Adaptive Controller for Single-Link Flexible Manipulators Based on Algebraic Identification and Generalized Proportional Integral Control

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Abstract—In this paper, we propose a fast online closed-loop identification method combined with an output-feedback controller of the generalized proportional integral (GPI) type for the control of an uncertain flexible robotic arm with unknown mass at the tip, including a Coulomb friction term in the motor dynamics. A fast nonasymptotic algebraic identification method developed in continuous time is used to identify the unknown system parameter and update the designed certainty equivalence GPI controller. In order to verify this method, several informative simulations and experiments are shown.

Index Terms—Adaptive control, algebraic estimation, flexible robots, generalized proportional integral (GPI) control.

I. INTRODUCTION

Flexible arm manipulators span a wide range of applications: space robots, nuclear maintenance, microsurgery, collision control, contouring control, pattern recognition, and many others. Surveys of the literature dealing with applications and challenging problems related to flexible manipulators may be found in [1] and [2].

The system, described by partial differential equations (PDEs), is a distributed-parameter system of infinite dimensions. Its nonminimum phase behavior makes it difficult to achieve high-level performance. Control techniques such as linear control [3], optimal control [4], adaptive control [5], sliding-mode control [6], neural networks [7], or fuzzy logic [8] deal with the control of flexible manipulators and the modeling based on a truncated (finite dimensional) model obtained from either the finite-element method (FEM) or assumed modes method. These methods require several sensors to be used in order to obtain an accurate trajectory tracking, and many of them also require the knowledge of all the system parameters to design properly the controller. We propose a new method, briefly explained in [9], to cancel the vibration of the flexible beam which gathers an online identification technique with a control scheme in a suited manner, with the only measures of both the motor angle obtained from an encoder and the coupling torque obtained from a pair of strain gauges as done in the work in [10]. In that paper, the nonlinearities effect in the motor dynamics, such as the Coulomb friction torque, required a compensation term as proposed in [11]. Robust control schemes minimized this effect [12]. However, this problem persists nowadays. Cicero et al. [13] used neural network to compensate this friction effect.

In this paper, we propose a generalized proportional integral (GPI) output-feedback control scheme which is found to be robust with respect to the effects of the unknown friction torque. Thus, friction models are not needed for its compensation. That controller was first uptaken by Marquez et al. [14] to control the velocity of a dc motor but that was internally unstable although the closed-loop system was asymptotically stable. We propose, by further manipulation of the integral reconstructor, an internally stable control scheme of the form of a classical compensator for an angular-position trajectory task of an uncertain flexible robot, which is found to be robust with respect to nonlinearities in the motor. The control scheme is truly an output-feedback controller; in addition, velocity measurements, which always introduce errors and noise, are not required. Nevertheless, the GPI control requires knowing some features inherent to the bar. Furthermore, the control robustness to payload changes depends on these unknowns. For this reason, to incorporate an identification technique, which allows us to estimate these parameter uncertainties, is necessary.

The objective of this paper is to develop a GPI controller combined with a fast online closed-loop identification of the unknown parameters of a flexible bar.

The authors are based on the fast-identification techniques that were recently proposed by Fliss et al. [15] (see also [16]) for the state and constant parameters estimation in a fast and reliable way in feedback-control systems (see also [17], [18]). Let us recall that those techniques are not asymptotic and do not need any statistical knowledge of the noise corrupting the data. In other words, to assume that the noise is Gaussian is not required. This assumption is common in other well-known methods like maximum likelihood or minimum least squares. Furthermore, this methodology has been successfully applied in [19] and [20] for signal processing.

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This paper is organized as follows. Section II describes the flexible-manipulator model. Section III is devoted to explain the GPI-controller design. In Section IV, the algebraic estimator mathematical development is explained. Section V describes the adaptive-control procedure. In Section VI, simulations of the adaptive-control system are shown and also the system behavior when high modes are not modeled and they affect the system dynamics. In Section VII, experimental results are presented. Section VIII is devoted to extend the application of the control method to a multilink flexible arm. Finally, Section IX is devoted to remark the main conclusions.

II. MODEL DESCRIPTION

A. Flexible-Beam Dynamics

The flexible slewing beam studied in this paper is considered to be a Euler–Bernoulli beam whose behavior is described by a PDE [21]. Its dynamics involves infinite vibration modes. Nonetheless, as the frequency of those modes increases, its amplitude decreases. This means that reduced models can be used, where only the low frequencies, usually more significant, are considered. In order to reduce the model, several approaches were proposed: 1) distributed parameters model where the infinite dimension is truncated to a finite number of vibration modes [3]; and 2) lumped parameters models where a spatial discretization leads to a finite-dimensional model. In this sense, the spatial discretization can be done by both a FEM [22] or a lumped-mass model [23].

A single-link flexible manipulator with tip mass is modeled, as developed in [23], that can rotate about the Z-axis perpendicular to the paper, as shown in Fig. 1. The axial deformation and the gravitational effect are neglected, because the mass of the flexible beam is floating over an air table which allows us to cancel the gravitational effect and the friction with the surface of the table. Since structural damping always increases the stability margin of the system, a design without considering damping may provide a valid but conservative result [24].

The real structure studied in this paper is made of carbon fiber, with high mechanical resistance and very small density. We study it under the hypothesis of small deformations with all its mass concentrated at the tip position because the mass of the load is bigger than that of the bar, then the mass of the beam can be neglected. In other words, the flexible beam vibrates with the fundamental mode; therefore, the rest of the modes are very far from the first one and they can be neglected. Thus, we only consider one mode of vibration. The main characteristic of this model is that the influence of the load changes can be modeled in a very easy manner, thus adaptive controller can be easily applied.

Based on these considerations, we propose the following model for the flexible beam:

\[ mL^2 \ddot{\theta}_t = c(\theta_m - \theta_t) \]  

where \( m \) is the unknown mass at the tip position, \( L \) and \( c = (3EI/L) \) are the length of the flexible arm and the stiffness of the bar, respectively, assumed to be perfectly known. The stiffness depends on the flexural rigidity \( EI \) and on the length of the bar \( L \). \( \theta_m \) is the angular position of the motor gear. \( \dot{\theta}_t \) and \( \dot{\theta}_m \) are the unmeasured angular position and angular acceleration of the tip, respectively.

B. DC-Motor Dynamics

A common electromechanical actuator, in many control systems, is constituted by the dc motor [25]. The dc motor used is supplied by a servo amplifier with a current inner loop control. We can write the dynamic equation of the system by using Newton’s second law

\[ ku = J\ddot{\theta} + \nu\dot{\theta} + \Gamma_c(\dot{\theta}_m, u) + \frac{\Gamma}{n} \]  

where \( J \) is the inertia of the motor [in kilograms square meters], \( \nu \) is the viscous friction coefficient [in Newton meters seconds], and \( \Gamma_c \) is the unknown Coulomb friction torque which affects the motor dynamics [in Newton meters]. This nonlinear-friction term is considered as a perturbation, depending only on the sign of the motor angular velocity. As a consequence, Coulomb’s friction, when \( \dot{\theta}_m \neq 0 \), follows the model:

\[ \hat{\Gamma}_c \cdot \text{sign}(\dot{\theta}_m) = \begin{cases} \hat{\Gamma}_{\text{Coul}}(\dot{\theta}_m > 0) \\ -\hat{\Gamma}_{\text{Coul}}(\dot{\theta}_m < 0) \end{cases} \]  

and when \( \dot{\theta}_m = 0 \)

\[ \hat{\Gamma}_c \cdot \text{sign}(u) = \begin{cases} \min(ku, \Gamma_{\text{Coul}})(u > 0) \\ \max(ku, -\Gamma_{\text{Coul}})(u < 0) \end{cases} \]  

with \( \Gamma_{\text{Coul}} \) as the static friction value which the motor torque must exceed to begin the movement. The parameter \( k \) is the known electromechanical constant of the motor servo amplifier system [in Newton meters per volt]. \( \dot{\theta}_m \) and \( \dot{\theta}_t \) are the angular acceleration of the motor [in radians per seconds squared] and the angular velocity of the motor [in radians per second], respectively. \( \Gamma \) is the coupling torque measured in the hub [in Newton meters], and \( n \) is the reduction ratio of the motor gear. \( u \) is the motor input voltage [in volts]. This variable is the control

1We denote by \( n \) the reduction ratio of the motor gear; thus, \( \theta_m = \dot{\theta}_m/n \), where \( \dot{\theta}_m \) is motor-shaft position.
variable of the system. This is the input to a servo amplifier
which controls the input current to the motor by means of an
internally PI current controller [see Fig. 2(a)]. This electrical
dynamics can be rejected because this is faster than the
mechanical dynamics of the motor. Thus, the servo amplifier
can be considered as a constant relation $k_{me}$ between the voltage
and the current to the motor: $i_m = V k_{c}$ [see Fig. 2(b)], where
$i_m$ is the armature circuit current and $k_{c}$ includes the gain of
the amplifier $k$ and $R$ as the input resistance of the amplifier
circuit.

The total torque given to the motor $\Gamma_T$ is directly propor-
tional to the armature circuit in the form $\Gamma_T = k_m i_m$, where
$k_m$ is the electromechanical constant of the motor. Thus, the
electromechanical constant of the motor servo amplifier system is
$k = k_c k_m$.

C. Complete-System Dynamics

The dynamics of the complete system, actuated by a dc
motor, is given by the following simplified model:

$$m L^2 \ddot{\theta}_t = c(\theta_m - \theta_t)$$

(5)

$$k u = J \ddot{\theta}_m + \nu \dot{\theta}_m + \Gamma_c + \Gamma$$

(6)

$$\Gamma = c(\theta_m - \theta_t).$$

(7)

Equation (5) represents the dynamics of the flexible beam;
(6) expresses the dynamics of the dc motor; and (7) stands for
the coupling torque measured in the hub and produced by the
translation of the flexible beam, which is directly proportional
to the stiffness of the beam and the difference between the
angles of the motor and the tip position, respectively.

III. GPI CONTROLLER

In Laplace transforms notation, the flexible-bar transfer func-
tion, obtained from (5), can be written as follows:

$$G b(s) = \frac{\theta_t(s)}{\theta_m(s)} = \frac{\omega^2}{s^2 + \omega^2}$$

(8)

where $\omega = (c/(m L^2))^{1/2}$ is the unknown natural frequency of
the bar due to the lack of precise knowledge of $m$. As done in
[10], the coupling torque can be canceled in the motor by means

![Fig. 2. (a) Complete amplifier scheme. (b) Equivalent amplifier scheme.](image)

![Fig. 3. Compensation of the coupling torque measured in the hub.](image)

of a compensation term. In this case, the voltage applied to the
motor is of the form

$$u = u_c + \frac{\Gamma}{k \cdot n}$$

(9)

where $u_c$ is the voltage applied before the compensation term.
The system in (6) is then given by

$$k u_c = J \ddot{\theta}_m + \nu \dot{\theta}_m + \Gamma_c$$

(10)

The controller to be designed will be robust with respect to the
unknown piecewise constant torque disturbances affecting the
motor dynamics $\Gamma_c$. Then, the perturbation-free system to be
considered is the following:

$$K u_c = J \ddot{\theta}_m + \nu \dot{\theta}_m$$

(11)

where $K = k/n$. To simplify the developments, let $A = K/J$ and
$B = \nu/J$. The dc-motor transfer function is then written as

$$G m(s) = \frac{\theta_m(s)}{u_c(s)} = \frac{A}{s(s + B)}.$$  

(12)

Fig. 3 shows the compensation scheme of the coupling torque
measured in the hub.

The regulation of the load position $\theta_L(t)$ to track a given
smooth reference trajectory $\theta^*_L(t)$ is desired. For the synthesis
of the feedback-control law, we are using only the measured
motor position $\theta_m$ and the measured coupling torque $\Gamma$.
One of the prevailing restrictions throughout our treatment of the
problem is our desire of not to measure, or compute on the basis
samplings, angular velocities of the motor shaft or of the load.

A. Outer Loop Controller

Consider the model of the flexible link, given in (5), and
suppose for a moment that we know the value of the unknown
parameter $\omega$. This subsystem is flat, with flat output given by $\theta_t$
(see [26]). The parameterization of $\theta_m$ in terms of $\theta_t$ is given,
in reduction-gear terms, by

$$\theta_m = \frac{m L^2}{c} \dot{\theta}_t + \frac{1}{\omega_0^2} \ddot{\theta}_t + \dot{\theta}_t.$$  

(13)

System (13) is a second-order system in which to regulate
the tip position of the flexible bar $\theta_t$ toward a given smooth
reference trajectory, $\theta^*_L(t)$ is desired, with $\theta_m$ acting as an aux-
iliary control input. Clearly, if there exists an auxiliary open-
loop control input $\theta^*_m(t)$ that ideally achieves the tracking of
controller could be proposed to be the following PID controller:

\[
\theta_m^*(t) = \frac{1}{\omega^2} \dot{\theta}_t^*(t) + \theta_t^*(t).
\]  

(14)

Subtracting (14) from (13), an expression in terms of the angular tracking error is obtained

\[
\dot{e}_{\theta_t} = \omega^2(e_{\theta_m} - e_{\theta_t}).
\]  

(15)

where \( e_{\theta_m} = \theta_m - \theta_m^*(t) \), \( e_{\theta_t} = \theta_t - \theta_t^*(t) \). Suppose, for a moment, that we are able to measure the angular-position tracking error \( e_{\theta_t} \), then the outer loop feedback incremental controller could be proposed to be the following PID controller:

\[
e_{\theta_m} = e_{\theta_t} + \frac{1}{\omega^2} \left[ -k_2 \dot{e}_{\theta_t} - k_1 e_{\theta_t} - k_0 \int_0^t e_{\theta_t}(\sigma) d\sigma \right].
\]  

(16)

In such a case, the closed-loop tracking error \( e_{\theta_t} \) evolves, governed by

\[
e_{\theta_t}^{(3)} + k_2 \dot{e}_{\theta_t} + k_1 e_{\theta_t} + k_0 e_{\theta_t} = 0.
\]  

(17)

The design parameters \( \{k_2, k_1, k_0\} \) are then chosen so as to render the closed-loop characteristic polynomial into a Hurwitz polynomial with desirable roots. However, in order to avoid tracking-error velocity measurements, we propose to obtain an integral reconstructor for the angular-velocity error signal \( \dot{e}_{\theta_t} \). We proceed by integrating (15) once and, later, by disregarding the constant error due to the tracking-error velocity initial conditions. The estimated error velocity \( \dot{e}_{\theta_t} \) can be computed in the following form:

\[
\dot{e}_{\theta_t}(t) = \dot{e}_{\theta_t}(0) + \omega^2 \int_0^t (e_{\theta_m}(\sigma) - e_{\theta_t}(\sigma)) d\sigma.
\]  

(18)

The integral reconstructor neglects the possibly nonzero initial condition \( \dot{e}_{\theta_t}(0) \), and hence, it exhibits a constant estimation error. When the reconstructor is used in the derivative part of the PID controller, the constant error is suitably compensated, owing to the integral control action of the PID controller. Substituting the integral reconstructor \( \dot{e}_{\theta_t} \) (18) by \( \dot{e}_{\theta_t} \) into the PID controller (16) and, after some rearrangements, we obtain

\[
(\theta_m - \theta_m^*) = \left[ \frac{\gamma_1 s + \gamma_0}{s + \gamma_2} \right] (\theta_t^* - \theta_t).
\]  

(19)

The tip angular position cannot be measured, but it certainly can be computed from the expression relating the tip position with the motor position and the coupling torque \( \Gamma \):

\[
\Gamma = c(\theta_m - \theta_t) = mL^2 \dot{\theta}_t.
\]  

(20)

Thus, the angular position \( \theta_t \) is readily expressed as

\[
\theta_t = \theta_m - \frac{1}{c} \Gamma.
\]  

(21)

Fig. 4 shows the feedback control scheme under which the outer loop controller would be actually implemented in practice. The outer loop system in Fig. 4 is exponentially stable. In order to specify the parameters \( \{\gamma_2, \gamma_1, \gamma_0\} \), we can choose to locate the closed-loop poles in the left half of the complex plane. All three poles can be located in the same point of the real line, \( s = -a \), being strictly positive, using the following polynomial equation:

\[
(s + a)^3 = s^3 + 3as^2 + 3a^2 s + a^3 = 0
\]  

(22)

where the parameter \( a \) represents the desired location of the poles. The characteristic equation of the closed-loop system is

\[
s^3 + \gamma_2 s^2 + \omega^2 (1 + \gamma_1) s + \omega^2 (\gamma_2 + \gamma_0) = 0.
\]  

(23)

Identifying each term of (22) with those of (23), the design parameters \( \{\gamma_2, \gamma_1, \gamma_0\} \) can be uniquely specified if \( \omega \) is known.

B. Inner Loop Controller

The angular position \( \theta_m \), generated as an auxiliary control input in the previous controller design step, is now regarded as a reference trajectory for the motor controller. We denote this reference trajectory by \( \theta_m^* \).

The dynamics of the dc motor, including the Coulomb friction term, is given by (10). The design of the controller to be robust with respect to this torque disturbance is desired.

The following feedback controller is proposed:

\[
e_t = \frac{\nu}{K} \dot{e}_{\theta_m} + \frac{J}{K} \left[ -k_3 \ddot{e}_{\theta_m} - k_2 e_{\theta_m} - k_1 \int_0^t e_{\theta_m}(\sigma) d\sigma \right] - k_0 \int_0^t \int_0^\sigma (e_{\theta_m}(\sigma_1)) d\sigma_1 d\sigma.
\]  

(24)
The following integral reconstructor for the angular-velocity error signal $\dot{\theta}_m$ is obtained:

$$\dot{\theta}_m = K \int_0^t e_\nu(\sigma)d(\sigma) - \nu J \dot{\theta}_m.$$  

(25)

Replacing $\dot{\theta}_m$ in (25) into (24) and, after some rearrangements, the feedback control law is obtained

$$(u_c - u^*_c) = \left[ \frac{\alpha_2 s^2 + \alpha_1 s + \alpha_0}{s(s + \alpha_3)} \right] (\dot{\theta}_m - \theta_m).$$  

(26)

The open-loop control $u^*_c(t)$ that ideally achieves the open-loop tracking of the inner loop is given by

$$u^*_c(t) = \frac{1}{A} \dot{\theta}_m^*(t) + \frac{B}{A} \theta_m^*(t).$$  

(27)

The inner loop system in Fig. 4 is exponentially stable. We can choose to place all the closed-loop poles in a desired location of the left half of the complex plane to design the parameters $\{\alpha_3, \alpha_2, \alpha_1, \alpha_0\}$. As done with the outer loop, all poles can be located at the same real value, and $\alpha_3, \alpha_2, \alpha_1$, and $\alpha_0$ can be uniquely obtained by equalizing the terms of the two following polynomials:

$$\begin{aligned}
(s + p)^4 &= s^4 + 4ps^3 + 6p^2 s^2 + 4p^3 s + p^4 = 0 \\
s^4 &= (s + \alpha_3 + B)s^3 + (\alpha_3 + B + \alpha_2 A)s^2 + \alpha_1 As + \alpha_0 A = 0
\end{aligned}$$  

(28) (29)

where the parameter $p$ represents the common location of all the closed-loop poles, this being strictly positive.

IV. IDENTIFICATION

As explained in the previous section, the control performance depends on the knowledge of the parameter $\omega$. In order to do this task, in this section, we analyze the identification issue, as well as the reasons of choosing the algebraic derivative method as estimator.

Identification of continuous-time system parameters has been studied from different points of view. The surveys led by Young in [27] and Unbehauen and Rao in [28] and the books of Sinha and Rao and Unbehauen and Rao in [29] and [30], respectively, describe most of the available techniques.

The different approaches are usually classified into two categories: 1) Indirect approaches: An equivalent discrete-time model to fit the data is needed. After that, the estimated discrete-time parameters are transferred to continuous time. 2) Direct approaches: In the continuous-time model, the original continuous-time parameters from the discrete-time data are estimated via approximations for the signals and operators.

In the case of the indirect method, a classical well-known theory is developed (see [31]). Nevertheless, these approaches have several disadvantages: 1) They require computationally costly minimization algorithms without even guaranteeing convergence. 2) The estimated parameters may not be correlated with the physical properties of the system. 3) At fast sampling rates, poles and zeros cluster near the $-1$ point in the $z$-plane.

Therefore, many researchers are doing a big effort following direct approaches (see [32]–[35], among others).

Unfortunately, identification of robotic systems is generally focused on indirect approaches (see [36], [37]), and as a consequence, the references using direct approaches are scarce. On the other hand, the existing identification techniques, included in the direct approach, suffer from poor speed performance. Additionally, it is well known that the closed-loop identification is more complicated than its open-loop counterpart (see [31]). These reasons have motivated the application of the algebraic derivative technique previously presented in the introduction.

In the next point, algebraic manipulations will be shown to develop an estimator which stems from the differential equations, analyzed in the model description, incorporating the measured signals in a suitable manner.

A. Algebraic Estimation of the Natural Frequency

In order to make more understandable the equation deduction, we suppose that the signals are noise free. The main goal is to obtain an estimation of $\omega^2$ as fast as possible, which we will denote by $\hat{\omega}_c$.

Proposition 4.1: The constant parameter $\omega^2$ of the noise-free system described by (5)–(7) can be exactly computed, in a nonsymptotic fashion, at some arbitrarily small time $t = \Delta > 0$, by means of the expression

$$\omega^2_{0c} = \begin{cases} 
\text{arbitrary} & \text{for } t \in [0, \Delta) \\
\frac{n_c(t)}{d_c(t)} & \text{for } t \in [\Delta, +\infty)
\end{cases}$$  

(30)

where $n_c(t)$ and $d_c(t)$ are the output of the time-varying linear unstable filter

$$\begin{aligned}
n_c(t) &= t^2 \dot{\theta}_l(t) + z_1 & d_c(t) &= z_3 \\
\dot{z}_1 &= z_2 - 4t \dot{\theta}_l(t) & \dot{z}_3 &= z_4 \\
\dot{z}_2 &= 2\theta_l(t) & \dot{z}_4 &= t^2 (\theta_m(t) - \theta_l(t)).
\end{aligned}$$  

(31)

Proof: Consider (5)

$$\dot{\theta}_l = \omega^2 (\theta_m - \theta_l).$$  

(32)

The Laplace transform of (32) is

$$s^2 \theta_l(s) - s \theta_l(0) - \dot{\theta}_l(0) = \omega^2 (\theta_m(s) - \theta_l(s)).$$  

(33)

Taking two derivatives with respect to the complex variable $s$, the initial conditions are cancelled

$$\frac{d^2 (s^2 \theta_l)}{ds^2} = \omega^2 \left( \frac{d^2 (\theta_m)}{ds^2} - \frac{d^2 (\theta_l)}{ds^2} \right).$$  

(34)

Employing the chain rule, we obtain

$$s^2 \frac{d^2 \theta_l}{ds^2} + 4s \frac{d \theta_l}{ds} + 2\theta_l = \omega^2 \left( \frac{d^2 (\theta_m)}{ds^2} - \frac{d^2 (\theta_l)}{ds^2} \right).$$  

(35)

Consequently, in order to avoid multiplications by positive powers of $s$, which are translated as undesirable time derivatives
in the time domain, we multiply the earlier expression by \(s^{-2}\). After some rearrangements, we obtain

\[
\omega^2 = \frac{d^2(\theta_0)}{dx^2} + 4s^{-1} \frac{d\theta_0}{dx} + 2s^{-2} \theta_1.
\]

(36)

Let \(\mathcal{L}\) denote the usual operational calculus transform acting on exponentially bounded signals with bounded left support (see [38]). Recall that \(\mathcal{L}^{-1}s^{\gamma}(\cdot) = (d/dt)\gamma(\cdot), \mathcal{L}^{-1}(d^\gamma/dx^\gamma)(\cdot) = (-1)^{\gamma} t^\gamma e^{\gamma t}(\cdot), \) and \(\mathcal{L}^{-1}(1/\lambda)(\cdot) = \int_0^t (\cdot) e^{-\lambda t} dt\). Taking this into account, we can translate (36) into the time domain

\[
\omega^2 = \frac{[t^2\theta_1(t) - 4 \int_0^t \sigma \theta_1(\sigma) d\sigma + 2 \int_0^t \theta_1(\lambda) d\lambda d\sigma]}{\int_0^t \int_0^\lambda \lambda^2 \theta_2(\lambda) d\lambda d\sigma - \int_0^\lambda \theta_2(\lambda) d\lambda d\sigma}.
\]

(37)

The time realization of (37) can be written via time-variant linear (unstable) filters

\[
\begin{align*}
n_e(t) &= t^2\theta_1(t) + z_1, \quad d_e(t) = z_3 \\
\dot{z}_1 &= z_2 - 4t\theta_1(t) \quad \dot{z}_2 = z_4 \\
\dot{z}_2 &= 2t\theta_1(t) \quad \dot{z}_4 = t^2(\theta_m(t) - \theta_1(t)).
\end{align*}
\]

(38)

The natural frequency estimator of \(\omega^2\) is given by

\[
\omega^2_{0e} = \begin{cases} 
\text{arbitrary} & \text{for } t \in [0, \Delta] \\
n_e(t) & \text{for } t \in [\Delta, +\infty)
\end{cases}
\]

(39)

where \(\Delta\) is an arbitrary small real number. Note that, for the time \(t = 0\), \(n_e(t)\) and \(d_e(t)\) are both zero. Therefore, the quotient is undefined for a small period of time. After a time \(t = \Delta > 0\), the quotient is reliably computed. Note that \(t = \Delta\) depends on the arithmetic processor precision and on the data acquisition card.

The unstable nature of the linear systems in perturbed Brunovsky’s form (38) is of no practical consequence on the determination of the unknown parameters since we have the following reasons: 1) Resetting of the unstable time-varying systems and of the entire estimation scheme is always possible and, specially, needed when the unknown parameters are known to undergo sudden changes to adopt new constant values. 2) Once the parameter estimation is reliably accomplished, after some time instant \(t = \Delta > 0\), the whole estimation process may be safely turned off.

Note that we only need to measure \(\theta_m\) and \(\Gamma\), since \(\theta_1\) is available according to (21). Unfortunately, the available signals \(\theta_m\) and \(\Gamma\) are noisy. Thus, the estimation precision yielded by the estimator in (30) and (31) will depend on the signal-to-noise ratio (SNR).

B. Unstructured Noise

We assume that \(\theta_m\) and \(\Gamma\) are perturbed by an added noise with unknown statistical properties. In order to enhance the SNR, we simultaneously filter the numerator and denominator by the same low-pass filter. Taking advantage of the estimator rational form in (37), the quotient will not be affected by the filters. This invariance is emphasized with the use of the different notations in frequency and time domain such as

\[
\omega^2_{0e} = \frac{n_f(t)}{d_f(t)} = \frac{F(s)n_e(t)}{F(s)d_e(t)}
\]

(40)

where \(n_f(t)\) and \(d_f(t)\) are the filtered numerator and denominator, respectively, and \(F(s)\) is the filter used. The choice of this filter depends on the \textit{a priori} available knowledge of the system. Nevertheless, if such a knowledge does not exist, pure integrations of the form \(1/s^p, p \geq 1\) may be utilized, where high-frequency noise has been assumed. This hypothesis has been motivated by recent developments in \textit{nonstandard analysis} toward a new nonstochastic noise theory (more details in [39]).

Finally, the parameter \(\omega^2\) is obtained by

\[
\omega^2_{0e} = \begin{cases} 
\text{arbitrary} & t \in [0, \Delta] \\
n_e(t) & t \in [\Delta, +\infty).
\end{cases}
\]

(41)

V. ADAPTIVE-CONTROL PROCEDURE

Fig. 4 shows the adaptive-control system implemented in practice in our laboratory. The estimator is linked up, from time \(t_0 = 0\), to the signals coming from the encoder \(\theta_m\) and the pair of strain gauges \(\Gamma\). Thus, the estimator begins to estimate when the closed loop begins to work, and then, we can obtain immediately the estimate of the parameter. When the natural frequency of the system is estimated at time \(t_1\), the switch \(s_1\) is switched on, and the control system is updated with this new parameter estimate. This is done in closed loop and at real time in a very short period of time. The updating of the control system is carried out by substituting \(\omega\) by the estimated parameter \(\omega_{0e}\) in (14) and (19). In fact, the feedforward term which ideally controls in open loop the inner loop subsystem \(u^e_m\) also depends on the \(\omega\) value because the variable \(\theta_m^e\) is obtained from the knowledge of the system natural frequency. Obviously, until the estimator obtains the true value of the natural frequency, the control system begins to work with an initial arbitrary value which we select \(\omega_{0i}\). Taking these considerations into account, the adaptive controller can be defined as follows.

For the outer loop, (14) is computed as

\[
\theta_m^e(t) = \frac{1}{x^2} \theta_1^e(t) + \theta_1^e(t).
\]

(42)

Moreover, (23) is computed as

\[
s^3 + \gamma_2 s^2 + x^2(1 + \gamma_1)s + x(\gamma_2 + \gamma_0) = 0.
\]

(43)

For the inner loop only changes the feedforward term in (27) which depends on the bounded derivatives of the new \(\theta_m^e(t)\) in (42).

The variable \(x\) is defined as

\[
x = \omega_{0i}, t < t_1
\]

(44)

\[
x = \omega_{0e}, t \geq t_1.
\]

(45)
VI. SIMULATIONS

The major problems associated with the control of flexible structures arise from the structure is a distributed parameter system with many modes, and there are likely to be many actuators [40]–[44]. We propose to control a flexible beam whose second mode is far away from the first one, with the only actuator being the motor and the only sensors as follows: an encoder to measure the motor position and a pair of strain gauges to estimate the tip position. The problem is that the high modal densities give rise to the well-known phenomenon of spillover [45], where contributions from the unmodeled modes affect the control of the modes of interest. Nevertheless, with the simulations as follows, we demonstrate that the hypothesis proposed before is valid, and the spillover effect is negligible.

In the simulations, we consider a saturation in the motor input voltage in a range of [−10, 10] [in volts]. The parameters used in the simulations are as follows: inertia \( J = 6.87 \cdot 10^{-5} \) [kg \( \cdot \) m\(^2\)], viscous friction \( \nu = 1.041 \cdot 10^{-3} \) [N \( \cdot \) m \( \cdot \) s], and electromechanical constant \( k = 0.21 \) [(N \( \cdot \) m)/V]. With these parameters, \( A \) and \( B \) of the transfer function of the dc motor in (12) can be computed as follows: \( A = 61.14 \) [N/(V \( \cdot \) kg \( \cdot \) s)] and \( B = 15.15 \) [(N \( \cdot \) s)/(kg \( \cdot \) m)]. The mass used to simulate the flexible-beam behavior is \( m = 0.03 \) [kg], the length \( L = 0.5 \) [m], and the flexural rigidity is \( EI = 0.264 \) [N \( \cdot \) m\(^2\)]. According to these parameters, the stiffness is \( c = 1.584 \) [N \( \cdot \) m], and the natural frequency is \( \omega = 14.5 \) [rad/s]. Note that we consider that the stiffness of the flexible beam is perfectly known; thus, the real natural frequency of the bar will be estimated as well as the tip position of the flexible bar. Nevertheless, it may occur that the value of the stiffness varies from the real value, and an approximation is then included in the control scheme. Such approximated value is denoted by \( c_0 \). In this case, we consider that the computation of the stiffness fits in with the real value, i.e., \( c = c_0 \).

A meticulous stability analysis of the control system under variations of the stiffness \( c \) is carried out in Appendix, where a study of the error in the estimation of the natural frequency is also achieved.

The sample time used in the simulations is \( 1 \cdot 10^{-3} \) [s].

The value of \( \hat{\Gamma}_c \) is 119.7 \( \cdot \) 10\(^{-3} \) [N \( \cdot \) m] taken in simulations is the true value estimated in real experiments. In voltage terms is \( (\hat{\Gamma}_c/k) = 0.57 \) [V].

In order to design the gains of the inner loop controller, the poles can be located in a reasonable location of the negative real axis. If closed-loop poles are located in, for example, -95, the transfer function of the controller from (26), that depends on the location of the poles in closed loop of the inner loop and the values of the motor parameters \( A \) and \( B \) as shown in (28) and (29), respectively, results in the following expression:

\[
\frac{u_c - u^*_c}{\theta_m - \theta_m^*} = \frac{798a^2 + 5.6 \cdot 10^4s + 1.3 \cdot 10^6}{s(s + 365)}. \tag{46}
\]

The feedforward term in (27), which depends on the values of the motor parameters, is computed in accordance with

\[
u^*_c = 0.02\ddot{\theta}_m(t) + 0.25\ddot{\theta}_m(t) \tag{47}\]

with a natural frequency of the bar given by an initial arbitrary estimate of \( \omega_{01} = 9 \) [rad/s]. The transfer function of the controller (19), which depends on the location of the closed-loop poles of the outer loop, -10 in this case, and the natural frequency of the bar as shown in (22) and (23), respectively, is given by the following expression:

\[
\frac{\theta_m - \theta_m^*}{\theta_l - \theta_l^*} = \frac{2.7s - 17.7}{s + 30}. \tag{48}
\]

The open-loop reference control input from (14) in terms of the initial arbitrary estimate of \( \omega_{01} \) is given by

\[
\theta^*_m(t) = \frac{1}{\omega^2_{01}}(\ddot{\theta}_l^*(t) + \ddot{\theta}_l^*(t)) = 12.3 \cdot 10^{-3}\ddot{\theta}_l^*(t) + \theta_l^*(t). \tag{49}
\]

The desired reference trajectory used for the tracking problem of the flexible arm is specified as a Bezier’s eighth-order polynomial. The online algebraic estimation of the unknown parameter \( \omega \), in accordance with (31), (40), and (41), is carried out in \( \Delta = 0.26 \) s [see Fig. 5(a)]. At the end of this small time interval, the controller is immediately replaced or updated by switching on the interruptor \( s_1 \) (see Fig. 4), with the accurate parameter estimate, given by \( \omega_{0e} = 14.5 \) [rad/s]. When the controller is updated, \( s_1 \) is switched off. Fig. 5(b) shows the trajectory tracking with the adaptive controller. Note that the trajectory of the tip and the reference are superimposed. The tip position \( \theta_1 \) tracks the desired trajectory \( \theta_1^* \) with no steady-state error [see Fig. 5(c)]. In this figure, the tracking error \( \theta_1^* - \theta_1 \) is shown. The corresponding transfer function of the new updated controller is then found to be

\[
\frac{\theta_m - \theta_m^*}{\theta_l - \theta_l^*} = \frac{0.3s - 25.7}{s + 30}. \tag{50}
\]

The open-loop reference control input \( \theta^*_m(t) \) from (14) in terms of the new estimate \( \omega_{0e} \) is given by

\[
\theta^*_m(t) = \frac{1}{\omega^2_{0e}}(\ddot{\theta}_l^*(t) + \ddot{\theta}_l^*(t)) = 4.7 \cdot 10^{-3}\ddot{\theta}_l^*(t) + \theta_l^*(t). \tag{51}
\]

The input control voltage to the dc motor is shown in Fig. 5(d), the coupling torque is shown in Fig. 5(e), and the Coulomb friction effect in Fig. 5(f). In Fig. 6, the motor angle \( \theta_m \) is shown.

A. Simulations of a Two-Mode Beam

In the next simulations, the behavior of the adaptive controller when this is applied to a flexible beam characterized by two modes of vibrations is shown. The adaptive controller previously presented is now used in a system modeled with two modes. This is obtained from a truncated lumped-mass model by considering the mass of the beam as not negligible. Thus, the reduced two-mode model is equivalent to a flexible beam with two concentrated masses: one in the tip position and the other in the middle of the bar (see [23]). In our case, the damping factor is also neglected. The two modes directly depend on the masses of the beam \( m_1 \) and the load \( m_2 \).
The simulations consist of making the adaptive-control system, explained in the previous sections, but now taking into account the effect that can be produced by modes in the system which have not been modeled. We consider \( m_2 = m = 0.03 \text{ [kg]} \) and change \( m_1 \) from 10\% up to 150\% of \( m_2 \) to bring the second mode closer to the first mode (see the second and the third columns of Table I to observe the manner that the modes change, where \( \omega_1 \) and \( \omega_2 \) are the first and second modes, in radians per second, respectively). Thus, the spillover effect is more notorious. The adaptive-control system begins to work with an arbitrary value of \( \omega \), which we select to be the same used in the previous simulation. This is \( \omega_0 = 9 \text{ [rad/s]} \); thus, we can compare the results. The estimator will estimate the value of the natural frequency at time \( t_1 \). This estimated value \( \omega_{0e} \) is used to update the GPI controller. The simulation results are shown in Figs. 7–9 and in Table I.

Fig. 7 shows the results of four simulations. Note that an estimation of the first mode is always obtained, but the time at which it is obtained vary from \( t_1 = 0.4 \text{ [s]} \) for \( m_1 = 10\%m_2 \) to \( t_1 = 0.9 \text{ [s]} \) for \( m_1 = 150\%m_2 \), and these estimates have an error \( \varepsilon_{0e} \text{ [%]} \), which also increases with the approximation of the second mode to the first one (see fifth column of Table I). Nevertheless, even when the second mode is very noticeable because of its proximity to the first mode, the system properly tracks the reference trajectory. In Fig. 8 are shown the trajectory tracking of the different simulations. The dotted vertical line represents the time at which the controller is updated. Obviously, the error in the trajectory tracking increases when the time \( t_1 \) and the error \( \varepsilon_{0e} \text{ [%]} \) in the estimation increase. Nevertheless, we can affirm that these errors and time estimation are small enough to make the system work properly; thus, we obtain an accurate trajectory tracking. The errors in the trajectory tracking are shown in Fig. 9. Note that the error increases when the second mode is near to the first one and that this oscillates in a very small margin around zero. The motor angle when the four tracking trajectories is carried out are shown in Fig. 10. Note that the signals are very smooth even when \( m_1 = 150\%m_2 \). This means that the inner loop does not excite the link in the second mode. Therefore, the use of large payloads in the tip is possible.\(^4\)

\(^4\) More information about the control of flexible robots manipulating large payloads may be found in [46] and [47].
In order to show, in a clearer manner, the error produced in the trajectory tracking, the error $\varepsilon_t = \sqrt{\sum_{t=0}^{t_f} (\theta_t^* - \theta_t)^2 T/t_f}$ is depicted in the seventh column of Table I, where $t_f$ is the final time of the simulations and $T$ is the sample time. In this case, $t_f = 5$ $[s]$ and $T = 1 \cdot 10^{-3}$ $[s]$.

VII. EXPERIMENTATION

In this section, the experimental platform and the experiment design are briefly explained. The identification and control method previously described are applied here to 1-DOF flexible robot.
Fig. 10. Motor angle during the trajectory tracking. (a) $\theta_m$ when $m_1 = 10\% m_2$. (b) $\theta_m$ when $m_1 = 50\% m_2$. (c) $\theta_m$ when $m_1 = 100\% m_2$. (d) $\theta_m$ when $m_1 = 150\% m_2$.

Fig. 11. Flexible robot prototype.

A. Experimental Platform Description

Fig. 11 shows a picture of the experimental platform constituted by a three-legged metallic structure to support a harmonic drive miniservo dc-motor RH-8D-6006-EOvAL-SP(N) which has a reduction relation characterized by $n = 50$. The frame makes it possible for the stable and free rotation of the motor around the vertical axis of the platform. The motor shaft is capable of turning either right or left around the $Z$-axis. A servo amplifier is used to supply voltage to the dc motor. This amplifier accepts control inputs from the computer in the range of $[-10, 10]$ [V]. A carbon-fiber flexible beam with a length 500 [mm] and stiffness of 1.6 [Nm] is embedded in the motor shaft. At the other end of the beam, there is a load with the shape of a disk with a diameter of 90 [mm] and mass of 0.028 [kg] which is considered unknown. The load freely rotates with respect to its vertical axis. It means that the torque produced by the load inertia does not influence the tip of the beam. The load floats on the surface of an air table, so the gravity effect and the friction of the load with the surface of the table are canceled. The sensor system is integrated by an encoder embedded in the motor which allows us to know the motor position with a precision of $7 \times 10^{-5}$ [rad] and a pair of strain gauges with a gauge factor of 2.16 and resistance of 120.2 [Ω]. The sample time for the processing of the signals was of $2 \times 10^{-3}$ [s].

<table>
<thead>
<tr>
<th>J [kgm$^2$]</th>
<th>v [Nm·s]</th>
<th>k [Nm/P]</th>
<th>n</th>
<th>A [N/kg]</th>
<th>B [N/kg·m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.87 x 10^{-5}</td>
<td>1.041 x 10^{-3}</td>
<td>0.21</td>
<td>50</td>
<td>61.14</td>
<td>15.15</td>
</tr>
</tbody>
</table>

Fig. 12. Periodogram of the flexible-beam oscillation.

B. Results

The values of the motor parameters are shown in Table II. These numerical values are used to implement the inner loop controller on the actual physical platform.

In order to obtain the natural frequency of the system $\omega$ to validate the estimation, a torque in the motor shaft was applied. Then, the tip of the beam oscillates. The oscillation is translated as a peak in the periodogram (see Fig. 12). Note that the maximization of periodogram is equivalent to the estimate by maximum likelihood of the frequency parameter of a sinusoidal signal corrupted with white noise.

The estimation provided by the peak of the periodogram, observed at the abscissa axis, is $f_0 \approx 2.4$ [Hz]; this is

3Recall that the periodogram of the signal $u(t)$, $t = 1, 2, \ldots, N$, is $|U_N(\omega)|^2$, where $U_N(\omega) = (1/\sqrt{N}) \sum_{t=1}^{N} u(t)e^{i2\pi kt/N}$, $k = 1, \ldots, N$, represents the discrete Fourier transform (DFT) for $\omega = 2\pi k/N$.
The system should be as fast as possible but care must be taken of possible saturations of the motor, which occur at 10 [V]. The inner loop controller is that designed in (46) with the parameters

\[
\alpha_0 = 1.3 \cdot 10^6 \quad \alpha_1 = 5.6 \cdot 10^4 \quad \alpha_2 = 798 \quad \alpha_3 = 365. \tag{52}
\]

The feedforward term in (27) is computed in accordance with (47). The parameters used for the flexible arm are \( c = 1.6 \) [Nm] and \( L = 0.5 \) [m]. The mass \( m \) is considered unknown. The poles for the outer loop design are located at \(-10\) in the real axis to assure that the outer loop is slower than the inner loop. With a natural frequency of the bar given by an initial arbitrary estimate, \( \omega_{0i} = 9 \) [rad/s]. The transfer function of the controller (19), which depends on the location of the closed-loop poles of the outer loop and the natural frequency of the bar as shown in (22) and (23), respectively, is the one designed in (48) with the following parameters:

\[
\gamma_0 = 17.7 \quad \gamma_1 = 2.7 \quad \gamma_2 = 30. \tag{53}
\]

The open-loop reference control input from (49) is computed with the initial arbitrary estimate of \( \omega_{0i} \).

The desired reference trajectory used for the tracking problem of the flexible arm is specified as a Bezier’s eighth-order polynomial. The online algebraic estimation of the unknown parameter \( \omega \), according to (31), (40), and (41), is carried out in \( \Delta = 0.5 \) s (see Fig. 13). At the end of this small time interval, the controller is immediately replaced, or updated, with the accurate parameter estimate given by \( \omega_{0e} = 15.2 \) [rad/s].

Fig. 14 shows the update of the controller and how, after the update of the controller (after the dashed line), the tip position \( \theta_t \) tracks the desired trajectory \( \theta_d \) with no steady-state error. The corresponding transfer function of this new controller is computed in accordance with the following new parameters:

\[
\gamma_0 = 25.7 \quad \gamma_1 = 0.3 \quad \gamma_2 = 30. \tag{54}
\]

The open-loop reference control input from (51) is computed with the new estimate \( \omega_{0e} \).

Note how the controller does not work fine and saturates the amplifier at \([10, -10]\) [V] before 0.5 [s]. At this time, the controller is updated with the online estimate \( \omega_{0e} \). After that, the controller rapidly eliminates the tracking error, and therefore, the input control voltage is smoothed and does not saturate the amplifier.

The motor angle is shown in Fig. 17. Note that, after the updating of the controller, the signal is very smooth.

A new experiment is carried out to validate the previous results. Now, a new initial arbitrary estimate of the natural frequency is introduced. This is \( \omega_{0i} = 20 \) [rad/s]. The real natural frequency is estimated with the algebraic method in approximately 0.5 s, and the value estimated is \( \omega_{0e} = 15.2 \) [rad/s] (see the first picture of Fig. 18). At this time, the controller is updated with this new accurate parameter estimate as done in

\[
\begin{align*}
\omega \approx 2.4 \cdot 2\pi \approx 15.1 \quad [\text{rad/s}] &. \quad \text{With the algebraic-identification method, the } \omega_{0e} \text{ value was } 15.22 \quad [\text{rad/s}] \text{ approximately. This demonstrates the correct performance of the method and the model used to design the adaptive-control system.}
\end{align*}
\]
Fig. 16. Control input voltage to the dc motor.

Fig. 17. Motor angle during the trajectory tracking.

Fig. 18. Results of the second experiment.

Fig. 19. Motor angle of the second experiment.

VIII. APPLICATION TO A MULTILINK FLEXIBLE ROBOT

The proposed method can be easily extended to multilink flexible robots, provided that all their mass is concentrated at the tip (lightweight flexible arms) and that the gravity is not acting (e.g., 2-DOF flexible arms moving on a horizontal plane or a 3-DOF robot in outer space). The dynamics of these arms can be expressed as (see [48])

\[ mC(\Theta)D^2P + P = P_u \]  \hspace{1cm} (55)

\[ \Gamma_{CT} = J^t(\Theta)C^{-1}(\Theta)(P_u - P) \]  \hspace{1cm} (56)

where \( m \) is the flexible-arm-tip mass and \( C(\Theta) \) is a compliance matrix which models the flexibility of the arm, with \( \Theta \) as the joint-angle column vector. This matrix relates the tip forces with the tip deformations. The tip position of the arm is expressed by the column vector \( P \) in Cartesian coordinates. The tip position of the arm assumed as rigid (obtained by applying the direct kinematic transform to the rigid robot using the joint angles measurements \( \Theta \)) is expressed by the column vector \( P_u \) in Cartesian coordinates. \( \Gamma_{CT} \) is the column vector which represents the coupling torques of the arm, produced in the joints. \( J \) is the Jacobian matrix of the arm (\( t \) represents the transposition of the matrix). Finally, \( D \) denotes the derivative operator.

From (56), the tip position of the flexible arm can be estimated because \( J, C(\Theta) \), and \( \Gamma_{CT} \) are known, with no \( m \) dependence

\[ P = P_u - \left[J^t(\Theta)C^{-1}(\Theta)\right] \Gamma_{CT}. \]  \hspace{1cm} (57)

By making \( C^{-1}(\Theta) = \lambda I + \varepsilon(\Theta) \), where \( \lambda \) is a scalar constant chosen to make \( \|\varepsilon(\Theta)\| \) as small as possible in all the robot workspace, and \( \varepsilon(\Theta) \) is a matrix that includes all the coupling and nonlinear effects of \( C^{-1}(\Theta) \), the multivariable equation (55) can be linearized and decoupled by substituting that in the before equation

\[ mD^2P + \lambda P = C^{-1}(\Theta)P_u - \varepsilon(\Theta)P. \]  \hspace{1cm} (58)

Note that the right side of this equation can be computed in real time and is independent of the tip mass; \( \Theta \) is measured (motor angles), \( P_u \) is computed from \( \Theta \), and \( P \) is estimated from (57).

If we denote \( P_m \) at this right side of (58), it is as follows:

\[ mD^2P + \lambda P = P_m. \]  \hspace{1cm} (59)

The GPI control proposed in this paper can be applied here by designing one GPI control for each degree of freedom. Because (58) is a multivariable equation, to estimate the mass of the tip with the algebraic estimator is only required in one of the equations that form (59), the rest are redundant to make the estimation.

IX. CONCLUSION

A two-stage GPI-controller design scheme is proposed in connection with a fast online closed-loop continuous-time estimator of the natural frequency of a flexible robot. This methodology only requires the measurement of the angular position.
of the motor and the coupling torque. Thus, the computation of angular velocities and bounded derivatives, which always introduces noise in the system and makes necessary the use of suitable low-pass filters, is not required. Among the advantages of this technique, we find the following advantages: 1) a control robust with respect to the Coulomb friction; 2) a direct estimation of the parameters without an undesired translation between discrete- and continuous-time domains; and 3) independent statistical hypothesis of the signal is not required, so closed-loop operation is easier to implement.

This methodology is well suited to face the important problem of control degradation in flexible arms as a consequence of payload changes. Its versatility and easy implementation make the controller suitable to be applied in more than 1-DOF flexible beams by applying the control law to each separated dynamics which constitute the complete system. The method proposed establishes the basis of this original adaptive control to be applied in more complex problems of flexible robotics.

Appendix
System Stability Analysis to Uncertainties in the Stiffness Estimation

In this paper, we assume that the stiffness ‘c’ of the flexible link is perfectly known because $c = (3EI/L)$. The stiffness depends on the flexural rigidity $EI$ and on the length of the beam $L$; both are constant and perfectly known. However, the value of $c$ used in practice and computed in the control scheme may slightly vary from the real value.

In this analysis, we consider that the value of the stiffness used to design the control method may have differences from the real value of the beam stiffness. Thus, if $c$ is the real value of the beam stiffness, $c_0$ is the approximated value computed in the control scheme.

The control scheme is designed in a general manner to make possible its use in major problems related to 1-DOF flexible arms, assuming that $\theta_t$ is perfectly known. Its knowledge can be provided by both the direct measure of the tip position and an accurate estimation from the coupling torque measurements. Next, we analyze the two possible cases.

A. Case A

Most of works in the literature use the direct measurement of $\theta_t$ to feedback the control scheme. This measure can be obtained by means of a camera system, a laser located in the hub and which directly measures the position of the tip, etc. In such cases, the control system described in this paper is characterized by a good performance with a negligible effect produced by not exact computation of the stiffness value $c$.

In this case, the behavior of the system is characterized as follows.

1) The natural frequency estimator is not affected by an approximation in the computation of $c$ [see (37)]. The estimator does not depend on this value but only depends on the direct measurements of the motor position $\theta_m$ and of the tip position $\theta_t$.

2) The outer loop feedback control scheme does not depend on the $c$ parameter. The controller in (16) and the feedforward term in (14) only depends on the estimation of $\omega$, and this is directly estimated from (37).

3) Nevertheless, an inaccurate approximation of $c$ in the inner loop affects the control system because the initial restriction in the control scheme is to separate the motor dynamics and the flexible-link dynamics by means of a coupling torque compensation term [see (9) and Fig. 3]. The compensation voltage to be injected in the motor can be expressed as

$$u_\Gamma = \frac{\Gamma}{kn}. \quad (60)$$

In the case that a sensor is explicitly used to measure the coupling torque $\Gamma$ apart from the sensor which directly measures the tip position, there is no effect caused by a variation of $c$ because this parameter does not appear in the adaptive control scheme. However, if no one sensor is used to measure the coupling torque and this value has to be estimated from motor and tip-position measurements through (7), the compensation term depends on $c$. Moreover, the coupling torque compensation term is as follows:

$$u_\Gamma = \frac{c}{kn}(\theta_m - \theta_t). \quad (61)$$

This term is affected by the flexible-bar stiffness; however, note that $c$ is divided by the motor gear-reduction ratio $n$, which is a very high value most of the time. Thus, the influence of an erroneous computation of $c$ is negligible. This could only affect direct-drive arms where gears are not used, but even in this case, we may expect that the effect produced by erroneous computation of $c$ is very small.

The rest of the inner loop control system is not affected by $c$. In the controller proposed in (24) and in the feedforward term in (27), such a parameter does not appear.

B. Case B

In this paper, the sensor system is integrated by a pair of strain gauges to measure the coupling torque, which is used to estimate the tip position to feedback the control law. In this case, the inner loop is not affected by variations in the computation of $c$ from its real value because, in the compensation of the coupling torque, its measure is directly included, and the stiffness does not appear in the inner loop control system law. Therefore, the system could be applied in direct-drive arms without any limitation because variations of $c$ does not affect the inner loop. However, the outer loop uses the estimation of $\theta_t$ given by (21) to feedback the controller.

1) Stability Analysis of the Closed-Loop System: Let us analyze the closed-loop system stability. Define $q = (1/c)$, $q_0 = (1/c_0)$, and $\Delta q = (q - q_0/q_0)$; let $\omega = \sqrt{T/qmL^2}$ define the unknown real natural frequency of the flexible beam; $\omega_l = \sqrt{T/q_0mL^2}$ the natural frequency of the bar if the stiffness of the flexible beam is approximated by $c_0$; $\omega_{le}$ the estimated
natural frequency of the flexible beam given by the algebraic estimator, and $\theta^*_t$ the estimated tip position of the flexible beam. Similarly to (21), we can obtain the dynamics for the estimated tip position

$$\theta^*_t = \theta_m - q_0 \Gamma.$$  \hspace{1cm} (62)

From the $\Delta q$ definition, $q_0 = q - \Delta q q_0$ can be replaced in the previous equation (62) to obtain an expression which contains the parameter $\Delta q$

$$\theta^*_t = \theta_m - \Gamma + \Delta q q_0 \Gamma.$$  \hspace{1cm} (63)

Moreover, bearing in mind the second equality of (20), another expression for $\theta^*_t$ is obtained after some rearrangements

$$\theta^*_t = \theta_t + \theta_t \frac{\Delta \theta_t}{\omega_0^2}.$$  \hspace{1cm} (64)

with $\omega_0 = (1/q_0 mL^2)$.

The equivalent control system is shown in Fig. 20; the controller gains $\gamma_0$, $\gamma_1$, and $\gamma_2$ are designed from a model with $\omega_{0e}$ [see (23)], which is equalized with a Hurwitz polynomial (22).

$$s^3 + \gamma_2 s^2 + \omega_{0e}^2 (1 + \gamma_1) s + \omega_{0e}^2 (\gamma_0 + \gamma_2) = (s + a)^3 = 0.$$  \hspace{1cm} (65)

Therefore, the gains are obtained to be

$$\gamma_0 = \frac{a^3}{\omega_{0e}^2} - 3a \quad \gamma_1 = \frac{3a^2}{\omega_{0e}^2} - 1 \quad \gamma_2 = 3a.$$  \hspace{1cm} (66)

The characteristic equation of the closed loop shown in Fig. 20 is the following:

$$1 + \frac{\gamma_1 s + \gamma_0}{s + \gamma_2} \frac{\omega^2}{s + \omega^2} \left(1 + \frac{\Delta q}{1 + \Delta q} \frac{1}{\omega^2 s^2}\right) = 0.$$  \hspace{1cm} (67)

After replacing the gains in (66) in (67), we obtain

$$\left(1 + \frac{3a^2 - \omega_{0e}^2}{\omega_{0e}^2} \frac{\Delta q}{1 + \Delta q}\right) s^3 + a \left(3 + \frac{a^2 - 3 \omega_{0e}^2}{\omega_{0e}^2} \frac{\Delta q}{1 + \Delta q}\right) s^2 + 3a^2 \frac{\omega_{0e}^2}{\omega_{0e}^2} s + a^3 \frac{\omega_{0e}^2}{\omega_{0e}^2} = 0.$$  \hspace{1cm} (68)

After rearrangements

$$a_0 s^3 + a_1 s^2 + a_2 s + a_3 = 0.$$  \hspace{1cm} (69)

with $a_0 = (\omega_{0e}^2 + 3a^2 \Delta q / \omega_{0e}^2 (1 + \Delta q))$, $a_1 = (3\omega_{0e}^2 + a^2 \Delta q / \omega_{0e}^2 (1 + \Delta q)) a$, $a_2 = (3a^2 \omega_{0e}^2 / \omega_{0e}^2)$, and $a_3 = (a^3 \omega_{0e}^2 / \omega_{0e}^2)$.

The Routh stability criterion establishes that all the coefficients of the polynomial in (69) must be positive [49]. The polynomial coefficients $a_2$ and $a_3$ always fulfill these restrictions. Nevertheless, in order to fulfill this condition, the coefficient $a_0$ requires that

$$\Delta q > -\frac{\omega_{0e}^2}{3a^2}.$$  \hspace{1cm} (70)

and the coefficient $a_1$ requires that

$$\Delta q > -\frac{3 \omega_{0e}^2}{a^2}.$$  \hspace{1cm} (71)

Because (70) is more restrictive than (71), we use (70) to condition the problem. With this, we assure that the polynomial (69) has all its coefficients positive.

The Routh table is depicted in Table III with $b_1 = (a_1 a_2 - a_0 a_3 / a_1)$. The Routh stability criterion establishes that the system is stable if the first column of Table III has all its elements positive. In this case, the restriction is given by the term $b_1$

$$b_1 = a^2 \omega_{0e}^2 \left[3 \left(3 \omega_{0e}^2 + a^2 \Delta q - (\omega_{0e}^2 + 3a^2 \Delta q) / \omega_{0e}^2 (3a^2 \omega_{0e}^2 + a^2 \Delta q)\right) \right] > 0.$$  \hspace{1cm} (72)

After solving (72), the condition is obtained: $8 \omega_{0e}^2 > 0$. Moreover, this is always fulfilled because the estimation of the natural frequency is always positive. Thus, the only stability condition is (70), and the designer can always fulfill it easily (note that the value $-(\omega_{0e}^2 / 3a^2)$ is negative) by properly reducing the value of $q_0$ (increasing the value of $c_0$), by moving further away the closed-loop poles given by an increase of the $a$ value, or even by approximating, in a conservative manner, $c_0$ to its smallest possible value fulfilling $\Delta q > 0$. This last value is enough to assure the system stability.

2) Analysis of the Estimator: The effect produced by variations in the computation of the stiffness in the natural frequency estimator is studied here.

See the estimator in (37), which can be expressed as

$$\omega_{0e}^2 = \frac{\left[\int_0^t (\theta_m(t) - \bar{\theta}_m(t)) \sigma \theta_t^e(\sigma) d\sigma + \int_0^t \int_0^\sigma \theta_t^e(\lambda) d\lambda d\sigma\right]}{\int_0^t \int_0^\sigma \lambda^2 (\theta_m(\lambda) - \bar{\theta}_m(\lambda)) d\lambda d\sigma}.$$  \hspace{1cm} (73)

From (62) and due to $\Delta q = (q - q_0)$, we obtain

$$\theta_m - \bar{\theta}_m = q_0 \Gamma = \frac{q}{1 + \Delta q} \Gamma.$$  \hspace{1cm} (74)
Moreover, due to the fact that
\[
\int_0^t \int_0^\sigma \lambda^2 \dot{z}(\lambda) d\lambda d\sigma = t^2 z(t) - 4 \int_0^t \sigma z(\sigma) d\sigma + 2 \int_0^t \int_0^\sigma z(\lambda) d\lambda d\sigma
\]
(75)
with \( z \) as whatever variable, we can obtain a new expression for the estimator. From (74), and bearing in mind (20)
\[
\theta_m - \hat{\theta}_m = (1 + \Delta \Phi)^{-1} q_m L^2 \ddot{\hat{\theta}} = \frac{1}{\omega^2 (1 + \Delta \Phi)} \ddot{\hat{\theta}}
\]
(76)
and also by using (64) to the numerator of (73) and by applying \( z = \hat{\theta}_m \), we can use the property (75) to the denominator of (73) to obtain the following expression for the estimator:
\[
\omega^2 _{0e} = \frac{t^2 \ddot{\hat{\theta}} - 4 \int_0^t \sigma \ddot{\theta}_\sigma(\sigma) d\sigma + 2 \int_0^t \int_0^\sigma \theta_\sigma(\lambda) d\lambda d\sigma}{\omega^2 (1 + \Delta \Phi)} \frac{t^2 \ddot{\hat{\theta}} - 4 \int_0^t \sigma \ddot{\theta}_\sigma(\sigma) d\sigma + 2 \int_0^t \int_0^\sigma \theta_\sigma(\lambda) d\lambda d\sigma}{\omega^2 (1 + \Delta \Phi)} \frac{t^2 \ddot{\hat{\theta}} - 4 \int_0^t \sigma \ddot{\theta}_\sigma(\sigma) d\sigma + 2 \int_0^t \int_0^\sigma \theta_\sigma(\lambda) d\lambda d\sigma}{\omega^2 (1 + \Delta \Phi)}
\]
(77)
Taking into account that \( \Delta \Phi = (q - q_0 / \omega_0) \), by multiplying and dividing such an expression by \( mL^2 \) and solving it for \( \omega^2 _{0e} \), we obtain that
\[
\omega^2 _{0e} = \omega^2 (1 + \Delta \Phi).
\]
(78)
This expression (78) can be substituted in (77) to obtain
\[
\omega^2 _{0e} = \omega^2 (1 + \Delta \Phi) + \Delta \Phi \Phi(t)
\]
(79)
with
\[
\Phi(t) = \frac{t^2 \ddot{\hat{\theta}} - 4 \int_0^t \sigma \ddot{\theta}_\sigma(\sigma) d\sigma + 2 \int_0^t \int_0^\sigma \theta_\sigma(\lambda) d\lambda d\sigma}{\int_0^t \int_0^\sigma \lambda^2 \theta_\sigma(\lambda) d\lambda d\sigma}.
\]
(80)
By multiplying and dividing the expression in (80) by \( mL^2 \), we obtain \( \Phi(t) \) as a function of \( \Gamma \) (note that \( \Gamma = mL^2 \ddot{\hat{\theta}} \))
\[
\Phi(t) = \frac{t^2 \ddot{\hat{\theta}} - 4 \int_0^t \sigma \ddot{\theta}_\sigma(\sigma) d\sigma + 2 \int_0^t \int_0^\sigma \theta_\sigma(\lambda) d\lambda d\sigma}{\int_0^t \int_0^\sigma \lambda^2 \Gamma \theta_\sigma(\lambda) d\lambda d\sigma}
\]
(81)
which can be properly obtained by computation of Brunovsky filters as \( \Phi(t) = (n(t)/d(t)) \)
\[
n(t) = t^2 \Gamma(t) + z_1 \quad d(t) = z_3
\]
\[
z_1 = z_2 - 4t \Gamma(t) \quad z_3 = z_4
\]
\[
z_2 = 2 \Gamma(t) \quad z_4 = t^2 \Gamma(t).
\]
(82)
We can obtain the relative error of the estimated \( \omega^2 _{0e} \) from (79)
\[
\left| \frac{\omega^2 _{0e} - \omega^2}{\omega^2} \right| = \Delta \Phi \left( 1 + \frac{\Phi(t)}{\omega^2} \right)
\]
(83)
or also
\[
\left| \frac{\omega^2 _{0e} - \omega^2}{\omega^2} \right| = \Delta \Phi \left( 1 + \frac{\Phi(t)}{\omega^2} \right).
\]
(84)

Because the estimation of the natural frequency is carried out in a very short period of time, the acceleration of the tip position \( \ddot{\hat{\theta}} \), which is directly related with the coupling torque, can be approximately considered as a constant value. If the estimation is produced at time instant \( t_e \), it is well supposed that, previously to that instant, a transitory effect happened, \( 0 < t < \Delta \), with \( \Delta < t_e \), in which the acceleration (coupling torque) changed from zero to the constant value considered. Thus, it can be proven that
\[
t^2 \Gamma - 4 \int_0^{t_e} \sigma \Gamma(\sigma) d\sigma + 2 \int_0^{t_e} \Gamma(\lambda) d\lambda d\sigma = \rho_1 t_e + \rho_0
\]
(85)
and
\[
\int_0^{t_e} \sigma \Gamma(\lambda) d\lambda d\sigma = \psi_4 t_e^4 + \psi_1 t_e + \psi_0
\]
(86)
with \( \rho, \rho_1, \psi_1, \psi_4, \) and \( \psi_4 \) constants. The parameter \( \Phi(t_e) \) can be expressed as
\[
\Phi(t_e) = \frac{\rho_1 t_e + \rho_n}{\psi_4 t_e^4 + \psi_1 t_e + \psi_0}.
\]
(87)
Moreover, \( \Phi(t_e) \) tends to the zero value when \( t_e \) increases. Therefore, \( \Phi(t) \) is bounded by a small constant, which is denoted by \( M \). The equation in (84) can be expressed as
\[
\left| \frac{\omega^2 _{0e} - \omega^2}{\omega^2} \right| \approx \Delta \Phi \left( 1 + \frac{M}{\omega^2} \right).
\]
(88)
Because \( M \) is a small limit and this is divided by the natural frequency of the bar to the power of two, that ratio
\[
\frac{M}{\omega^2} \approx 0.
\]
(89)
Then, the relative error of the estimation of the natural frequency of the flexible bar is approached by the following expression:
\[
\left| \frac{\omega^2 _{0e} - \omega^2}{\omega^2} \right| \approx |\Delta \Phi|.
\]
(90)

REFERENCES


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