leq b\), we get

\[
|(\mathcal{F}_2 y)(t_2) - (\mathcal{F}_2 y)(t_1)| \leq \left| \int_{t_1}^{t_2} t^{(1-v)/(1-\mu)}(T_2 x)(t_2) - t^{(1-v)/(1-\mu)}(T_2 x)(t_1) \, ds \right|,
\]

and

\[
\|\mathcal{F}_2 y\|_{\mathcal{C}(I, X)} \leq M \mathcal{N}(\mu).
\]

By condition (H3), one can reduce that \(\lim_{t_2 \to t_1} I_1 = 0\). Noting that

\[
\left[ t^{(1-v)/(1-\mu)}(t_1 - s)^{\mu-1} - t^{(1-v)/(1-\mu)}(t_2 - s)^{\mu-1} \right] m(s) \leq t^{(1-v)/(1-\mu)}(t_1 - s)^{\mu-1} m(s),
\]

we have

\[
\mathcal{F}_2 y \in \mathcal{B}(I, X).
\]
and \( \int_0^t \left[ t^{(1-\nu)(1-\mu)}(t_1 - s)^{\mu-1} - t_2^{(1-\nu)(1-\mu)}(t_2 - s)^{\mu-1} \right] m(s)ds \rightarrow 0 \) as \( t_2 \rightarrow t_1 \),

then one can deduce that \( \lim_{\epsilon \rightarrow 0} I_2 = 0 \).

For \( \epsilon > 0 \) be enough small, we have

\[
I_3 \leq t_1^{(1-\nu)(1-\mu)} \int_0^{t_1} (t_1 - s)^{\mu-1} |m(s)|ds \leq 2M \frac{1}{\Gamma(\mu)} \int_0^{t_1} (t_1 - s)^{\mu-1} m(s)ds \leq I_{31} + I_{32} + I_{33}.
\]

where

\[
I_{31} = r \frac{1}{M} \sup_{s \in [0, t_1 - \epsilon]} |P_\mu(t_2 - s) - P_\mu(t_1 - s)|,
\]

\[
I_{32} = 2M \frac{1}{\Gamma(\mu)} \int_0^{t_1} (t_1 - s)^{\mu-1} m(s)ds - \int_{t_1 - \epsilon}^{t_1} (t_1 - \epsilon)^{(1-\nu)(1-\mu)}(t_1 - s)^{\mu-1} m(s)ds,
\]

\[
I_{33} = 2M \frac{1}{\Gamma(\mu)} \int_0^{t_1} \left[ (t_1 - \epsilon)^{(1-\nu)(1-\mu)}(t_1 - s)^{\mu-1} - t_2^{(1-\nu)(1-\mu)}(t_2 - s)^{\mu-1} \right] m(s)ds.
\]

By Proposition 2.15, it is easy to see that \( I_{31} \rightarrow 0 \) as \( t_2 \rightarrow t_1 \). Similar to the proof that \( I_1, I_2 \) tend to zero, we get \( I_{32} \rightarrow 0 \) and \( I_{33} \rightarrow 0 \) as \( \epsilon \rightarrow 0 \). Thus, \( I_3 \) tends to zero independently of \( y \in \mathcal{B}_r(J) \) as \( t_2 \rightarrow t_1 \), \( \epsilon \rightarrow 0 \). Therefore, \( |(\mathcal{T} y)(t_2) - (\mathcal{T} y)(t_1)| \rightarrow 0 \) independently of \( y \in B_r(J) \) as \( t_2 \rightarrow t_1 \), which means that \( \{\mathcal{T} y : y \in \mathcal{B}_r(J)\} \) is equicontinuous.

Therefore, \( \mathcal{T} \) is equicontinuous. This completes the proof. \( \Box \)

**Theorem 3.2.** Assume that (H1)–(H3) hold, then \( \mathcal{T} \) maps \( \mathcal{B}_r(J) \) into \( \mathcal{B}_r(J) \), and is continuous on \( \mathcal{B}_r(J) \).

**Proof.**

**Step I.** \( \mathcal{T} \) maps \( \mathcal{B}_r(J) \) into \( \mathcal{B}_r(J) \). For any \( y \in \mathcal{B}_r(J) \), let \( x(t) = t^{\nu-1}y(t) \). Then \( x \in \mathcal{B}_r'(J') \). For \( t \in J \), by Proposition 2.16, we have

\[
|((\mathcal{T} y)(t))| \leq |t^{(1-\nu)(1-\mu)}S_{\mu}(t)x_0| + t^{(1-\nu)(1-\mu)} \frac{1}{\Gamma(\mu)} \int_0^t K_\mu(t - s)f(s, x(s))ds,
\]

\[
\leq \frac{M|x_0|}{\Gamma(\nu(1-\mu) + \mu)} + M \frac{t^{(1-\nu)(1-\mu)}}{\Gamma(\mu)} \int_0^t (t - s)^{\mu-1} f(s, x(s))ds,
\]

\[
\leq M \left( \frac{|x_0|}{\Gamma(\nu(1-\mu) + \mu)} + \sup_{x \in J'} \left( \frac{t^{(1-\nu)(1-\mu)}}{\Gamma(\mu)} \int_0^t (t - s)^{\mu-1} m(s)ds \right) \right) \leq r.
\]

Hence, \( \|\mathcal{T} y\| \leq r \), for any \( y \in \mathcal{B}_r(J) \).

**Step II.** \( \mathcal{T} \) is continuous in \( \mathcal{B}_r(J) \). For any \( y_m, y \in \mathcal{B}_r(J) \), \( m = 1, 2, \ldots \), with \( \lim_{m \to \infty} y_m = y \), we have

\[
\lim_{m \to \infty} y_m(t) = y(t) \quad \text{and} \quad \lim_{m \to \infty} t^{(1-\nu)(1-\mu)}y_m(t) = t^{(1-\nu)(1-\mu)}y(t), \quad \text{for} \ t \in J'.
\]

Then by (H2), we have

\[
f(t, x_m(t)) = f(t, t^{(1-\nu)(1-\mu)}y_m(t)) \to f(t, t^{(1-\nu)(1-\mu)}y(t)) = f(t, x(t)), \quad \text{as} \ m \to \infty,
\]

where \( x_m(t) = t^{(1-\nu)(1-\mu)}y_m(t) \) and \( x(t) = t^{(1-\nu)(1-\mu)}y(t) \).

On the one hand, using (H3), we get for each \( t \in J' \),

\[
(t - s)^{\mu-1}f(s, x_m(s)) - f(s, x(s)) \leq (t - s)^{\mu-1}2m(s), \quad \text{a.e. in} \ 0, t.
\]

On the other hand, the function \( s \to (t - s)^{\mu-1}2m(s) \) is integrable for \( s \in [0, t) \) and \( t \in J' \). By Lebesgue dominated convergence theorem, we have

\[
\int_0^t (t - s)^{\mu-1}f(s, x_m(s)) - f(s, x(s))ds \to 0, \quad \text{as} \ m \to 0.
\]

For \( t \in J \),
\[
|| (\mathcal{F} y_m)(t) - (\mathcal{F} y)(t) || \leq t^{(1-\gamma)(1-\mu)} \left| \int_0^t K_{\mu}(t-s)f(s,x_m(s))ds \right|
\]
\[
\leq \frac{M_2 t^{(1-\gamma)(1-\mu)}}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1}f(s,x_m(s))ds \to 0, \quad as \ m \to \infty.
\]

Therefore, \( \mathcal{F} y_m \to \mathcal{F} y \) pointwise on \( J \) as \( m \to \infty \), by which Theorem 3.1 implies that \( \mathcal{F} y_m \to \mathcal{F} y \) uniformly on \( J \) as \( m \to \infty \) and so \( \mathcal{F} \) is continuous. This completes the proof. \( \square \)

**Theorem 3.3.** Assume that (H1)-(H4) hold, then the Cauchy problem (1.1) has at least one mild solution in \( B^\mu_\gamma(f) \).

**Proof.** Let \( y_m(t) = t^{(1-\gamma)(1-\mu)}S_{\mu}(t)\chi_0 \) for all \( t \in J \) and \( y_m = \mathcal{F} y_m \). Consider the set \( \mathcal{H} = \{ y_m : m = 0, 1, 2, \ldots \} \), and we will prove set \( \mathcal{H} \) is relatively compact.

It follows from Theorems 3.1 and 3.2 that \( \mathcal{H} \) is uniformly bounded and equicontinuous on \( J \). Next, we only prove that for any \( t \in J \), set \( \mathcal{H}(t) = \{ y_m(t) : m = 0, 1, 2, \ldots \} \) is relatively compact in \( X \).

Under the condition (H4), by Lemmas 2.7 and 2.9, for any \( t \in J \) we have
\[
\alpha(\mathcal{H}(t)) = \alpha(\{ y_m(t) \}_{m=0}^\infty) = \alpha(\{ y_0(t) \} \cup \{ y_m(t) \}_{m=1}^\infty) = \alpha(\{ y_m(t) \}_{m=1}^\infty),
\]
and
\[
\alpha(\{ y_m(t) \}_{m=1}^\infty) = \alpha(\{ (\mathcal{F} y_m)(t) \}_{m=0}^\infty) = \alpha\left( \left\{ t^{(1-\gamma)(1-\mu)}S_{\mu}(t)x_0 + t^{(1-\gamma)(1-\mu)} \int_0^t K_{\mu}(t-s)f(s,x_m(s))ds \right\}_{m=0}^\infty \right)
\]
\[
\leq 2 \frac{M_2 t^{(1-\gamma)(1-\mu)}}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1}f(s,\{ s^{(1-\gamma)(1-\mu)}y_m(s) \}_{m=0}^\infty)ds
\]
\[
\leq 2 \frac{M_1 t^{(1-\gamma)(1-\mu)}}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1}f(s,\{ s^{(1-\gamma)(1-\mu)}y_m(s) \}_{m=0}^\infty)ds
\]
\[
\leq 2 \frac{M_1 t^{(1-\gamma)(1-\mu)}}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1}\alpha(\{ y_m(s) \}_{m=0}^\infty)ds,
\]
then
\[
\alpha(\mathcal{H}(t)) \leq 2 \frac{M_1 t^{(1-\gamma)(1-\mu)}}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1}\alpha(\mathcal{H}(s))ds.
\]

Therefore, by Lemmas 2.8 and 2.10, we obtain that \( \alpha(\mathcal{H}(0)) = 0 \), then \( \mathcal{H}(t) \) is relatively compact. Consequently, it follows from Ascoli-Azela Theorem that set \( \mathcal{H} \) is relatively compact, i.e., there exists a convergent subsequence of \( \{ y_m \}_{m=0}^\infty \). With no confusion, let \( \lim_{m \to \infty} y_m = y^* \in B^\mu_\gamma(f) \).

Thus, by continuity of the operator \( \mathcal{F} \), we have
\[
y^* = \lim_{m \to \infty} y_m = \lim_{m \to \infty} \mathcal{F} y_{m-1} = \mathcal{F} \left( \lim_{m \to \infty} y_{m-1} \right) = \mathcal{F} y^*.
\]
which implies the Cauchy problem (1.1) has at least a mild solution. This completes the proof. \( \square \)

**Remark 3.4.** The assumption (H2) is replaced by the following assumption:
(H2') there exists a constant \( \mu_1 \in (0, \mu) \) and \( m \in L^\mu(f, \mathbb{R}^n) \) such that
\[
|f(t,x)| \leq m(t), \quad for \ all \ x \in X \ and \ almost \ all \ t \in J.
\]
Then we have the following Corollary.

**Corollary 3.5.** Assume that (H1), (H2'), (H3) and (H4) hold, then the Cauchy problem (1.1) has at least one mild solution in \( B^\mu_\gamma(f) \).

**Remark 3.6.** Obviously, our results can be applied to the evolution equations with Riemann-Liouville fractional derivative and Caputo fractional derivative.

4. Conclusions

The paper is concerned with existence of mild solution of evolution equation with Hilfer fractional derivative which generalized the famous Riemann–Liouville fractional derivative. Here, we do not require that $C_0$ semigroup $(Q(t))_{t>0}$ is compact. By noncompact measure method and Ascoli–Arzela Theorem, we obtain some sufficient conditions to ensure the existence of mild solution. Our results are new and more general to known results.

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References


