On fractional impulsive equations of Sobolev type with nonlocal condition in Banach spaces

K. Balachandran, S. Kiruthika, J.J. Trujillo

Department of Mathematics, Bharathiar University, Coimbatore-641 046, India
Departamento de Análisis Matemático, Universidad de La Laguna, 38271 La Laguna, Tenerife, Spain

ARTICLE INFO

Keywords:
Fractional differential equations
Sobolev type
Nonlocal condition
Impulsive conditions
Fixed point theorems

ABSTRACT

The objective of this paper is to establish the existence of solutions of nonlinear impulsive fractional integrodifferential equations of Sobolev type with nonlocal condition. The results are obtained by using fractional calculus and fixed point techniques.

© 2011 Elsevier Ltd. All rights reserved.

1. Introduction

In recent years, fractional calculus has attracted many physicists, mathematicians and engineers and notable contributions have been made to both theory and applications of fractional differential equations. In fact fractional differential equations are considered as an alternative model to nonlinear differential equations [1]. Fractional differential equations draw a great application in nonlinear oscillations of earthquakes [2], many physical phenomena such as seepage flow in porous media and in fluid dynamic traffic models. Fractional derivatives can eliminate the deficiency of continuum traffic flow.

The most important advantage of using fractional differential equations in these and other applications [3] is their nonlocal property. It is well known that the integer order differential operator is a local operator but the fractional order differential operator is non-local. This means that the next state of a system depends not only upon its current state but also upon all of its historical states. This is probably the most relevant feature for making this fractional tool useful from an applied standpoint and interesting from a mathematical standpoint and in turn led to the sustained study of the theory of fractional differential equations [4]. The existence of solutions of abstract differential equations is investigated in [5] whereas the existence of solutions of fractional differential equations by using fixed point techniques have been discussed by several authors [6–10].

Brill [11] and Showalter [12] investigated the existence problem for semilinear Sobolev type equations in Banach spaces. The Sobolev type semilinear integrodifferential equation serves as an abstract formulation of partial integrodifferential equation which arise in various applications such as in the flow of fluid through fissured rocks [13], thermodynamics and shear in second order fluids and so on. Balachandran et al. [14] established the existence of solutions for Sobolev type semilinear integrodifferential equation whereas Balachandran and Uchiyama [15] studied the existence of solutions of nonlinear integrodifferential equations of Sobolev type in Banach spaces. The problem of existence of solutions of evolution equations with nonlocal condition was initiated by Byszewski [16] and subsequently studied by several authors for different kinds of problems [17,18]. On the other hand, the study of impulsive differential equations has attracted a great deal of attention in fractional dynamics and its theory has been treated in several works [18–21]. The differential equations involving impulsive effects appear as a natural description of observed evolution phenomena introduction of the basic
2. Preliminaries

We need some basic definitions and properties of fractional calculus which are used in this paper. Let $X$ and $Y$ be Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|_{Y}$ respectively and $\mathbb{R}_{+} = [0, \infty)$. Suppose $f \in L_1(\mathbb{R}_{+})$. Let $C(J, X)$ be the Banach space of continuous functions $x(t)$ with $x(t) \in X$ for $t \in J = [0, a]$ and $\|x\|_{C(J, X)} = \max_{t \in J} \|x(t)\|$. Also consider the Banach space

$$P C(J, X) = \{u : f \to X : u \in C((t_k, t_{k+1}], X), k = 0, \ldots, m \text{ and there exist} \ u(t_k^+) \text{ and } u(t_k^-), \ k = 1, \ldots, m \text{ with } \ u(t_k^+) = u(t_k^-)\},$$

with the norm $\|u\|_{PC} = \sup_{t \in J} \|u(t)\|$. Set $J' := [0, a] \setminus \{t_1, \ldots, t_m\}$.

**Definition 2.1.** The Riemann–Liouville fractional integral operator of order $\alpha > 0$, of function $f \in L_1(\mathbb{R}_{+})$ is defined as

$$I_{0+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} f(s) \, ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

**Definition 2.2.** The Riemann–Liouville fractional derivative of order $\alpha > 0$, $n - 1 < \alpha < n$, $n \in \mathbb{N}$, is defined as

$$(\mathcal{D}^n I_{0+} f)(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \left( \int_{0}^{t} (t-s)^{n-\alpha-1} f(s) \, ds \right),$$

where the function $f(t)$ has absolutely continuous derivatives up to order $(n - 1)$.

The Riemann–Liouville fractional derivatives have singularity at zero and the fractional differential equations in the Riemann–Liouville sense require initial conditions of special form lacking physical interpretation [4], but Caputo defined the fractional derivative in the following way, overcome such specific initial conditions.

**Definition 2.3.** The Caputo fractional derivative of order $\alpha > 0$, $n - 1 < \alpha < n$, is defined as

$$C D_{0+}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} (t-s)^{n-\alpha-1} f^n(s) \, ds,$$

where the function $f(t)$ has absolutely continuous derivatives up to order $(n - 1)$. If $0 < \alpha < 1$, then

$$C D_{0+}^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{0}^{t} f'(s) \frac{1}{(t-s)^\alpha} \, ds,$$

where $f'(s) = Df(s) = \frac{df(s)}{ds}$ and $f$ is an abstract function with values in $X$.

For basic facts about fractional integrals and fractional derivatives and in particular the properties of the operators $I_{0+}^\alpha$ and $C D_{0+}^\alpha$ one can refer to the books [23–27].

Consider the following nonlinear fractional impulsive integrodifferential equation of Sobolev type of the form

$$C D_{0+}^q (Bu)(t) + Au(t) = f(t, u(t)) + \int_{0}^{t} h(t, s, u(s)) \, ds, \quad t \in J = [0, a], \ t \neq t_k$$

$$\Delta u|_{t=t_k} = I_k(u(t_k^-)), \quad u(0) = u_0,$$

where $0 < q < 1, A$ and $B$ are a linear operator with domains contained in a Banach space $X$ and ranges contained in a Banach space $Y$ and the operators $A : D(A) \subset X \to Y$ and $B : D(B) \subset X \to Y$ satisfy the following hypotheses:

(H1) $A$ and $B$ are closed linear operators,
(H2) $D(B) \subset D(A)$ and $B$ is bijective,
(H3) $B^{-1} : Y \to D(B)$ is compact,
(H4) $B^{-1} A : X \to D(B)$ is continuous.

The nonlinear operators $f : J \times X \to Y$ and $h : \Omega \times X \to Y$ are given abstract functions, $l_k : X \to Y$, $k = 1, 2, \ldots, m$ and $u_0 \in X, 0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = a$, $\Delta u|_{t=t_k} = u(t_k^-) - u(t_k^+), u(t_k^+) = \lim_{h \to 0^+} u(t_k + h)$ and $u(t_k^-) = \lim_{h \to 0^-} u(t_k + h)$ represent the right and left limits of $u(t)$ at $t = t_k$. Here $\Omega = \{(t, s) : 0 \leq s \leq t \leq a\}$. It is
easy to prove that the Eq. (2.1) is equivalent to the integral equation

\[
\frac{d}{dt}u(t) = \sum_{i=1}^{k} B^{-1}I_i(u(t^-)) + \int_{0}^{t} \left( f(s, u(s)) + \int_{0}^{t} h(s, \tau, u(\tau))d\tau \right) ds,
\]

where \( u(t) \) satisfies

\[
\begin{align*}
&\left\{ \begin{array}{l}
\frac{d}{dt}u(t) - \frac{1}{\Gamma(q)} \int_{0}^{t} (t - s)^{q-1}B^{-1}Au(s)ds \\
+ \frac{1}{\Gamma(q)} \int_{0}^{t} (t - s)^{q-1}B^{-1} \left( f(s, u(s))ds + \int_{0}^{t} h(s, \tau, u(\tau))d\tau \right) ds,
\end{array} \right. \\
&\quad \text{if } t \in [0, t_1],
\end{align*}
\]

\[
\begin{align*}
&\left\{ \begin{array}{l}
\frac{d}{dt}u(t) - \frac{1}{\Gamma(q)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-1}B^{-1}Au(s)ds - \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1}B^{-1}Au(s)ds \\
+ \frac{1}{\Gamma(q)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-1}B^{-1} \left( f(s, u(s))ds + \int_{0}^{t} h(s, \tau, u(\tau))d\tau \right) ds \\
+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1}B^{-1} \left( f(s, u(s))ds + \int_{0}^{t} h(s, \tau, u(\tau))d\tau \right) ds \\
+ \sum_{i=1}^{k} B^{-1}I_i(u(t^-))
\end{array} \right. \\
&\quad \text{if } t \in (t_k, t_{k+1}].
\end{align*}
\]

By a local solution of the abstract Cauchy problem (2.1), we mean an abstract function \( u \) such that the following conditions are satisfied:

(i) \( u \in PC(f, X) \) and \( u \in D(A) \) on \( J' \);

(ii) \( \frac{d}{dt}u \) exists and continuous on \( J' \), where \( 0 < q < 1 \);

(iii) \( u \) satisfies Eq. (2.1) on \( J' \) and satisfies the conditions \( \Delta u|_{t=t_k} = I_k(u(t^-)) \), \( u(0) = u_0 \in X \) or that it is equivalent \( u \) satisfying the integral Eq. (2.2).

We assume the following conditions to prove the existence of a solution of the Eq. (2.1):

(H5) The functions \( I_k : X \rightarrow Y \) are continuous and there exists a constant \( L > 0 \), such that

\[
\|I_k(u) - I_k(v)\|_Y \leq L\|u - v\|_X,
\]

for each \( u, v \in X \) and \( k = 1, 2, \ldots, m \).

(H6) \( f : J \times X \rightarrow Y \) is continuous and there exists a constant \( L_1 > 0 \), such that

\[
\|f(t, u) - f(t, v)\|_Y \leq L_1\|u - v\|_X,
\]

for all \( u, v \in X \).

(H7) \( h : \Omega \times X \rightarrow Y \) is continuous and there exists a constant \( L_2 > 0 \), such that

\[
\left\| \int_{0}^{t} [h(t, s, u) - h(t, s, v)]ds \right\|_Y \leq L_2\|u - v\|_X,
\]

for all \( u, v \in X \).

For brevity let us take \( \gamma = \frac{\gamma}{\Gamma(q+1)} \) and \( R = \|B^{-1}A\| \), \( R^* = \|B^{-1}\| \), \( N = \max_{t \in \Omega} \|f(t, 0)\| \), \( N* = \max_{t \in \Omega} \left( \|\int_{0}^{t} h(t, s, 0)ds\| \right) \).

3. Main results

**Theorem 3.1.** If the hypotheses (H1)–(H7) are satisfied and if \( \gamma (m+1)(R + R^*(L_1+L_2)) + mR^*L \leq \frac{1}{2} \), then the problem (2.1) has a unique solution continuous on \( J \).

**Proof.** Let \( Z = PC(f, X) \). Define the mapping \( \Phi : Z \rightarrow Z \) by

\[
\Phi(u)(t) = u_0 - \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1}B^{-1}Au(s)ds - \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1}B^{-1}Au(s)ds
\]

\[
+ \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1}B^{-1} \left( f(s, u(s)) + \int_{0}^{t} h(s, \tau, u(\tau))d\tau \right) ds
\]

\[
+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1}B^{-1} \left( f(s, u(s)) + \int_{0}^{t} h(s, \tau, u(\tau))d\tau \right) ds + \sum_{0 < t_k < t} B^{-1}I_k(u(t^-))
\]

and we have to show that \( \Phi \) has a fixed point. This fixed point is then a solution of the Eq. (2.1). Choose \( r \geq 2(\|u_0\| + \gamma (m+1)R^*(N + N^*)) \). Then we can show that \( \Phi B_r \subset B_r \), where \( B_r := \{u \in Z : \|u\| \leq r\} \). From the assumptions we have

\[
\left\| \Phi(u)(t) \right\| \leq \|u_0\| + \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1}\|B^{-1}A\| \|u(s)\|ds + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1}\|B^{-1}A\| \|u(s)\|ds
\]
\[ + \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \| B^{-1} \| \left( \| f(s, u(s)) \| + \left\| \int_0^s h(s, \tau, u(\tau)) \, d\tau \right\| \right) \, ds \]
\[ + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} \| B^{-1} \| \left( \| f(s, u(s)) \| + \left\| \int_0^s h(s, \tau, u(\tau)) \, d\tau \right\| \right) \, ds + \sum_{0 < t_k < t} \| B^{-1} \| \| I_k(u(t^-_k)) \| \]
\[ \leq \| u_0 \| + \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \| B^{-1} A \| \| u(s) \| ds + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} \| B^{-1} A \| \| u(s) \| ds \]
\[ + \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \| B^{-1} \| \left( \| f(s, u(s)) - f(s, 0) \| + \| f(s, 0) \| \right) ds \]
\[ + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} \| B^{-1} \| \left( \| f(s, u(s)) - f(s, 0) \| + \| f(s, 0) \| \right) ds \]
\[ + \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \| B^{-1} \| \left( \left\| \int_0^s [h(s, \tau, u(\tau)) - h(s, \tau, 0)] \, d\tau \right\| + \left\| \int_0^s h(s, \tau, 0) \, d\tau \right\| \right) ds \]
\[ + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} \| B^{-1} \| \left( \left\| \int_0^s [h(s, \tau, u(\tau)) - h(s, \tau, 0)] \, d\tau \right\| + \left\| \int_0^s h(s, \tau, 0) \, d\tau \right\| \right) ds \]
\[ + \sum_{0 < t_k < t} \| B^{-1} \| \| I_k(u(t^-_k)) \| \]
\[ \leq \| u_0 \| + \frac{d^q}{\Gamma(q + 1)} \left( (m + 1) r (R + R^* (L_1 + L_2)) + (m + 1) R^*(N + N^*) \right) + mr L \]
\[ \leq \| u_0 \| + r \left( \gamma (m + 1) (R + R^* (L_1 + L_2)) + mr L \right) + \gamma (m + 1) R^*(N + N^*) \]
\[ \leq r. \]

Thus, \( \Phi \) maps \( B_r \) into itself. Now, for \( u, v \in Z \), we have
\[ \| \Phi u(t) - \Phi v(t) \| \leq \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \| B^{-1} \| \| u(s) - v(s) \| ds \]
\[ + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} \| B^{-1} A \| \| u(s) - v(s) \| ds \]
\[ + \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \| B^{-1} \| \| f(s, u(s)) - f(s, v(s)) \| ds \]
\[ + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} \| B^{-1} \| \| f(s, u(s)) - f(s, v(s)) \| ds \]
\[ + \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \| B^{-1} \| \left\| \int_0^s [h(s, \tau, u(\tau)) - h(s, \tau, v(\tau))] \, d\tau \right\| ds \]
\[ + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} \| B^{-1} \| \left\| \int_0^s [h(s, \tau, u(\tau)) - h(s, \tau, v(\tau))] \, d\tau \right\| ds \]
\[ + \sum_{0 < t_k < t} \| B^{-1} \| \| I_k(u(t^-_k)) - I_k(v(t^-_k)) \| \]
\[ \leq \left( \frac{d^q}{\Gamma(q + 1)} (m + 1) (R + R^* (L_1 + L_2)) + mr L \right) \| u - v \|
\[ \leq \left( \gamma (m + 1) (R + R^* (L_1 + L_2)) + mr L \right) \| u - v \|.

Hence \( \Phi \) is a contraction mapping and therefore there exists a unique fixed point \( u \in B_r \) such that \( \Phi u(t) = u(t) \). Any fixed point of \( \Phi \) is a solution of Eq. (2.1). \( \square \)
Now we discuss the existence of solution of the fractional impulsive Sobolev type Eq. (2.1) with nonlocal condition of the form
\[ u(0) + g(u) = u_0 \]  \hspace{1cm} (3.2)
where \( g: \mathcal{P}C(J, X) \rightarrow X \) is a given function which satisfies the following condition.
(H8) \( g: \mathcal{P}C(J, X) \rightarrow X \) is continuous and there exists a constant \( G > 0 \), such that
\[ \|g(u) - g(v)\| \leq G\|u - v\|_{\mathcal{P}C} \quad \text{for} \quad u, v \in \mathcal{P}C(J, X). \]

**Theorem 3.2.** If the hypotheses (H1)–(H8) are satisfied and if \( \gamma (m + 1) (R^r + R^s (L_1 + L_2)) + mR^s L + G \leq \frac{1}{2} \) then the problem (2.1) with nonlocal condition (3.2) has a unique solution continuous on \( J \).

**Proof.** We want to prove that the operator defined by \( \Psi: Z \rightarrow Z \) by
\[
\Psi u(t) = u_0 - g(u) - \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_k}^{t} (t - s)^{q-1}B^{-1}Au(s)ds - \frac{1}{\Gamma(q)} \int_{t}^{t} (t - s)^{q-1}B^{-1}Au(s)ds + \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_k}^{t} (t_k - s)^{q-1}B^{-1}\left(f(s, u(s)) + \int_{0}^{s} h(s, \tau, u(\tau))d\tau\right)ds + \sum_{0 < t_k < t} B^{-1}I_k(u(t_k^-)) \hspace{1cm} (3.3)
\]
has a fixed point. This fixed point is then a solution of the Eqs. (2.1) and (3.2). Choose \( r \geq 2(\|u_0\| + \|g_0\| + \gamma (m + 1)R^s (N + N^*)) \). Then we can easily show that \( \Psi B_r \subset B_r \).
\[
\|\Psi u(t) - \Psi v(t)\| \leq \frac{\gamma (m + 1) (R^r + R^s (L_1 + L_2)) + G + mR^s L}{2} \|u - v\|
\]

The result follows by the application of the contraction mapping principle. \( \square \)

**Krasnoselskii Theorem [28].** Let \( \delta \) be a closed convex nonempty subset of a Banach space \( X \). Let \( \mathcal{P}, \mathcal{Q} \) be two operators such that
(i) \( \mathcal{P}x + \mathcal{Q}y \in \delta \) whenever \( x, y \in \delta \);
(ii) \( \mathcal{P} \) is a contraction mapping;
(iii) \( \mathcal{Q} \) is compact and continuous.

Then there exists \( z \in \delta \) such that \( z = \mathcal{P}z + \mathcal{Q}z \).

Now, we assume the following conditions instead of (Hf) and apply the above fixed point theorem.

(H9) \( f: J \times X \rightarrow Y \) is continuous and there exists a continuous function \( \mu \in L^1(J) \) such that \( \|f(t, u)\| \leq \mu(t) \), for all \( (t, u) \in J \times X \).

(H10) \( h: J \times X \rightarrow Y \) is continuous and there exists a continuous function \( \mu^* \in L^1(J) \) such that \( \int_{0}^{t} h(t, s, u)ds \leq \mu^*(t) \), for all \( (t, s) \in \Omega, u \in X \).

**Theorem 3.3.** Assume that (H1)–(H4), (H6)–(H9) hold. If \( G + \gamma (m + 1)R^r + mR^s L < 1 \), then the fractional evolution (2.1) with nonlocal condition (3.2) has a solution on \( J \).

**Proof.** Choose \( r \geq \frac{\|u_0\| + \|g_0\| + \gamma (m + 1)\mu_0 + 1}{1 - (G + \gamma R^r (m + 1)R^s L)\mu_1} \) where \( \mu_0 = \sup_{t \in J} \mu(t) \), \( \mu_1 = \sup_{t \in J} \mu^*(t) \) and define the operators \( \mathcal{P} \) and \( \mathcal{Q} \) on \( B_r \) as
\[
\mathcal{P}u(t) = u_0 - g(u) - \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_k}^{t} (t_k - s)^{q-1}B^{-1}Au(s)ds - \frac{1}{\Gamma(q)} \int_{t}^{t} (t - s)^{q-1}B^{-1}Au(s)ds + \sum_{0 < t_k < t} B^{-1}I_k(u(t_k^-)) \hspace{1cm} \text{and}
\]
\[ Q u(t) = \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_k}^{t_k} (t_k - s)^{q-1} B^{-1} \left( f(s, u(s)) + \int_{0}^{s} h(s, \tau, u(\tau)) d\tau \right) ds + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} B^{-1} \left( f(s, u(s)) + \int_{0}^{s} h(s, \tau, u(\tau)) d\tau \right) ds. \]

For any \( u, v \in B_r \), we have

\[ \| P u(t) + Q v(t) \| \leq \| u_0 \| + \| g(u) - g(0) \| + \| g(0) \| + \frac{1}{\Gamma(q)} \sum_{0 < t_k < r} \int_{t_k}^{t_k} (t_k - s)^{q-1} \| B^{-1} \| u(s) \| ds + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} \| B^{-1} \| u(s) \| ds + \frac{1}{\Gamma(q)} \sum_{0 < t_k < r} \int_{t_k}^{t_k} (t_k - s)^{q-1} \| B^{-1} \| \left( \| f(s, u(s)) \| + \| f(s, u(s)) \| + \| f(s, u(s)) \| \right) ds + \sum_{0 < t_k < r} \| B^{-1} \| \| I_k(u(t_k)) \| \leq \| u_0 \| + \| g(0) \| + \gamma R^*(m + 1)(\mu_0 + \mu_1) + r \left( G + R^* (\gamma (m + 1)) R + mL \right) \leq r. \]

Hence, we deduce that \( \| P u + Q v \| \leq r \).

Next, for any \( t \in J, u, v \in X \) we have

\[ \| P u(t) - P v(t) \| \leq \| g(u) - g(v) \| + \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_k}^{t_k} (t_k - s)^{q-1} \| B^{-1} \| u(s) - v(s) \| ds + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} \| B^{-1} \| u(s) - v(s) \| ds + \sum_{0 < t_k < t} \| B^{-1} \| \| I_k(u(t_k)) - I_k(v(t_k)) \| \leq G \| u - v \| + \gamma (m + 1) R^* \| u - v \| + m R^* \| u - v \| \leq \left( G + R^* (\gamma (m + 1) R + mL) \right) \| u - v \|. \]

And since \( G + R^* (\gamma (m + 1) R + mL) < 1 \), then \( P \) is a contraction mapping.

Now, let us prove that \( Q \) is continuous and compact.

Let \( \{ u_n \} \) be a sequence in \( B_r \), such that \( u_n \to u \) in \( B_r \). Then

\[ f(s, u_n(s)) \to f(s, u(s)), \quad h(s, \tau, u_n(\tau)) \to h(s, \tau, u(\tau)) \quad n \to \infty \]

because the function \( f \) is continuous on \( J \times X \) and \( h \) is continuous on \( \Omega \times X \). Now, for each \( t \in J \), we have

\[ \| Q u_n(t) - Q u(t) \| \leq \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_k}^{t_k} (t_k - s)^{q-1} \| f(s, u_n(s)) - f(s, u(s)) \| ds + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} \| f(s, u_n(s)) - f(s, u(s)) \| ds + \frac{1}{\Gamma(q)} \sum_{0 < t_k < t} \int_{t_k}^{t_k} (t_k - s)^{q-1} \| B^{-1} \| \int_{0}^{s} [h(s, \tau, u_n(\tau)) - h(s, \tau, u(\tau))] d\tau ds + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} \| B^{-1} \| \int_{0}^{s} [h(s, \tau, u_n(\tau)) - h(s, \tau, u(\tau))] d\tau ds \to 0 \quad \text{as} \ n \to \infty. \]

Consequently, \( \lim_{n \to \infty} \| Q u_n(t) - Q u(t) \| = 0 \). In other words, \( Q \) is continuous.
Let us now note that $Q$ is uniformly bounded on $B_r$. This follows from the inequality

$$
\|Q(u(t))\| \leq \frac{1}{\Gamma(q)} \sum_{0 < \eta < \xi < \cdots < \xi_{r-1}} \int_{\xi_{r-1}}^{\xi} \left( (t_k - s)^{q-1} \left\| B^{-1} \right\| \left( \|f(s, u(s))\| + \left\| \int_0^s h(s, \tau, u(\tau))d\tau \right\| \right) ds + \frac{1}{\Gamma(q)} \int_t^{\xi} (t - s)^{q-1} \left\| B^{-1} \right\| \left( \|f(s, u(s))\| + \left\| \int_0^s h(s, \tau, u(\tau))d\tau \right\| \right) ds \leq \gamma R^*(m + \eta_0 + \mu_1).
$$

We first prove that \{Q(u(t)) : u \in B_r\} is relatively compact in $X$, for all $t \in J$ (see [29]). Obviously, \{Q(u(0)) : u \in B_r\} is compact. Fix $t \in (0, a)$ and for each $\epsilon \in (0, t)$, and $u \in B_r$, define the operator $Q^\epsilon$ by

$$
Q^\epsilon u(t) = \frac{1}{\Gamma(q)} \sum_{0 < \eta < \xi < \cdots < \xi_{r-1}} \int_{\xi_{r-1}}^{\xi} \left( (t_k - s)^{q-1} \left\| B^{-1} \right\| \left( \|f(s, u(s))\| + \left\| \int_0^s h(s, \tau, u(\tau))d\tau \right\| \right) ds + \frac{1}{\Gamma(q)} \int_t^{\epsilon} (t - s)^{q-1} \left\| B^{-1} \right\| \left( \|f(s, u(s))\| + \left\| \int_0^s h(s, \tau, u(\tau))d\tau \right\| \right) ds.
$$

Since the operator $B^{-1}$ is compact in $X$ for $t > 0$, then the sets \{Q^\epsilon u(t) : u \in B_r\} are relatively compact in $X$ for all $t \in (0, a]$ and since it is compact at $t = 0$ we have relative compactness in $X$ for all $t \in J$. Moreover, by using (H9) and (H10)

$$
\|Q(u(t) - Q^\epsilon u(t))\| \leq \frac{1}{\Gamma(q)} \sum_{0 < \eta < \xi < \cdots < \xi_{r-1}} \int_{\xi_{r-1}}^{\xi} \left( (t_k - s)^{q-1} \left\| B^{-1} \right\| \left( \|f(s, u(s))\| + \left\| \int_0^s h(s, \tau, u(\tau))d\tau \right\| \right) ds + \frac{1}{\Gamma(q)} \int_t^{\epsilon} (t - s)^{q-1} \left\| B^{-1} \right\| \left( \|f(s, u(s))\| + \left\| \int_0^s h(s, \tau, u(\tau))d\tau \right\| \right) ds \leq R^*(m + 1)(\mu_0 + \mu_1)\epsilon^q.
$$

From this we deduce that \{Q(u(t)) : u \in B_r\} is relatively compact in $X$ for all $t \in (0, a]$ and since it is compact at $t = 0$ we have the relative compactness in $X$ for all $t \in J$. Now, let us prove that $Q^\epsilon u, u \in B_r$ is equicontinuous. The functions $Q^\epsilon u(t), u \in B_r$, are equicontinuous at $t = 0$. Let $u \in B_r, 0 < t_1 < t_2 \leq a$ we have

$$
\|Q^\epsilon u(t_2) - Q^\epsilon u(t_1)\| \leq \frac{1}{\Gamma(q)} \int_0^{t_1} \left( (t_2 - s)^{q-1} \left\| B^{-1} \right\| \left( \|f(s, u(s))\| + \left\| \int_0^s h(s, \tau, u(\tau))d\tau \right\| \right) ds + \frac{1}{\Gamma(q)} \int_0^{t_1} \left( (t_1 - \eta)^{q-1} \left\| B^{-1} \right\| \left( \|f(s, u(s))\| + \left\| \int_0^s h(s, \tau, u(\tau))d\tau \right\| \right) ds \leq \frac{1}{\Gamma(q)} \int_0^{t_1} \left( (t_2 - s)^{q-1} - (t_1 - s)^{q-1} \right) \left\| B^{-1} \right\| \left( \|f(s, u(s))\| + \left\| \int_0^s h(s, \tau, u(\tau))d\tau \right\| \right) ds + \frac{1}{\Gamma(q)} \int_0^{t_1} \left( (t_2 - s)^{q-1} - (t_1 - s)^{q-1} \right) \left\| B^{-1} \right\| \left( \|f(s, u(s))\| + \left\| \int_0^s h(s, \tau, u(\tau))d\tau \right\| \right) ds \leq R^*(\mu_0 + \mu_1)\epsilon^q.
$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero. Thus we have proved that $Q (B_r)$ is relatively compact for $t \in J$. By Arzelà-Ascoli’s theorem, $Q$ is compact. Hence by the Krasnoselskii theorem there exists a solution of the problem (2.1) with nonlocal condition (3.2). □

4. Examples

Consider the following nonlinear fractional impulsive integrodifferential equation of Sobolev type of the form

$$
\frac{d^q u(t)}{dt^q} + \frac{1}{30} u(t) = \frac{e^{-|u(t)|}}{(29 + e^t)(1 + |u(t)|)} + \int_0^t e^{-\frac{1}{3}(|u|)} ds, \quad t \in J,
$$

$$
\Delta u|_{t=\frac{1}{2}} = \frac{|u|\left(\frac{1}{2}\right)}{8 + |u|\left(\frac{1}{2}\right)},
$$

$$
 u(0) = u_0,
$$

where $0 < q \leq 1$. Take $J := [0, 1]$. 
Set
\[ B = I, \]
\[ A = \frac{1}{30}. \]
\[ f(t, u) = \frac{e^{-t}|u(t)|}{(29 + e^t)(1 + |u(t)|)}. \]
\[ \int_0^t h(t, s, u(s))ds = \int_0^t e^{-\frac{t}{6}}u(s)ds \quad t \in J, u \in X. \]

Let \( u, v \in X \) and \( t \in J \). Then we have
\[
\| f(t, u) - f(t, v) \| = \frac{e^{-t}}{(29 + e^t)} \frac{|u - v|}{(1 + u)} \leq \frac{1}{30}|u - v| \quad \text{and}
\]
\[
\| h(u) - h(v) \| = \frac{|u|}{8 + u} - \frac{|v|}{8 + u} = \frac{8|u - v|}{(8 + u)(8 + v)} \leq \frac{1}{8}|u - v|. \]

Hence the conditions (H1)–(H7) hold with \( L = \frac{1}{4}, L_1 = \frac{1}{30} \) and \( L_2 = \frac{1}{9}. \)

Choose \( m = 1 \). We shall check that condition
\[
\gamma (m + 1)(R + R^*(L_1 + L_2)) + mR^*L \leq \frac{1}{2},
\]
is satisfied. Indeed
\[
\gamma (m + 1)(R + R^*(L_1 + L_2)) + mR^*L \leq \frac{1}{2} \iff \Gamma(q + 1) > \frac{8}{9}, \quad (4.4)
\]
which is satisfied for some \( q \in (0, 1) \). Then by Theorem 3.1 the problems (4.1)–(4.3) has a unique solution on \([0,1]\) for the values of \( q \) satisfying (4.4).

Acknowledgements

The second author is thankful to the University Grants Commission, New Delhi for awarding the Basic Scientific Research Fellowship and the third author to MICINN of Spain (grant MTM2010-16499) and to the National Science Fund of Bulgaria (grant D-ID 02/25/2009).

References


