Differential Inequalities for Functional Perturbations of First-Order Ordinary Differential Equations

J. J. NIETO
Departamento de Análisis Matemático, Facultad de Matemáticas
Universidad de Santiago de Compostela
Santiago de Compostela, 15782 Spain

(Received January 2001; accepted March 2001)

Communicated by R. P. Agarwal

Abstract—The theory of differential inequalities plays a central role in the qualitative and quantitative study of differential equations. In this paper, we present several comparison results for a class of functional differential equations of first order with periodic boundary value conditions. The inequalities obtained are, generally speaking, of the following type: \( P v \leq 0 \) implies that \( v \leq 0 \), where \( P \) is a functional differential operator subject to some boundary conditions, and \( v \) is an element of a prescribed space of functions.

We first obtain several new results for the linear problem. Then, we consider a nonlinear differential equation as a functional perturbation of the original differential equation and give different comparison results. Our results improve and generalize previous estimates described in the literature. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords—Differential inequalities, Comparison result, Functional differential equation, Periodic boundary value problem.

1. INTRODUCTION

There exists extensive literature on the theory of differential inequalities ranging from some classical books [1-4] to some recent and interesting monographs such as, for instance, [5,6].

Comparison results are used to study qualitative properties of differential equations. It includes existence and uniqueness results, existence of positive solutions, regularity of solutions, stability of solutions, and even convergence of numerical methods. It is, therefore, very important to have at our disposal a variety of results relative to differential inequalities.

In this paper, we consider a periodic boundary value problem for a first-order ordinary differential equation subject to functional perturbations of different types. Thus, we study
\[
v'(t) + mv(t) + [p(v)](t) = \sigma(t), \quad \text{a.e. } t \in I = [0, T],
\]
\[
v(0) = v(T) + \lambda. \tag{1}
\]
where $m, \lambda$ are real constants, $T > 0$, $\sigma \in L^1(I)$, and

$$
p : L^1(I) \to L^1(I)
$$

is an application not necessarily linear nor continuous. Problem (1) is then a functional perturbation of the following linear boundary value problem:

$$
v'(t) + mv(t) = \sigma(t), \quad \text{a.e. } t \in I = [0, T],
$$

$$
v(0) = v(T) + \lambda.
$$

We show the validity of several comparison principles relative to problems (1),(2) which are either new or generalizing previous results described in the literature.

2. LINEAR PERIODIC BOUNDARY VALUE PROBLEM

For $m \neq 0$, there exists a unique solution of problem (2) given by

$$
v(t) = \int_0^T g_m(t, s)\sigma(s) \, ds + \lambda h_m(t),
$$

where

$$
g_m(t, s) = \begin{cases}
  \frac{e^{-m(t-s)}}{1 - e^{-mT}}, & 0 < s < t < T, \\
  \frac{e^{-m(T+t-s)}}{1 - e^{-mT}}, & 0 < t < s < T,
\end{cases}
$$

and

$$
h_m(t) = \frac{e^{-mt}}{1 - e^{-mT}}, \quad t \in I.
$$

This unique solution $v \in W^{1,1}(I)$, that is, $v$ is absolutely continuous on $I$. We point out that for $m > 0$ Green’s function satisfies that

$$
9m(t, s) > 0, \quad (t, s) \in I \times I.
$$

In consequence, we have the following comparison principles for the boundary value problem (2):

$$
\lambda \leq 0, \quad \sigma \leq 0, \quad \text{a.e. } I \implies v \leq 0 \text{ on } I,
$$

and

$$
\lambda \geq 0, \quad \sigma \geq 0, \quad \text{a.e. } I \implies v \geq 0 \text{ on } I.
$$

We note that, for $m > 0$, we have the following relations:

$$
\min_{(t,s) \in I \times I} g_m(t, s) = \frac{1}{e^{mT} - 1}, \quad \max_{(t,s) \in I \times I} g_m(t, s) = \frac{1}{1 - e^{-mT}}.
$$

REMARK. If $m < 0$, then $g_m < 0$ on $I \times I$, and, in this case, we have a dual result of (4),(5)

$$
\lambda \leq 0, \quad \sigma \leq 0, \quad \text{a.e. } I \implies v \geq 0 \text{ on } I.
$$

Therefore, we consider only the case $m > 0$ since the results and arguments for $m < 0$ are similar.

If $\lambda > 0$, result (4) is not valid.

EXAMPLE. Let

$$
v(t) = -t + \frac{1}{m}, \quad m > 0.
$$

Thus, $v(0) = 1/m > -T + 1/m = v(T)$ and $v(t) + mv(t) = -1 + m(-t + 1/m) \leq 0$. However, $v(0) > 0$ and obviously $v \leq 0$ on $I$ is not true.

This example shows that, for $\lambda > 0$, we need an additional condition to guarantee that $v \leq 0$ on $I$. In this direction, we note that, for any $\lambda$, if $v(0) \leq 0$, then $v \leq 0$ on $I$. 

Proposition 1. Let $m > 0$, $\lambda \in \mathbb{R}$, and $\sigma \in L^1(I)$ with $\sigma \leq 0$ a.e. $I$. If $v \in W^{1,1}(I)$ is a solution of (2) and $v(0) \leq 0$, then $v \leq 0$ on $I$.

Proof. If $\max_{t \in I} v(t) = v(t_0) > 0$, then $t_0 > 0$ and there exists $t_1 \in (0, t_0)$ such that $v(t) > 0$, $t \in (t_1, t_0)$, $v(t_1) = 0$. Hence, $v'(t) < 0$, a.e. $t \in (t_1, t_0)$, and $v$ is monotone nonincreasing on the interval $(t_1, t_0)$ which is in contradiction with the fact that $v(t_1) = 0$ and $v(t_0) > 0$.

This result gives us several useful consequences. We can consider the case where $\lambda$ is arbitrary and also the situation when $\sigma \leq 0$ a.e. on $I$ is not satisfied, but we still obtain that $v \leq 0$ on $I$, of course, under some additional condition.

Theorem 1. Consider problem (2) with $m > 0$, $\lambda \in \mathbb{R}$, and $\sigma \in L^1(I)$. If

$$\int_0^T e^{-m(T-s)}\sigma(s)\,ds + \lambda \leq 0,$$  \hspace{1cm} (6)

then $v \leq 0$ on $I$.

Proof. Using the integral representation (3) and (6), we have that

$$v(0) = \int_0^T g_m(0, s)\sigma(s)\,ds + \lambda h_m(0) - \frac{1}{1 - e^{-mT}} \left( \int_0^T e^{-m(T-s)}\sigma(s)\,ds + \lambda \right) \leq 0.$$  \hspace{1cm}

Now, using Proposition 1, we obtain that $v \leq 0$ on $I$.

Remark. This theorem generalizes the result of (4) since $\lambda \leq 0$ and $\sigma \leq 0$ a.e. on $I$ imply that condition (6) holds, and hence, $v \leq 0$ on $I$.

It also includes some other results. For instance, the following consequence is precisely Theorem 2.1 in [7].

Corollary 1. Let $m > 0$, $v \in W^{1,1}(I)$, and suppose that there exists $a \in L^1(I)$, $a \geq 0$ a.e. $t \in I$ such that

$$v'(t) + mv(t) + a(t) \leq 0, \quad \text{a.e. } t \in I, v(0) - v(T) = \lambda,$$

and

$$\int_0^T e^{-m(T-s)}a(s)\,ds \geq \lambda.$$  \hspace{1cm} (7)

Then, $v \leq 0$ on $I$.

Proof. For a.e. $t \in I$, we have that $v'(t) + mv(t) = \sigma(t)$ with $\sigma(t) = \xi(t) - a(t)$, $\xi \in L^1(I)$, $\xi \leq 0$ a.e. $t \in I$. Now, condition (7) implies (6), and hence, $v \leq 0$ on $I$.

If $a$ is constant, we then obtain either Lemma 1.2.2(i) of [8] or Corollary 2.1 of [7]. When $a$ is linear, we also generalize results of [7,9].

We now present some other comparison results for problem (2) depending on the sign of $\lambda$.

Theorem 2. Let $m > 0$, $\lambda \leq 0$, and $\sigma \in L^1(I)$. If

$$e^{mT} \int_0^T \sigma^+(s)\,ds \leq \int_0^T \sigma^-(s)\,ds,$$  \hspace{1cm} (8)

then $v \leq 0$ on $I$.

Proof. For any $t \in I$, we have that

$$v(t) \leq \int_0^T g_m(t, s)\sigma(s)\,ds = \int_0^T g_m(t, s)\sigma^+(s)\,ds - \int_0^T g_m(t, s)\sigma^-(s)\,ds$$

$$\leq \frac{1}{1 - e^{-mT}} \left( \int_0^T \sigma^+(s)\,ds - e^{-mT} \int_0^T \sigma^-(s)\,ds \right) \leq 0,$$

since (8) holds.
THEOREM 3. Let $m > 0$, $\lambda > 0$, and $\sigma \in L^1(I)$. If

$$e^{mT} \left( \lambda + \int_0^T \sigma^+(s) \, ds \right) \leq \int_0^T \sigma^-(s) \, ds,$$

then $v \leq 0$ on $I$.

PROOF. For $t \in I$, we see that

$$v(t) \leq \frac{1}{1 - e^{-mT}} \int_0^T \sigma^+(s) \, ds - \frac{e^{-mT}}{1 - e^{-mT}} \int_0^T \sigma^-(s) \, ds + \frac{\lambda e^{-mT}}{1 - e^{-mT}} \leq 0.$$

3. FUNCTIONAL PERTURBATIONS

Let $p : L^1(I) \to L^1(I)$ and consider the functional perturbation (1) of the linear problem (2). We do not impose any further condition on the perturbation $p$. For convenience, we recall here the problem

$$v'(t) + mu(t) + [p(v)](t) = e(t), \quad a.e. \quad t \in [0, T], \quad v(0) = v(T) + \lambda. \quad (10)$$

THEOREM 4. Let $m > 0$, $\lambda \leq 0$, and $\sigma \leq 0$ a.e. on $I$. Suppose that $p$ satisfies the following hypothesis.

There exists a real constant $n$ such that

$$m - n > 0 \quad \text{and} \quad -p(w) \leq nw, \quad a.e. \quad w \in L^1(I). \quad (11)$$

If $v$ is a solution of (10), then $v \leq 0$ on $I$.

PROOF. We can write for a.e. $t \in I$,

$$v'(t) + (m - n)v(t) = \sigma(t) - [p(v)](t) - \lambda v(t) \leq 0.$$

Hence, $v \leq 0$ on $I$ since $m - n > 0$. If (11) does not hold, then the result is not valid.

EXAMPLE. Take $p(w) = -mw$, $\sigma(t) = 0$, $t \in I$. Then, problem (10) is

$$v'(t) = 0, \quad t \in I, \quad v(0) = v(T).$$

Thus, $v(t) = c, \quad t \in I$ is a solution for any constant $c$. In particular, we see that $v \leq 0$ on $I$ is not valid for $c > 0$.

We emphasize that in this last theorem we do not affirm that (10) has a solution neither it is unique. To obtain existence and approximate solutions, we need additional conditions. Results in this direction will appear elsewhere.

Note that the result of Theorem 4 includes the case of linear perturbations

$$p(w) = \mu w, \quad \mu \in \mathbb{R}.$$ 

In this case, we have that problem (10) is

$$v'(t) + (m + \mu)v(t) = \sigma(t), \quad a.e. \quad t \in I, \quad v(0) = v(T) + \lambda,$$

and condition (11) is obviously satisfied if $m + \mu > 0$.

To deal with perturbations of a more general form, we introduce some new conditions. We consider perturbations $p$ verifying

$$p(w) \in C(I), \quad p(w) \in L^\infty(I). \quad (12)$$

For $w \in L^\infty(I)$, we define, as usual, the essential infimum of $w$ on $I$ as the least upper bound of constants $\beta$ such that $w(t) \geq \beta$ a.e. on $I$, and it is denoted by

$$\text{ess inf}_{t \in I} w(t).$$
THEOREM 5. Consider problem (10) where \(\lambda \leq 0, \sigma \leq 0\) a.e. on \(I\), and the functional perturbation satisfies (12) and there exists \(n \geq 0\) such that

\[
\text{for every } \tau \in I, \ w \in C(I), \ \text{ess } \inf_{t \in [0, \tau]} [p(w)](t) \geq n \cdot \min_{t \in [0, \tau]} w(t), \quad (13)
\]

and

\[
\frac{n(e^{m\tau} - 1)}{m} < 1. \quad (14)
\]

If \(v\) is a solution of (10), then \(v \leq 0\) on \(I\).

PROOF. As in the demonstration of Theorem 4, if the conclusion is not valid, then there exists \(t_0 \in I\) such that \(\max_{t \in I} v(t) = v(t_0) > 0\). If \(v(0) < 0\), then we can find \(t_2 \in (0, T]\) with \(v(t) < 0, \ t \in [0, t_2]\), and \(v(t_2) = 0\). Now, choose \(t_3 \in [0, t_2]\) such that

\[
\min_{t \in [0, t_2]} v(t) = v(t_3) < 0.
\]

For a.e. \(t \in I\), we know that \(v'(t) + mv(t) \leq -[p(v)](t)\). In consequence, using (13), for a.e. \(t \in [t_3, t_2]\), we have that

\[
v'(t) + mv(t) \leq -[p(v)](t) \leq \text{ess sup}_{t \in [t_3, t_2]} \{-[p(v)](t)\} \leq \text{ess sup}_{t \in [0, t_2]} \{-[p(v)](t)\} = -\text{ess inf}_{t \in [0, t_2]} [p(v)](t) \leq -n \cdot \min_{t \in [0, t_2]} v(t) = -nv(t_3).
\]

Hence,

\[
[v'(t) + mv(t)]e^{mt} \leq -nv(t_3)e^{mt}, \quad \text{for a.e. } t \in [t_3, t_2].
\]

We integrate this last inequality on the interval \([t_3, t_2]\) to obtain

\[
v(t_2)e^{mt_2} - v(t_3)e^{mt_3} \leq -nv(t_3)e^{mt_3} \quad \frac{e^{mt_2} - e^{mt_3}}{m}.
\]

Taking into account that \(v(t_2) = 0\) and that \(v(t_3) < 0\), we see that

\[
e^{mt_3} \leq \frac{n(e^{mt_2} - e^{mt_3})}{m}
\]

and

\[
1 \leq \frac{n[e^{m(t_2-t_3)} - 1]}{m} \leq \frac{n(e^{mT} - 1)}{m},
\]

which contradicts estimate (14).

Now suppose that \(v(0) \geq 0\). In the case that \(v \geq 0\) on \(I\), we have for a.e. \(t \in I\) that

\[
\text{ess sup}_{t \in I} [p(v)](t) = \text{ess sup}_{t \in I} \{-[p(v)](t)\} \leq -n \min_{t \in I} v(t) \leq 0.
\]

Hence, \(v\) is nonincreasing on \(I\). This implies that \(v\) is constant since \(v(0) \leq v(T)\). Hence, \(v(t) = v(t_0) > 0\), and from (22), we obtain that

\[
0 < mv(t_0) \leq -n \min_{t \in I} v(t) \leq 0,
\]

which is not possible. Therefore, there exists \(t_1 \in (0, T)\) such that

\[
\min_{t \in I} v(t) = v(t_1) < 0.
\]
As before, a.e. on the interval \([t_1, T]\), we can write
\[
    v'(t) + mv(t) \leq -[p(v)](t) \leq \text{ess sup}_{t \in [t_1, T]} \{-[p(v)](t)\} \leq \text{ess sup}_{t \in I} \{-[p(v)](t)\} = -\text{ess inf}_{t \in I} [p(v)](t) \leq -n \cdot \min_{t \in I} v(t) = -nv(t_1),
\]
and
\[
    [v'(t) + mv(t)] e^{mt} \leq -nv(t_1) e^{mt}.
\]
Integrating on the subinterval \([t_1, T]\), we arrive at
\[
    v(T)e^{mT} - v(t_1)e^{mt_1} \leq -nv(t_1)\frac{e^{mT} - e^{mt_1}}{m} \leq -nv(t_1)\frac{e^{mT} - 1}{m}.
\]
Noting that \(v(T) \geq 0\), we have
\[
    -v(t_1) \leq -v(t_1)e^{mt_1} \leq -nv(t_1)\frac{e^{mT} - 1}{m},
\]
which is again a contradiction to (14). This concludes the proof.

Now, consider problem (10) with the perturbation given by
\[
    [p(v)](t) = nv(\theta(t)), \tag{16}
\]
where \(n \geq 0\), and \(\theta : I \rightarrow I\). Note that this perturbation satisfies (12) if, for instance, \(\theta\) is continuous.

In this case, the problem to consider is
\[
    v'(t) + mv(t) + nv(\theta(t)) = \sigma(t), \quad \text{a.e. } t \in I = [0, T],
\]
\[
    v(0) = v(T) + \lambda. \tag{17}
\]

It is evident that if \(\theta\) satisfies
\[
    \theta(t) \leq t, \quad \text{for a.e. } t \in I, \tag{18}
\]
then the functional perturbation given by (16) satisfies (13). Therefore, the result of Theorem 5 is applicable to the functional problem (17). We thus have the following corollary.

**Corollary 2.** Consider problem (17) with \(n \geq 0\), \(\lambda \leq 0\), \(\sigma \leq 0\) a.e. on \(I\), and \(\theta : I \rightarrow I\) such that (12) and (18) hold. Assume that estimate (14) holds. Then any solution \(v\) of (17) satisfies \(v \leq 0\) on \(I\).

In [10], we proved the following maximum principle for problem (17) when \(\sigma \in C(I)\), but the proof is identical for \(\sigma \in L^1(I)\) and we thus omit the details.

**Theorem 6.** (See [10, Theorem 3.1].) Consider problem (17) with \(\theta : I \rightarrow I\) continuous and satisfying (18). Let \(v \in W^{1,1}(I)\) be a solution of (17) with \(\sigma \leq 0\) a.e. on \(I\), and \(\lambda \leq 0\). If
\[
    nT e^{mT} \leq 1, \tag{19}
\]
then \(v \leq 0\) on \(I\).

Thus, Theorem 5 and its Corollary 2 generalize and improve Theorem 6 since estimate (14) is sharper than estimate (19). Indeed, for any \(m > 0\) it is easily seen that
\[
    \frac{1}{Te^{mT}} < \frac{m}{e^{mT} - 1}.
\]
REFERENCES