THE INCREMENTAL RATIO BASED CAUSAL FRACTIONAL CALCULUS

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The generalized incremental ratio fractional derivative is revised and its main properties deduced. It is shown that in the case of analytic functions it enjoys some interesting properties like: linearity and causality and has a semi-group structure. Some simple examples are presented. The enlargement of the set of functions for which the group properties of the fractional derivative are valid is done. With this it is shown that some well-known results are valid in a more general set-up. Some examples are presented.

Keywords: Grünwald-Letnikov derivative, incremental ratio, distribution.

1. Introduction

Fractional calculus is an over 300 years old mathematical field that has been attracting the attention of scientists and engineers. In fact, in recent years fractional calculus has been rediscovered and applied in an increasing number of fields, namely in the areas of electromagnetism, control engineering, and signal processing. The number of physical and engineering processes that are best described by fractional differential equations has also been increasing from last 90th decade. This led to the increment of the study of Fractional Calculus and development of new theories and tools that produced its enrichment and had as consequence a somehow semi-chaotic state of the art. In fact, there are several definitions that lead to different results, making difficult the establishment of a compatible framework with the classic calculus. Grünwald-Letnikov, Riemann-Liouville, Caputo, Hadamard, Marchaud, Liouville are some of the known
definitions [Diaz & Osler, 1974; Dugowson, 1994; Kilbas et al., 2006; Miller & Ross, 1993; Nishimoto, 1989; Ortigueira, 2004, 2006; Podlubny, 1999; Samko et al., 1993; Li & Zhao, 2011]. Till recently the Riemann-Liouville derivative was favoured relatively to the others, mainly because it is the more natural definition of fractional derivative join to the Grünwald-Letnikov definition, and the rigorous study presented in the excellent references [Miller & Ross, 1993; Samko et al., 1993]. However in recent years Caputo derivative call the attention and received users preference due to the initial conditions problem [Ortigueira & Coito, 2010]. Although from a purely mathematical point of view it is legitimate to accept and even use one or all, from the point of view of applications the situation could be different, in many cases, although the different authors use one or other fractional derivative definition to present their models, at the end when they want obtain a numerical approximation of their results or validate the models they use often, as it is natural the Grünwald-Letnikov derivative to replace the fractional derivative that they have used, so we suggest use directly such derivative or some variants of it to be used in the applied science and engineering [Ortigueira et al., 2005, 2010; Ortigueira & Trujillo, 2009]. In previous papers [Ortigueira, 2004, 2006] some contributions were made towards the goal of obtaining a definition of fractional derivative suitable for our interests, but with a wide generality and compatibility with classic definition under suitable restrictions of the framework of our specific problem. In [Ortigueira et al., 2005] we addressed this problem and proposed a solution based on a reasoning that led to the use of the Grünwald-Letnikov and the so called generalised functions [Podlubny, 1999], forward and backward derivatives. However, we met ourselves in a somehow uncomfortable situation, because it was not clear the relation with the classic Calculus. We tried to overcome this problem, but the results were not conveniently explored to lead to a completely compatible scheme. One of our goals in this manuscript will be to solve the mentioned problem. We will start from the fractional incremental ratio to define a general fractional derivative and show that this leads to the well known Grünwald-Letnikov (G-L) derivatives as introduced by Grünwald in 1867 and independently by Letnikov in 1868. Any way, we do not explore in this paper all the possibilities of such general definition.

We will show that these derivatives impose causality: one is causal and the other anti-causal. For the general derivative we will prove its semi-group properties and deduce some other interesting features. We will show that it is compatible with classic derivative that appears here as a special case. We will compute the derivatives of some useful functions. In particular, we obtain derivatives of exponentials, causal exponentials, causal powers and logarithms. We will proof the generalized Leibniz formula and the integration by parts for the new definitions of fractional derivatives of G-L as main results in section 2, also in this section we present the way we followed from the fractional differences to the fractional derivative defined in the complex plane. The semi-group properties are considered in section 3. We present a general formula and two special cases valid for real variable functions. We exemplify with the exponential function and present the forward and backward derivatives as special cases valid for real functions in section 4. We must remark that our calculations are being done formally, but they are true for a wide set of suitable kind of functions.

Remark 1.1. In this paper we deal with a multivalued expression \( z^a \). As is well known, to define a function we have to fix a branch cut line and choose a branch (Riemann surface). It is a common procedure to choose the negative real half-axis as branch cut line. In what follows we will assume that we adopt the principal branch and assume that the obtained function is continuous above the branch cut line. With this, we will write \((-1)^a = e^{j\pi a}\)

2. The Fractional derivative

Definition 2.1. To generalize the known notion of fractional derivatives we introduce the general formulation of the incremental ratio valid for any order, real or complex. In the following we will consider the real
The incremental ratio based causal fractional calculus

Similarly to the classic case, we define fractional derivative by the limit of the fractional incremental ratio

$$D_{\theta}^\alpha f(z) = e^{-j\alpha \theta} \lim_{|h| \to 0} \frac{\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(z-kh)}{|h|^\alpha}, \quad (1)$$

where $h = e^{j\theta}$ is a complex number, with $\theta \in (-\pi, \pi]$.

The above defined derivative is a general incremental ratio based derivative that generalizes classical Grünwald-Letnikov fractional derivative. To understand and give an interpretation to the above formula, assume that $z = t \in \mathbb{R}$ is a time and that $h$ is real, $\theta = 0$ or $\theta = \pi$. If $\theta = 0$, only the present and past values are being used (2), while, if $\theta = \pi$, only the present and future values are used (see (4)). This means that if we look at (1) as a linear system, the first case is causal, while the second is anti-causal\(^1\) [Ortigueira et al., 2005; Ortigueira, 2004, 2006].

In general, if $\theta = 0$, we call (1) the forward Grünwald-Letnikov\(^2\) derivative, which is well known; however its properties are not so well studied:

$$D_f^\alpha f(z) = \lim_{|h| \to 0^+} \frac{\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(z-kh)}{h^\alpha}. \quad (2)$$

In particular, if we assume that $f(t) = 0$ for $t < a$ and let $[\cdot]$ be the ”integer part of the argument”, we obtain from (2).

$$D_f^\alpha f(z) = \lim_{|h| \to 0^+} \frac{\sum_{k=0}^{[\frac{z-a}{h}]} (-1)^k \binom{\alpha}{k} f(z-kh)}{h^\alpha}, \quad (3)$$

that is, the formulation we find frequently [Miller & Ross, 1993; Podlubny, 1999; Kilbas et al., 2006].

$$D_{-\theta}^\alpha f(z) = e^{-j\pi \alpha} \lim_{|h| \to 0^+} \frac{\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(z+kh)}{h^\alpha}. \quad (4)$$

It is interesting to remark that the approach to fractional calculus stated in (2) and (4) was proposed first by Liouville [Dugowson, 1994].

2.1. Existence

It is not a simple task to formulate the weakest conditions that ensure the existence of the fractional derivatives (1), (2) and (4), although we can give some necessary conditions for their existence. To study the existence conditions for the fractional derivatives we must care about the behaviour of the function along the half straight-line $z \pm nh$ with $n \in \mathbb{Z}^+$. If the function is zero for $\Re(e(z)) < a \in \mathbb{R}$ (respectively, when $\Re(e(z)) > a$) the forward (backward) derivative exists at every finite point of $f(z)$. In the general case, we must have in mind the behavior of the binomial coefficients. They verify

$$\left| \binom{\alpha}{k} \right| \leq \frac{A}{k^{\alpha+1}}, \quad (5)$$

\(^1\)We will return to this matter later.

\(^2\)The terms forward and backward are used here in agreement to the way the time flows, from past to future or the reverse.
meaning that \( f(z) \frac{A_k^{\alpha+1}}{k^{\alpha+1}} \) must decrease, at least as \( f(z) \frac{A_k^{\alpha+1}}{k^{\alpha+1}} \) when \( k \) goes to infinite. For example considering the forward case, if \( \alpha > 0 \), it is enough that \( f(z) \) be bounded in the left half plane, but if \( \alpha < 0 \), \( f(z) \) must decrease to zero to obtain a convergent series. In particular, this suggests that \( \Re(e(h)) > 0 \) and \( \Re(e(h)) < 0 \) should be adopted for right and left functions, respectively in agreement with Liouville reasoning [Dugowson, 1994]. In particular, they should be used for the functions such that \( f(z) = 0 \) for \( \Re(e(z)) < 0 \) and \( f(z) = 0 \) for \( \Re(e(z)) > 0 \), respectively. Such issue is very interesting, since we conclude that the existence of the fractional derivative depends only on what happens in one half complex plane, left or right. Consider \( f(z) = \frac{z^2}{\sqrt{2}} \) with a suitable branch cut line. If \( \beta > \alpha \), we conclude immediately that \( D_{\alpha}^f(z) \) defined for every \( z \in \mathbb{C} \) does not exist, unless \( \alpha \) been a positive integer, because the summation in (1) is divergent.

2.2. Properties

We are going to present the main properties of the derivative above presented.

**Linearity.** The linearity property of the fractional derivative is evident from the above formula. In fact, we have

\[
D_{\alpha}^0 [f(z) + g(z)] = D_{\alpha}^0 f(z) + D_{\alpha}^0 g(z).
\]

(6)

**Causality.** The causality property was already referred above and can also be obtained easily. We only have to use (2), or (4). Assume that \( z = t \in \mathbb{R} \) and that \( f(t) = 0 \), for \( t < 0 \), we conclude immediately from (2) that \( D_{\alpha}^0 f(t) = 0 \) for \( t < 0 \). For the anti-causal case, the situation is similar.

**Scale change.** Let \( f(z) = g(az) \), where \( a \) is a constant. From (1), we have:

\[
D_{\alpha}^0 g(az) = \lim_{h \to 0} \sum_{k=0}^{\infty} (-1)^k \frac{\alpha}{k} \frac{g(az - kah)}{h^\alpha} = a^\alpha \lim_{h \to 0} \sum_{k=0}^{\infty} (-1)^k \frac{\alpha}{k} \frac{g(az - kah)}{(ah)^\alpha} = a^\alpha D_{\alpha}^0 g(\tau) |_{\tau = az}.
\]

(7)

**Time reversal.** If \( f(z) = g(-z) \), we obtain from the property we just deduced above, but consider a negative constant and working with the main branch if the negative powers appear bellow that:

\[
D_{\alpha}^0 g(-z) = (-1)^\alpha \lim_{h \to 0} \sum_{k=0}^{\infty} (-1)^k \frac{\alpha}{k} \frac{g(-z + kh)}{(-h)^\alpha} = (-1)^\alpha D_{\alpha}^0 g(\tau) |_{\tau = -z},
\]

(8)

in agreement with (2) and (4). This means that the time reversal converts the forward derivative into the backward and vice-versa.

**Time shift.** The derivative operator is shift invariant:

\[
D_{\alpha}^0 g(z - a) = D_{\alpha}^0 g(\tau) |_{\tau = z - a}.
\]

(9)
Derivative of a product. We are going to compute the derivative of the product of two functions: \( f(t) = \varphi(t) \cdot \psi(t) \) assumed to be defined for \( t \in \mathbb{R} \), by simplicity, although the result we will obtain is valid for \( t \in \mathbb{C} \), excepting over an eventual branch cut line. Assume that one of them is analytic in a given region. From (1), and working with increments we can write

\[
\Delta^\alpha f(z) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} \varphi(z - kh) \psi(z - kh). \tag{10}
\]

But, as

\[
\Delta^N f(z) = \sum_{k=0}^{N} (-1)^k \binom{N}{k} f(z - kh), \tag{11}
\]

and

\[
f(z - kh) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \Delta^i f(z), \tag{12}
\]

then

\[
\Delta^\alpha f(z) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} \varphi(z - kh) \sum_{i=0}^{k} (-1)^i \binom{k}{i} \Delta^i \psi(z), \tag{13}
\]

which can be transformed in

\[
\Delta^\alpha f(z) = \sum_{i=0}^{\infty} (-1)^i \Delta^i \psi(z) \sum_{k=i}^{\infty} (-1)^k \binom{k}{i} \binom{\alpha}{k} \varphi(z - kh). \tag{14}
\]

However, since

\[
\binom{k+i}{i} \binom{\alpha}{k+i} = \binom{\alpha}{i} \binom{\alpha-i}{k} \tag{15}
\]

then, replacing (15) in (14), we obtain

\[
\Delta^\alpha f(z) = \sum_{i=0}^{\infty} \binom{\alpha}{i} \Delta^i \psi(z) \sum_{k=0}^{\infty} (-1)^k \binom{\alpha-i}{k} \varphi(z - kh - ih). \tag{16}
\]

Therefore

\[
\frac{\Delta^\alpha f(z)}{h^\alpha} = \sum_{i=0}^{\infty} \binom{\alpha}{i} \Delta^i \psi(z) \left( \sum_{k=0}^{\infty} (-1)^k \binom{\alpha-i}{k} \varphi(z - kh - ih) \right) \tag{17}
\]

Computing the limit as \( h \to 0 \) of (17), we obtain the derivative of the product:

\[
D_0^\alpha [\varphi(t) \psi(t)] = \sum_{n=0}^{\infty} \binom{\alpha}{n} \varphi^{(n)}(t) \psi^{(\alpha-n)}(t), \tag{18}
\]

that is, the generalized Leibniz rule. We must remark that the above formula is commutative if both functions are analytic. If only one of them is analytic, it is not commutative. We must remark that the noncommutativity of this rule seems natural, since we only require analyticity to one function. It is a situation very similar to the one we find when defining the product of generalized functions and its derivatives.
The deduction of (18) we presented here differs from others presented in literature [Samko et al., 1993; Miller & Ross, 1993; Podlubny, 1999]. As it is clear when \( \alpha = N \in \mathbb{Z}^+ \) we obtain the classic Leibniz rule. When \( \alpha = -1 \), we obtain a very interesting formula for computing the primitive of the product of two functions:

\[
D^{-1}[\varphi(t)\psi(t)] = \sum_{n=0}^{\infty} (-1)^n \varphi^{(n)}(t)\psi^{(-n-1)}(t),
\]

(19)

To exemplify the use of the formula (19), let \( f(t) = \frac{t^n}{n!}e^{at} \). Put \( \phi(t) = \frac{t^n}{n!} \) and \( \psi(t) = e^{at} \). As \( \phi^{(k)}(t) = \frac{t^{n-k}}{(n-k)!} \), while \( k \leq n \) and \( \psi^{(-k-1)}(t) = a^{-k-1}e^{at} \). Then:

\[
D^{-1}f(t) = e^{at} \sum_{k=0}^{n} (-1)^k a^{-k-1} \frac{t^{n-k}}{(n-k)!}.
\]

(20)

If we put \( \phi(t) = f(t) \) and \( \psi(t) = 1 \), \( t \in \mathbb{R} \), we have:

\[
D^{-1}f(t) = \sum_{n=0}^{\infty} (-1)^n f^{(n)}(t) \frac{t^{n+1}}{(n+1)!}
\]

(21)

similar to the McLaurin formula.

**Integration by parts.** The so called integration by parts relates both causal and anti-causal derivatives and can be stated as:

\[
\int_{-\infty}^{+\infty} g(t)D_{\alpha}^\beta f(t) \, dt = (-1)^{\alpha} \int_{-\infty}^{+\infty} f(t)D_{\alpha}^\beta g(t) \, dt
\]

(22)

where we assume that both integrals exist. To obtain this formula, we only have to use (2) inside the integral and perform a variable change

\[
\int_{-\infty}^{+\infty} \lim_{h \to 0^+} \sum_{k=0}^{\infty} (-1)^k \frac{\alpha}{k} f(t - kh)g(t) \, dt = \int_{-\infty}^{+\infty} \lim_{h \to 0^+} \sum_{k=0}^{\infty} (-1)^k \frac{\alpha}{k} g(t + kh)f(t) \, dt,
\]

(23)

leading immediately to (22) using (4). This result is slightly different from the one find in current literature due to our definition of backward derivative.

Of course, the remark 1.1 have been consider here.

3. **Group structure of the fractional derivative**

**Additivity and Commutativity.** We are going to apply (1) twice for two orders. We have

\[
D_{\alpha}^\beta \left[D_{\alpha}^\beta f(t)\right] = D_{\alpha}^\beta \left[D_{\alpha}^\beta f(t)\right] = D_{\alpha}^{\alpha+\beta} f(t)
\]

(24)

To prove this statement we start from (1), and work with suitable functions so we can interchange the series because with such functions the series is absolutely convergent in the corresponding interval. Therefore we
write:

\[
D^{\alpha}_{\theta} \left[ D^{\beta}_{\theta} f(t) \right] = \lim_{h \to 0} \frac{\sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k \left[ \sum_{n=0}^{\infty} \binom{\beta}{n} f(t - kh - nh) \right]}{h^{\alpha+\beta}} \\
= \lim_{h \to 0} \frac{\sum_{n=0}^{\infty} \binom{\beta}{n} (-1)^n \left[ \sum_{k=0}^{\infty} \binom{\alpha}{k} f(t - kh - nh) \right]}{h^{\alpha+\beta}} \\
= \lim_{h \to 0} \frac{\sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} \binom{\alpha}{m-n} \binom{\beta}{n} \right) (-1)^m f(t - mh)}{h^{\alpha+\beta}} \\
= \lim_{h \to 0} \frac{\sum_{m=0}^{\infty} \binom{\alpha + \beta}{m} (-1)^m f(t - mh)}{h^{\alpha+\beta}} = D^{\alpha+\beta} f(t),
\]

if we take into account the following relation

\[
\sum_{n=0}^{m} \binom{\alpha}{m-n} \binom{\beta}{n} = \binom{\alpha + \beta}{m}. \tag{26}
\]

**Associativity.** This property comes easily from the above results. In fact, it is easy to show that

\[
D^{\gamma}_{\theta} \left[ D^{\alpha+\beta}_{\theta} f(t) \right] = D^{\gamma+\alpha+\beta}_{\theta} f(t) = D^{\alpha+\beta+\gamma}_{\theta} f(t) = D^{\gamma+\alpha}_{\theta} \left[ D^{\beta+\gamma}_{\theta} f(t) \right]. \tag{27}
\]

**Neutral element.** If we put \( \beta = -\alpha \) in (24) we obtain:

\[
D^{\alpha}_{\theta} \left[ D^{-\alpha}_{\theta} f(t) \right] = D^{\alpha-\alpha}_{\theta} f(t) = f(t) \tag{28}
\]
or again by (24)

\[
D^{-\alpha}_{\theta} \left[ D^{\alpha}_{\theta} f(t) \right] = D^{\alpha-\alpha}_{\theta} f(t) = f(t), \tag{29}
\]

This is very important because it states the existence of inverse.

**Inverse element.** From the last result we conclude that there is always an inverse element: for every \( \alpha \) order derivative, there is always a \( -\alpha \) order derivative. This seems to be contradictory with our knowledge from the classic calculus where the Nth order derivative has N primitives. To be more concrete assume that we have a real function such that its \( \alpha \) order fractional derivative given by (2) exists. Therefore, we do not need to add a specific formula to calculate the primitive of a function. To exemplify such fact, we consider \( f(t) = e^t, \) \( t \in \mathbb{R}. \) As shown bellow, its derivative is also \( e^t, \) \( t \in \mathbb{R}, \) as well as its anti-derivative, provided that they are computed using (2). This situation is also illustrated in formulas (19) and (21). They give us the primitive without needing to join any constant. Another example is given in the following section. This forces us to be consistent and careful with the used language. So, when \( \alpha \) is positive we will speak of derivative. When \( \alpha \) is negative, we will use the term anti-derivative or primitive (not integral). This could clarify the situation and solves the problem created by Riemann when he introduced the complimentary polynomials and show how to obtain the “proper primitives” of [Kreml, 2006].
4. Simple examples

Example 4.1. The exponential

Let us apply the above definitions to the function \( f(z) = e^{sz}, \ z \in \mathbb{C} \). The convergence of (1) is dependent of \( s \) and of \( h \). Let \( h > 0 \), the series in (2) becomes

\[
e^{sz} \sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k e^{-kh}
\]

(30)

As it is well known [Diaz & Osler, 1974], the binomial series

\[
\sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k e^{-kh}
\]

is convergent to the main branch of

\[
g(s) = (1 - e^{-sh})^\alpha
\]

(32)

provided that \( |e^{-sh}| < 1 \), that is if \( \Re(e(s)) > 0 \). This means that the branch cut line of \( g(s) \) must be in the left hand half complex plane. Then

\[
D^\alpha_f z = \lim_{h \to 0^+} \frac{(1 - e^{-sh})^\alpha}{h^\alpha} e^{sz} = |h|^\alpha e^{j\theta} e^{sz}
\]

(33)

valid iff \( \theta \in (-\pi, \pi) \) which corresponds to be working with the principal branch of \( (\cdot)^\alpha \) and assuming a branch cut line in the left hand complex half plane.

Now, consider the series in (4) with \( f(z) = e^{sz} \). Proceeding as above, we obtain another binomial series:

\[
\sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k e^{kh}
\]

(34)

that is convergent to the main branch of

\[
f(s) = (1 - e^{sh})^\alpha
\]

(35)

provided that \( \Re(e(s)) < 0 \). This means that the branch cut line of \( f(s) \) must be in the left hand half complex plane. We will assume to work in the principal branch and that \( h(s) \) is continuous from above. Here we must remark that in \( (\cdot)^\alpha \) we are again in the principal branch but we are assuming a branch cut line in the right hand complex half plane.

We obtain directly:

\[
D^\alpha_b z = |s|^\alpha e^{j\theta} e^{sz}
\]

(36)

valid iff \( \theta \in (-\pi, \frac{3\pi}{2}) \).

We must be careful in using the above results. In fact, in a first glance, we could be led to use it for computing the derivatives of functions like \( \cos(z) \), \( \cos(z) \), \( \sinh(z) \) and \( \cosh(z) \). But if we have in mind our reasoning we can conclude immediately that those functions do not have finite derivatives if \( z \in \mathbb{C} \). In fact, they use simultaneously the exponentials \( e^z \) and \( e^{-z} \) whose derivatives cannot exist simultaneously, as we just saw. These results can be used to generalize a well known property of the Laplace transform. If we return back to equation (2) and apply the bilateral Laplace transform

\[
F(s) = \int_{-\infty}^{+\infty} f(t) e^{-st} dt
\]

(37)
to both sides [Ortigueira & Trujillo, 2009] we conclude that:

$$LT\left[D^\alpha f(t)\right] = s^\alpha F(s) \quad (\text{Re}(s) > 0)$$

(38)

where in $s^\alpha$ we assume the principal branch and a cut line in the left half plane. With equation (4) we obtain:

$$LT\left[D^\alpha f(t)\right] = s^\alpha F(s) \quad (\text{Re}(s) < 0)$$

(39)

where now the branch cut line is in the right half plane. These results have a system interpretation: there are two systems (differintegrators) with the same expression for the transfer function $H(s) = s^\alpha$. We will not compute the impulse responses here. The $s = j$ case was treated before [Ortigueira & Trujillo, 2009]. We showed that we have:

$$(jw)^\alpha = |w|^\alpha \cdot \begin{cases} 
e e^{j\frac{\alpha \pi}{2}}, \text{ if } \omega > 0 \\
\ne e^{-j\frac{\alpha \pi}{2}}, \text{ if } \omega < 0
\end{cases}$$

(40)

in the causal case and

$$(jw)^\alpha = |w|^\alpha \cdot \begin{cases} 
e e^{j\frac{\alpha \pi}{2}}, \text{ if } \omega > 0 \\
\ne e^{j3\frac{\alpha \pi}{2}}, \text{ if } \omega < 0
\end{cases}$$

(41)

**Example 4.2. The general power function**

We start by computing the fractional derivative of the constant function. Let then $f(t) = 1$ for every $t \in \mathbb{R}$ and $\alpha \in \mathbb{R} - \mathbb{Z}^-$. From (1) we have:

$$D^\alpha = \lim_{t \to 0} \frac{\sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k}{k^\alpha} = \begin{cases} 0, \text{ if } \alpha > 0 \\
\infty, \text{ if } \alpha < 0
\end{cases}$$

(42)

To prove it, we are going to consider the partial sum of the series

$$\sum_{k=0}^{n} \binom{\alpha}{k} (-1)^k = \frac{\alpha - 1}{n} \binom{\alpha + n + 1}{n} \Gamma(1 - \alpha) \Gamma(n + 1) \to \frac{1}{\Gamma(1 - \alpha)} > \frac{1}{n^\alpha}.$$  (43)

As $n \to \infty$ [Samko et al., 1993] we obtain the limits shown in (42). So, the $\alpha$ order fractional derivative of 1 is the null function. If $\alpha < 0$, the limit is infinite. So there is no fractional “primitive” of a constant.

This means that when working in the context defined by (1) two functions with the same fractional derivative are equal.

The example we just treated allows us to obtain an interesting result: “there are no fractional derivatives of the power function defined in $\mathbb{R}$”. In fact, suppose that there is a fractional derivative of $t^n$, $t \in \mathbb{R}$, $n \in \mathbb{N}^+$. We must have:

$$D^\alpha t^n = n!D^\alpha D^{-n}1 = D^{-n}D^\alpha 1.$$  (44)

This means that we must be careful when trying to generalize the Taylor series. We conclude also that we cannot compute the fractional derivative of a function by using directly its Taylor expansion.

The same result could be obtained directly from (1). It is enough to remark that a power function tends to infinite when the argument tends to $-\infty$. 
5. The Distributional Fractional Derivative

Here we will generalize the concept of fractional derivative in order to guarantee that all the properties of the above defined derivative remain valid even with generalized functions. In particular, the group properties should be valid:

\[ D^{\gamma} \left[ D^{\alpha + \beta} f(t) \right] = D^{\gamma + \alpha + \beta} f(t) = D^{\alpha + \beta + \gamma} f(t) = D^\alpha \left[ D^{\beta + \gamma} f(t) \right] \]

(45)

with neutral element

\[ D^\alpha \left[ D^{-\alpha} f(t) \right] = D^0 f(t) = f(t) \quad \text{and} \quad D^{-\alpha} \left[ D^\alpha f(t) \right] = D^0 f(t) = f(t) \]

(46)

In some derivatives these properties do not remain valid, since the commutativity is not valid. Here we will extend the validity of formula (2) by a suitable generalized function definition.

The above properties are valid provided that all the involved derivatives exist. This may not happen in a lot of situations; for example, the derivative of the power function causal or not. As these functions are very important we will consider them with detail. Meanwhile let us see how we can enlarge the validity of the above formulas. Let us consider formula (2) and a function \( f(t) \) such that there exists \( D^\alpha f(t) \) but it may be discontinuous. In principle, we cannot assure that we can apply (1) to obtain \( D^{\alpha + \beta} f(t) \). To solve the problem, we define distribution as an integer order derivative of a continuous function [Ferreira, 1997]: \( f(t) = D^n g(t) \), where \( n \) is a positive integer and \( g(t) \) is continuous and with continuous fractional derivative of order \( \alpha + \beta \). In this case, we can write:

\[ D^{\alpha + \beta} f(t) = D^{\alpha + \beta} D^n g(t) = D^n D^{\alpha + \beta} g(t) \]

(47)

So, we obtained the desired derivative by integer order derivative computation of the fractional derivative. The other properties are consequence of this one. Exemplifying will clarify the situation. From the point of view of the Laplace transform (LT), the above relations show that the derivative rule remain valid for any real order: \( LT[f^{(\alpha)}(t)] = s^\alpha F(s) \).

The results obtained allow us to obtain the derivative of any order of the continuous function \( p(t) = t^\beta u(t) \), with \( \beta > 0 \); \( u(t) \) is the Heaviside unit step. As a continuous function, it is indefinitely (integer order) derivable in distributional sense. To compute the fractional derivative of \( p(t) \), the easiest way is to use the Laplace transform (LT). As well known, the LT of \( p(t) \) is \( P(s) = \frac{\Gamma(\beta + 1)}{s^{\beta+1}} \), for \( \Re(s) > 0 \). The transform of the fractional derivative of order \( \alpha \) is given by: \( s^\alpha F(s) \). So,

\[ D^\alpha t^\beta u(t) = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha + 1)} t^{\beta - \alpha} u(t) \]

(48)

that generalizes the integer order formula for any \( \alpha \in \mathbb{R} \) generalizing the current result [Kilbas et al., 2006; Miller & Ross, 1993; Podlubny, 1999; Samko et al., 1993]. To obtain it, we use the rule of the derivative of the product. With \( \beta > 0 \), \( \rho(t) \) is a continuous function. So, we can compute the Nth order derivative to obtain a distribution. The derivative of \( u(t) \) is \( \delta(t) \) that appears multiplied by a power that is zero at \( t = 0 \): \( D \left( t^\beta u(t) \right) = \beta t^{\beta - 1} u(t) + t^\beta \delta(t) = \beta t^{\beta - 1} u(t) \). However, repeating the procedure the second term is no longer null, but we remove it to give us the finite part. With these considerations we conclude that (5) remains valid provided that \( \alpha \in \mathbb{R} \) and \( \beta \in \mathbb{R} - Z^- \). In particular, we have: \( D^\alpha u(t) = \frac{1}{\Gamma(1 - \alpha)} t^{-\alpha} u(t) \) and, from it, \( \delta(t) = \frac{\Gamma(-\alpha)}{\Gamma(-\alpha)} t^{-\alpha} u(t) \), valid for positive non-integer orders. To generalise the above result for \( \beta \in Z^- \), we going to study the causal logarithm \( \lambda(t) = \log(t) u(t) \). We start from relation (5) and compute the derivative of both sides relative to \( \beta \) to obtain:

\[ D^\alpha \left[ t^\beta \log(t) u(t) \right] = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha + 1)} t^{\beta - \alpha} \left[ \log(t) + \psi(\beta + 1) - \psi(\beta - \alpha + 1) \right] \]

(49)
where we represented by $\psi$ the logarithmic derivative of the gamma function. With $\beta = 0$ we obtain the derivative of the causal logarithm and from it, by integer order derivation, we obtain after some manipulations

$$D^\alpha t^{-N} u(t) = \frac{D^N [\log(t) u(t)]}{(-1)^{N-1} (N-1)!}$$

$$= -\frac{(\alpha)_N}{(N-1)! \Gamma(-\alpha + 1)} t^{-\alpha - N} u(t) \left[ \log(t) - \gamma - \psi(-\alpha + 1) - \sum_{n=1}^{N} \frac{1}{1 - \alpha - n} \right],$$

valid for $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{Z}^-$ where $\gamma = -\psi(1) = -\Gamma'(1)$ is the Euler-Mascheroni constant.

6. Conclusions

In this paper we proposed a new look at the fractional derivative by taking as starting point the Grünwald-Letnikov definition that we had generalized before. We showed that, for analytic functions, it enjoys some nice and useful properties and a semi-group structure. We presented some examples.

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References

REFERENCES


Forthcoming Articles (have been scheduled for publication)  Accepted Articles with Assigned DOI (an incomplete list)

<table>
<thead>
<tr>
<th>ID</th>
<th>Article Title</th>
<th>Corresponding Author</th>
</tr>
</thead>
<tbody>
<tr>
<td>IJBC-D-11-00444</td>
<td>A Spectral Radius Estimate and Entropy of Hypercubes</td>
<td>Michal Misiolekiewicz</td>
</tr>
<tr>
<td>IJBC-D-11-00435</td>
<td>Multiple Solutions of a Generalized Singular Perturbed Bratu Problem</td>
<td>Yuri Kaznetsov</td>
</tr>
<tr>
<td>IJBC-D-11-00415</td>
<td>Calculation of Julia Sets by Equipotential Point Algorithm</td>
<td>Yuan-Yuan Sun</td>
</tr>
<tr>
<td>IJBC-D-11-00408</td>
<td>Hopf Bifurcation in a Calcium Oscillation Model and its Control: Frequency Domain Approach</td>
<td>Yu Chang</td>
</tr>
<tr>
<td>IJBC-D-11-00398</td>
<td>Bifurcations of Traveling Wave Solutions in a Microstructured Solid Model</td>
<td>Jinlin Li</td>
</tr>
<tr>
<td>IJBC-D-11-00396</td>
<td>Partial control of Escapes in Chaotic Scattering</td>
<td>Jesus Seoane</td>
</tr>
<tr>
<td>IJBC-D-11-00389</td>
<td>Eight Limit Cycles Around a Center in Quadratic Hamiltonian System with Third-order Perturbation</td>
<td>Pei Yu</td>
</tr>
<tr>
<td>IJBC-D-11-00383</td>
<td>Scale-free LT Codes</td>
<td>Francis Lau</td>
</tr>
<tr>
<td>IJBC-D-11-00382</td>
<td>Horseshoes in a Chaotic System with only One Stable Equilibrium</td>
<td>Xiao-Song Yang</td>
</tr>
<tr>
<td>IJBC-D-11-00381</td>
<td>Periodic Orbits Associated with the Libration Points of the Homogeneous Rotating Gravitating Triaxial Ellipsoid</td>
<td>Eusebius J Doedel</td>
</tr>
<tr>
<td>IJBC-D-11-00380</td>
<td>Analytical Period-m Solutions and Chaos in Nonlinear Systems</td>
<td>Albert Luo</td>
</tr>
<tr>
<td>IJBC-D-11-00378</td>
<td>A New Data Rate Adaption Communications Scheme for Code-shifted Differential Chaos Shift Keying Modulation</td>
<td>Weikai Xu</td>
</tr>
<tr>
<td>IJBC-D-11-00372</td>
<td>Geometric Limits of Mandelbrot and Julia Sets under Degree Growth</td>
<td>Suzanne Boyd</td>
</tr>
<tr>
<td>IJBC-D-11-00362</td>
<td>Analysis of Grain-scale Measurements of Sand Using Kinematical Complex Networks</td>
<td>David Walker</td>
</tr>
<tr>
<td>IJBC-D-11-00360</td>
<td>Chaotification of Linear Impulsive Differential Systems with Applications</td>
<td>Qi-Gui Yang</td>
</tr>
<tr>
<td>IJBC-D-11-00359</td>
<td>Asymptotic Expansions of Melnikov Functions and Limit Cycle Bifurcations</td>
<td>Miaoan Han</td>
</tr>
<tr>
<td>IJBC-D-11-00358</td>
<td>Saddle-node Bifurcation Cascades and Associated Travelling Waves in Weakly Coupling CML</td>
<td>Dolores Sotelo Hen</td>
</tr>
<tr>
<td>IJBC-D-11-00357</td>
<td>Dynamics of a Prey-Dependent Digestive Model with State-Dependent Impulsive Control</td>
<td>Zhaosheng Feng</td>
</tr>
<tr>
<td>IJBC-D-11-00356</td>
<td>Multi-scales Dynamics of Two Coupled Non-smooth Systems</td>
<td>Alierez Ture Savad</td>
</tr>
<tr>
<td>IJBC-D-11-00353</td>
<td>Conventional and Extended Time-delayed Feedback Controlled Zero-crossing Digital Phase Locked Loop</td>
<td>Tammy Banetjee</td>
</tr>
<tr>
<td>IJBC-D-11-00352</td>
<td>Wavelet Bifurcation Analysis of Dynamical Systems: A Case Study in Oscillations of Chana Coralina Transmembrane Potential</td>
<td>Eugene B Postniko</td>
</tr>
<tr>
<td>IJBC-D-11-00345</td>
<td>On the Performance of Linear Adaptive Filters driven by the Ergodic Chaotic Logistic Map</td>
<td>Andreas Mueller</td>
</tr>
<tr>
<td>IJBC-D-11-00342</td>
<td>Metastable Systems as Random Maps</td>
<td>Abrahim Royaksky</td>
</tr>
<tr>
<td>IJBC-D-11-00340</td>
<td>Local Bifurcations of a Quasiperiodic Orbit</td>
<td>Soumitro Banerjee</td>
</tr>
<tr>
<td>IJBC-D-11-00339</td>
<td>Chaos in Coupled Clocks</td>
<td>Tomasz Kapitanian</td>
</tr>
<tr>
<td>IJBC-D-11-00333</td>
<td>Generalized Extreme Value Distribution Parameters as Dynamical Indicators of Stability</td>
<td>Davide Faranda</td>
</tr>
<tr>
<td>IJBC-D-11-00332</td>
<td>Self-organised Structure Formation in Organised Microstructuring by Laser-Jet-Etching</td>
<td>Peter Jorg Phath</td>
</tr>
<tr>
<td>IJBC-D-11-00331</td>
<td>Chaos of the Propagating Pulse Wave in a Ring of Six-coupled Bistable Oscillators</td>
<td>Tetsuro Endo</td>
</tr>
<tr>
<td>IJBC-D-11-00327</td>
<td>A Rikitake Type System with Quadratic Control</td>
<td>Tudor Binarz</td>
</tr>
<tr>
<td>IJBC-D-11-00321</td>
<td>Chaotification of Non-autonomous Discrete Dynamical Systems</td>
<td>Qilin Huang</td>
</tr>
<tr>
<td>IJBC-D-11-00319</td>
<td>Rational First Integrals for Polynomial Vector Fields on Algebraic Hypersurfaces of R^4(n+1)</td>
<td>Jaume Lidbre</td>
</tr>
<tr>
<td>IJBC-D-11-00315</td>
<td>Synchronization of Chaos and the Transition to Wave Turbulence</td>
<td>Ricardo Viana</td>
</tr>
<tr>
<td>IJBC-D-11-00313</td>
<td>Parametric Analysis of Bifurcation and Chaos in a Periodically Driven Horizontal Impact Pair</td>
<td>Albert Luo</td>
</tr>
<tr>
<td>IJBC-D-11-00308</td>
<td>Exact Travelling Wave Solutions of the Kudryashov-Sinelshchikov Equation and Their Bifurcations</td>
<td>Jinbin Li</td>
</tr>
<tr>
<td>IJBC-D-11-00290</td>
<td>Polynomial First Integrals for the Chen and Lü Systems</td>
<td>Claudia Valls</td>
</tr>
<tr>
<td>IJBC-D-11-00289</td>
<td>Detecting Stretch-and-fold Mechanism in Chaotic Dynamics</td>
<td>Yutaka Shimada</td>
</tr>
<tr>
<td>IJBC-D-11-00288</td>
<td>Synchronization as a Process of Sharing and Transferring Information</td>
<td>Erik Boltt</td>
</tr>
<tr>
<td>IJBC-D-11-00283</td>
<td>Panhoni's Game Model to Find Numerically the Stable Attractors of a Tumor Growth Model</td>
<td>Marius-F. Dunca</td>
</tr>
<tr>
<td>IJBC-D-11-00277</td>
<td>Mehrab Maps: One-dimensional Piecewise Nonlinear Chaotic Maps</td>
<td>Shahram Etemadi Borujeni</td>
</tr>
<tr>
<td>IJBC-D-11-00276</td>
<td>An Initial Study of the Flow Around an Aerosol at High Reynolds Numbers Using Continuation</td>
<td>Christopher Wales</td>
</tr>
<tr>
<td>IJBC-D-11-00267</td>
<td>Highly Accurate Doubter Generator for Memristor-based Analog Memories</td>
<td>Changjie Yang</td>
</tr>
<tr>
<td>IJBC-D-11-00266</td>
<td>The Exact Travelling Wave Solutions and their Bifurcations in the Gardiner and Gardiner-KP Equations</td>
<td>Cuncai Hua</td>
</tr>
<tr>
<td>IJBC-D-11-00265</td>
<td>Shrimp: Occurrence, Scaling and Relevance</td>
<td>Ruedi Stoop</td>
</tr>
</tbody>
</table>
On Quantum Vortices and Trajectories in Particle

Finite Difference Methods for Fractional Differential

The Incremental Ratio Based Causal Fractional Calculus

An Excitable Electronic

Characterization of Effects of Quantization, Delay and Internal Resistances

Influence of Sampling

Complex Dynamics of Fractional Dynamics in Experiences on an Efficient Control of Multifractional Property Analysis of Human Sleep EEG

A Fractional-order Spectra and Their Application in the Theory of Viscoelasticity

and Their Application in the Theory of Viscoelasticity

Existence Results for Fractional Boundary Value Problem Pattern Formation in Fractional Reaction-Diffusion Systems with Multiple Heterodimensional Cycles under Transversality Condition

Properties of Popp's Attractor

Chaos in Cylindrical Piecewise Smooth Reversible Dynamical Systems at a Self-Organization of Surface Fitting and Error

Functional Analysis for Generalized Hopf Bifurcation in a Frequency Domain Classification and Exact Formulation

Two-fold Singularity

alpha < 2

alpha < 2

Two-sided Boundary Conditions

both Fractional and Dry Friction Type of Dissipation

Properties of Popp's Attractor

Chaotic Attractors inside Band-merging Scenario in a ZAD-controlled Accelerator Maps

Scaling Properties for the Chaotic Dynamics of a Financial Indexes

Analysis Using Fractal Interpolation

Chaos of a Hybrid three Species Food Chain with Time Delayed Intervention

Existence Results for Fractional Boundary Value Problem via Critical Point Theory

Finite Difference Schemes for Variable-order Time Fractional Diffusion Equation

On Stability of Commensurate Fractional Order Systems

Stochastic Hopf Bifurcation of Quasi-integrable Hamiltonian Systems with Fractional Derivative Damping

Fractional-order Spectra and Their Application in the Theory of Viscoelasticity

A Fractional-order Universal High-gain Adaptive Stabilizer

Multifractal Property Analysis of Human Sleep EEG Signals

Efficient Control of Accelerator Maps

Experiences on an Internet Link Characterization and Networked Control of

Universal High-gain Adaptive Stabilizer

Critical Exponents and Scaling Properties for the Chaotic Dynamics of a Particle in a Time-dependent Potential Barrier

Fractional Dynamics in Financial Indexes

The Effect of Weak Dissipation in Two-dimensional Mapping

Regularization of Tunneling Rates with Quantum Chaos

Complex Dynamics of Semiconductor Quantum Dot Lasers Subject to Delayed Optical Feedback

Influence of Sampling Length and Sampling Interval on Calculating the Fractal Dimension of Chaotic Attractors

Effects of Quantization, Delay and Internal Resistances in Digitally ZAD-controlled Buck Converter

Characterization of Chaotic Attractors inside Band-merging Scenario in a ZAD-controlled Buck Converter

A-Standard Finite Difference Scheme for Two-Sided Space-Fractional Partial Differential Equations

Chaotic Detector for BPSK Signals in Very Low SNR Conditions

An Excitable Electronic Circuit as a Sensory Neuron Model

The Incremental Ratio Based Causal Fractional Calculus

Finite Difference Methods for Fractional Differential Equations

Existence and Continuation Theories of Riemann-Liouville Type Fractional Differential Equations

Quantum Vortices and Trajectories in Particle Diffraction

On Impact Scripts with both Fractional and Dry Friction Type of Dissipation

How to Approximate the Fractional Derivative of Order 1 < alpha < 2

Multi-resonance and Enhanced Synchronization in Stochastically Chaotic Ratchets

On the Dynamics of Eleven Mammals

Chains of Rotational Tori and Filamentary Structures Close to High Multiplicity Periodic Orbits in a 3D Galactic Potential