A PLU-factorization of rectangular matrices by the Neville elimination

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Abstract

In this paper we prove that Neville elimination can be matricially described by elementary matrices. A PLU-factorization is obtained for any \( n \times m \) matrix, where \( P \) is a permutation matrix, \( L \) is a lower triangular matrix (product of bidiagonal factors) and \( U \) is an upper triangular matrix. This result generalizes the Neville factorization usually applied to characterize the totally positive matrices. We prove that this elimination procedure is an alternative to Gaussian elimination and sometimes provides a lower computational cost.

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1. Introduction

A real matrix is called totally positive if all its minors are nonnegative. These matrices have become increasingly important in approximation theory and other fields. For a comprehensive survey of this subject from an algebraic point of view, complete with historical references, see [1].

Fiedler and Markham obtained, in [2,3], a factorization for totally nonsingular matrices that satisfy the properties consecutive-column CC and consecutive-row CR, using the Neville elimination. From this elimination process Gasca and Peña obtained an LU factorization for matrices satisfying the without row exchange WR condition. In both cases the Neville elimination can be performed without row exchange.
In this paper, taking into account that there exist totally positive matrices such that they do not satisfy condition WR, we generalize some of these results for matrices such that they do not satisfy conditions CC, CR or WR and are not necessarily regular matrices.

In Section 2 we recall the Neville elimination process and in Section 3 we are going to obtain a matricial description of this process for matrices of size \( n \times m \). We obtain a PLU factorization of a matrix of size \( n \times m \), where \( P \) is a permutation matrix of size \( n \times n \), \( L \) is a lower triangular matrix (product of bidiagonal factors) of size \( n \times n \) and \( U \) is an upper triangular matrix of size \( n \times m \).

Finally, from the remarks of Gasca and Peña in [5], we define a class of matrices where Neville elimination has a lower computational cost than Gaussian elimination.

2. Neville elimination

The essence of Neville elimination is to produce zeros in a column of a matrix by adding to each row an appropriate multiple of the previous one (instead of using a fixed row with a fixed pivot as in Gaussian elimination). Eventual reorderings of the rows of the matrix may be necessary.

More precisely, we recall the Neville elimination process [4] for any \( n \times m \) matrix \( A = (a_{ij}) \). Let \( \tilde{A}_1 := (\tilde{a}_{ij}^1) \) be such that \( \tilde{a}_{ij}^1 = a_{ij} \), \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \). If there are zeros in the first column of \( \tilde{A}_1 \), we carry the corresponding rows down to the bottom in such a way that the relative order among them is the same as in \( \tilde{A}_1 \). We denote the new matrix by \( A_1 \). If we have not carried any row down to the bottom, then \( A_1 := \tilde{A}_1 \). In both cases, let \( i_1 := 1 \). The method consist in constructing a finite sequence of matrices \( A_k \) such that, for each \( A_k \), the submatrix formed by its \( k-1 \) initial columns is an upper echelon form matrix. In fact, if \( A_t = (a_{ij}^t) \) then we introduce zeros in its \( t \)th column below the position \((i_t, t)\), thus forming the \( n \times m \) matrix

\[
\tilde{A}_{t+1} = (\tilde{a}_{ij}^{t+1}),
\]

where, for any \( j \) such that \( 1 \leq j \leq m \), we have

\[
\tilde{a}_{ij}^{t+1} := a_{ij}^t, \quad i = 1, 2, \ldots, i_t,
\]

\[
\tilde{a}_{ij}^{t+1} := a_{ij}^t - \frac{a_{it}^t}{a_{i-1,t}^t} a_{i-1,t}^t \quad \text{if} \quad a_{i-1,t}^t \neq 0, \quad i_t < i \leq n,
\]

\[
\tilde{a}_{ij}^{t+1} := a_{ij}^t \quad \text{if} \quad a_{i-1,t}^t = 0, \quad i_t < i \leq n.
\]

Observe that with our assumptions, \( a_{i-1,t}^t = 0 \) implies \( a_{it}^t = 0 \). Then we define

\[
i_{t+1} := \begin{cases} 
  i_t & \text{if} \quad a_{i_t,t}^t(= \tilde{a}_{i_t,t}^{t+1}) = 0, \\
  i_t + 1 & \text{if} \quad a_{i_t,t}^t(= \tilde{a}_{i_t,t}^{t+1}) \neq 0.
\end{cases}
\]
If $\bar{A}_{t+1}$ has zeros in the $(t+1)$th column in the row $i_{t+1}$ or below it, we will carry these rows down as we have done with $\bar{A}_1$. The matrix obtained in this way will be denoted by $A_{t+1} = (a_{ij}^{t+1})$. Of course, if there is no row that has been carried down, then $A_{t+1} := \bar{A}_{t+1}$. After a finite number of steps we get $\bar{A}_{t-1}, A_{t-1}$, and $A_t = \bar{U}(\bar{t} \leq m+1)$, where $U$ is an upper echelon form matrix. In this process the element
\[ p_{ij} := a_{ij}^t, \quad 1 \leq j \leq m, \quad i_j \leq i \leq n, \]
is called the $(i, j)$ pivot of the Neville elimination of $A$ and the number
\[ m_{ij} := \begin{cases} a_{ij}^t/a_{i-1,j}^t & \text{if } a_{i-1,j}^t \neq 0, \\ 0 & \text{if } a_{i-1,j}^t = 0, \end{cases} \quad 1 \leq j \leq m, \quad i_j < i \leq n, \]
the $(i, j)$ multiplier of the Neville elimination of $A$. We observe that $m_{ij} = 0$ if and only if $a_{ij}^t = 0$.

3. Matricial description of Neville elimination

The Neville elimination process, for an $n \times m$ matrix, can be matricially described by elementary and permutation matrices. We shall use similar notations to [5]. Let $E_{ij}(\alpha), 1 \leq i \neq j \leq n$, be the elementary triangular matrix whose $(r, s)$ entry $1 \leq r, s \leq n$, is given by

\[
E_{ij}(\alpha) = \begin{cases} 
1 & \text{if } r = s, \\
\alpha & \text{if } (r, s) = (i, j), \\
0 & \text{elsewhere}.
\end{cases}
\]

Note that if $i > j$ the matrix $E_{ij}(\alpha)$ is a lower triangular matrix. We are interested in the matrices $E_{i+1,i}(\alpha)$ which for simplicity will be denoted, as in [5], by $E_{i+1}(\alpha)$. They are bidiagonal and lower triangular, and given by

\[
E_{i+1}(\alpha) = \begin{bmatrix}
1 & 1 & 1 \\
& 1 & 1 \\
& & \ddots & 1 \\
& & & \alpha & 1 \\
& & & & \ddots & 1 \\
& & & & & & 1
\end{bmatrix}.
\]

We denote by $P_{ij}$ the elementary permutation matrix whose $(r, s)$ entry, $1 \leq r, s \leq n$ is given by
\[ \begin{align*}
1 & \text{ if } r = s, r \neq i, j, \\
1 & \text{ if } (r, s) = (i, j), \\
1 & \text{ if } (r, s) = (j, i), \\
0 & \text{ otherwise.}
\end{align*} \]

So they are given explicitly by

\[ P_{ij} = \begin{bmatrix}
1 & 1 & & \\
& 0 & 1 & \\
& & \ddots & 1 \\
& & & \ddots & 0 \\
& & & & 1
\end{bmatrix}. \]

To obtain a matricial description of Neville elimination, we define the following matrix.

**Definition 3.1.** From the matrices \( P_{ij} \) we denote by \( \Pi_j \) the following permutation matrix:

\[ \Pi_j = P_{n-1,n-2} \cdots P_{j+1,j} P_{j,n}. \] (1)

Note that the product \( \Pi_j A \) carries the row \( j \) down, reordering the remaining rows of the matrix \( A \).

The following lemma describes elementary properties of the matrices \( \Pi_j \).

**Lemma 3.1.** Every matrix \( \Pi_j \) satisfies the following properties:

(i) \( \Pi_j \) is a nonsingular matrix and \( \Pi_j^{-1} = P_{j,n} P_{j+1,j} \cdots P_{n-1,n-2} \).

(ii) Let \( A \) be a matrix of size \( n \times m \). If \( \bar{f}_i \) denote its \( i \)th row, \( i = 1, 2, \ldots, n \), then

\[ \Pi_j A = \begin{bmatrix}
\bar{f}_1 \\
\vdots \\
\bar{f}_{j-1} \\
\bar{f}_{j+1} \\
\vdots \\
\bar{f}_n
\end{bmatrix}. \]

(iii) \( \Pi_{n-1} = P_{n,n-1} \).
For a matrix $A$, of size $n \times m$, the Neville elimination process can be written in the following way:

If there are zeros in the first column of $\tilde{A}_1 = A$, the corresponding rows are carried down to the bottom in such a way that the relative order among them is the same as in $\tilde{A}_1$. The new matrix is denoted by $A_1$, that is

$$A_1 = \Pi^1_{n-1} \cdots \Pi^1_1 \tilde{A}_1,$$

where

$$
\Pi^1_j = I_n \quad \text{if} \quad \tilde{a}^1_{j1} \neq 0, \\
\Pi^1_j = \Pi_j \quad \text{if} \quad \tilde{a}^1_{j1} = 0.
$$

In the next step we make zeros in the first column, and we obtain

$$\tilde{A}_2 = E_n(-m_{n1}) \cdots E_2(-m_{21}) A_1.$$

Again, if there are zeros in the second column of $\tilde{A}_2$, the corresponding rows are carried down to the bottom so that the relative order among them is the same as in $\tilde{A}_2$. The new matrix is denoted by $A_2$, that is

$$A_2 = \Pi^2_{n-1} \cdots \Pi^2_1 \tilde{A}_2$$

$$= \prod_{j=2}^{n-1} \Pi^2_j E_n(-m_{n1}) \cdots E_2(-m_{21}) \prod_{j=1}^{n-1} \Pi^1_j A.$$

In general, the method consists of constructing a finite sequence of matrices as follow

$$A = \tilde{A}_1 \longrightarrow A_1 \longrightarrow \tilde{A}_2 \longrightarrow A_2 \longrightarrow \cdots \longrightarrow \tilde{A}_n \longrightarrow U,$$

where $U$ is an upper echelon form matrix. We observe that $\tilde{A}_i = A_i$ when there are not row exchanges, and

$$
\Pi^i_j = I_n \quad \text{if} \quad \tilde{a}^i_{ji} \neq 0, \\
\Pi^i_j = \Pi_j \quad \text{if} \quad \tilde{a}^i_{ji} = 0.
$$

Like [5] we denote by $F_i$ the following lower triangular matrix

$$F_i = \begin{bmatrix}
1 & 0 & 1 & \cdots & \\
0 & 1 & \cdots & \\
\cdots & \cdots & \cdots & \cdots & 1 \\
-m_{i+1,i} & 1 & \cdots & \\
-\cdots & \cdots & \cdots & \cdots & \cdots & 1 \\
E_{i+1,i}(-m_{i+1,i}) \cdots E_{n,n-1}(-m_{ni}) & 1
\end{bmatrix}$$
and

$$K_i = \Pi_{i-1}^i \cdots \Pi_1^i,$$

where $\Pi_j^i$ is defined in (2). So we can establish the following result.

**Theorem 3.1.** Let $A$ be a matrix of size $n \times m$. The Neville elimination process for $A$ can be described as

$$F_{n-1}^n K_{n-1}^n \cdots F^2 K_2 F_1 K_1 A = U,$$

where $K_i, i = 1, 2, \ldots, n - 1$, is a permutation matrix and $F_i, i = 1, 2, \ldots, n - 1$, is a lower triangular matrix.

4. **PLU decomposition of a matrix $A$**

If $A$ is a matrix of size $n \times m$, we are interested to realign, in an adequate way, the matrices $F_i$ and $K_i$ in order to obtain a **PLU** decomposition of $A$, where $P$ is a permutation matrix of size $n \times n$, $L$ is a lower triangular matrix of size $n \times n$ and $U$ is an upper echelon form matrix of size $n \times m$.

We can observe that the elementary matrices $P_{ij}$ and $E_{ij}(\alpha)$ do not satisfy the commutative property, but it is easy to prove the following properties.

**Lemma 4.1.** For any matrices $E_{ij}(\alpha)$ and $\Pi_k$ we have the following statements:

(a) If $k > i$, then $\Pi_k E_{ij}(\alpha) = E_{ij}(\alpha) \Pi_k$.

(b) If $k = i$, then $\Pi_k E_{ij}(\alpha) = E_{nj}(\alpha) \Pi_j$.

(c) If $k < i$, we distinguish:

(c1) If $k = j$, then $\Pi_k E_{ij}(\alpha) = E_{i-1,n}(\alpha) \Pi_j$.

(c2) If $k \neq j$, we have

(*) for $j < k < i$, $\Pi_k E_{ij}(\alpha) = E_{i-1,j}(\alpha) \Pi_k$.

(*) for $j > k$, $\Pi_k E_{ij}(\alpha) = E_{i-1,j-1}(\alpha) \Pi_k$.

Note that we do not deal with the case $k = j$ because we apply the matrix $E_{ij}$ when the $(i, j)$ position is a pivot of the Neville elimination and therefore we do not need to use the matrix $\Pi_j$. So, for $i > j$, $\Pi_k E_{ij}(\alpha) = \Pi_k E_{ij}(\alpha)$, with $l > m$.

Therefore we can realign the matrices $\Pi_k$ and $E_{i-1}(\alpha)$ in order to obtain the desired **PLU** decomposition. We obtain the following result.

**Theorem 4.1.** For a matrix $A$ of size $n \times m$, the Neville elimination process can be described in the following way:

$$E_{i_1j_1}(\alpha_1) \cdots E_{i_nj_n}(\alpha_n) K_{n-1} \cdots K_1 A = U,$$

where $E_{ipjp}(\cdot), p = 1, 2, \ldots, n, i_p > j_p$, is a lower triangular matrix, $K_i, i = 1, 2, \ldots, n - 1$ is a permutation matrix and $U$ is an $n \times m$ upper echelon form matrix.
Proof. If we apply Lemma 4.1 to Theorem 3.1 we obtain this result. □

Taking into account that $E_{ij}(\alpha)^{-1} = E_{ij}(-\alpha)$ and $K_i^{-1}$ is a permutation matrix also, it is easy to prove the following result.

Corollary 4.1. A matrix $A$ of size $n \times m$ can be factorized as $A = P L U$, where $P$ is a permutation matrix of size $n \times n$, $L$ is a lower triangular matrix of size $n \times n$ and $U$ is an upper echelon form matrix of size $n \times m$. 

Proof. According to Theorem 4.1

\[
A = (K_{n-1} \cdot \cdots \cdot K_1)^{-1} (E_{i_1 j_1}(\alpha_1) \cdots E_{i_n j_n}(\alpha_n))^{-1} U
\]

= $K_1^{-1} \cdot \cdots \cdot K_{n-1}^{-1} E_{i_n j_n}(-\alpha_n) \cdots E_{i_1 j_1}(-\alpha_1) U,$

and

$L = E_{i_n j_n}(-\alpha_n) \cdots E_{i_1 j_1}(-\alpha_1),$

$P = K_1^{-1} \cdot \cdots \cdot K_{n-1}^{-1},$

where $L$ is a lower triangular matrix and $P$ is a permutation matrix. □

Example 4.1. Consider the following $4 \times 4$ matrix

\[
A = \begin{bmatrix}
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1
\end{bmatrix}.
\]

By applying, in this order, the elementary matrices $\Pi_2$, $\Pi_2$, $E_{21}(-1)$, $E_{32}(1)$ and $\Pi_3$ we obtain

\[
U = \begin{bmatrix}
1 & 1 & 1 & 0 \\
0 & -1 & -1 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

So,

$\Pi_3 E_{32}(1) E_{21}(-1) \Pi_2 \Pi_2 A = U,$

and by applying Lemma 4.1

$E_{42}(1) E_{21}(-1) \Pi_3 \Pi_2 \Pi_2 A = U,$

and

$A = \Pi_2^{-1} \Pi_2^{-1} \Pi_3^{-1} E_{21}(1) E_{42}(-1) U = PLU.$
Note that when $A$ is a nonsingular matrix and satisfies the WR condition defined in [4,5] the Neville elimination can be performed without row exchanges and we have the factorization $LU$ obtained by Gasca and Peña in [5].

In [5], a class of regular matrices is defined, where the Neville elimination process has a lower computational cost than Gaussian elimination. Now, we extend this result for a class of matrices not necessarily regular. So, if

$$A = \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix},$$

where $L = E_{i+k}(\alpha_{i+k}) \cdots E_i(\alpha_i)$, of size $k \times k$, the Gauss elimination process requires $k(k-1)/2$ steps and $k-1$ for Neville elimination.

**Example 4.2.** Consider the following matrix of size $5 \times 7$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -4 & 4 & 1 & 0 & 0 & 0 & 0 \\ -8 & 8 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$ 

By Gauss elimination process we need the elementary matrix $E_{21}(1)$, $E_{31}(4)$, $E_{41}(8)$, $E_{32}(-4)$, $E_{42}(-8)$ and $E_{43}(-2)$ to obtain

$$U = \begin{bmatrix} I_4 & 0 \\ 0 & 0 \end{bmatrix}.$$ 

By Neville elimination we need $E_{43}(-2)$, $E_{32}(-4)$ and $E_{21}(1)$ to obtain the same matrix $U$.

In general, for matrices whose structure is

$$A = \begin{bmatrix} L & 0 \\ 0 & X \end{bmatrix},$$

where $X = I_p$, $X = 0$, $X = L$, etc., the Neville elimination need less elementary matrices than Gaussian elimination.

**Example 4.3.** Consider the following totally positive matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

which does not satisfy the WR condition. By applying, in this order, the elementary matrices $E_{21}(-1)$ and $P_{32}$, we obtain
The PLU decomposition of $A$ is

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}. $$

We can observe that $U$ is a totally positive matrix but $L$ is not.

The authors intend to apply the results of this paper to generalized totally positive matrices.

References