Marking Estimation of Fuzzy Timed Marked Graphs

Juan Carlos González-Castolo and Ernesto López-Mellado

Abstract—This paper addresses state estimation of discrete event systems (DES) modeled by Fuzzy Timed Petri Nets (FTPN). A definition of FTPN in which fuzzy sets, associated to places, represent the ending time uncertainty of activities is presented; then the fuzzy state equation composed by a set of matrix expressions is developed, allowing computing the marking evolution of DES exhibiting cyclic behavior. Results focusing on marked graphs are discussed and illustrated through an academic example.

I. INTRODUCTION

State estimation of dynamic systems is a resort often used when not all the state variables can be directly measured; observers are the entities providing the system state from the knowledge of its internal structure and its (partially) measured behavior. The problem of discrete event systems (DES) state estimation has been addressed using a sensor based approach [6], [10] in which the marking of a Petri net (PN) model describing a DES is progressively computed from the evolution of its inputs and outputs. Also in [15] the optimal sensor placement is studied. In that works the approximation on the non measurable places may be far from the actual marking.

The state of a systems can be also inferred using the knowledge on the duration of activities. However this task becomes complex when, besides the absence of sensors, the durations of the operations are uncertain; in this situation the observer obtains and revise a belief that approximates the current system state. Consequently this approach is useful for non critical applications of the state monitoring and feedback.

The uncertainty of activities duration in DES can be handled using fuzzy PN (FPN) [5], [3], [14], [2], [4]; this PN extension has been applied to knowledge modeling [7], [9], [12], planning [13], reasoning [8] and controller design [1], [16].

In that works the proposed techniques include the computation of imprecise markings; however the class of models dealt does not include strongly connected PN for the modeling of cyclic behavior. In this paper we address the problem of state estimation of DES by calculating the fuzzy marking of a FTPN when it evolves through p-invariants; the degradation of the estimated marking is characterized, and a set of matrix expressions for the recursive computing the current fuzzy marking is developed.

II. BACKGROUND

A. Possibility Theory

In theory of possibility, a fuzzy set \( \tilde{a} \) is used to delimit ill-known values or for representing values characterized by symbolic expressions. The set is defined as \( \tilde{a} = (a_1, a_2, a_3, a_4) \) such that \( a_1, a_2, a_3, a_4 \in \mathbb{R}^+ \), \( a_1 \leq a_2 \) and \( a_3 \leq a_4 \). The fuzzy set \( \tilde{a} \) delimits the run time as follows:

- The ranges values \( \tilde{a}_0 \in (a_1, a_2) \) and \( \tilde{a}_e \in (a_3, a_4) \) indicate that the activity is possibly executed: \( \alpha(\tau) \in (0, 1) \).
- When \( \tau \in \tilde{a}_0 \), the function \( \alpha(\tau) \) grows towards 1, which means that the possibility of stopping increases. When \( \tau \in \tilde{a}_e \), the membership function \( \alpha(\tau) \) decreases towards 0, representing that there is a reduction of the possibility of stopping.
- The values \( (0, a_1] \) mean that the activity is running.
- The values \( [a_4, +\infty) \) mean that the activity is stopped.
- The values \( \tilde{a} \in [a_2, a_3] \) represent full possibility, that is \( \alpha(\tau) = 1 \), represents that is certain that activity is stopped.

A fuzzy set \( \tilde{a} \) is referred indirectly by the function \( \alpha(\tau) \) or the characterization \( (a_1, a_2, a_3, a_4) \). For simplicity, the fuzzy possibility distribution of the time is described with a trapezoidal form or triangular form.

Definition 1: Let \( \tilde{a} \) and \( \tilde{b} \) be two trapezoidal fuzzy sets where \( \tilde{a} = (a_1, a_2, a_3, a_4) \) and \( \tilde{b} = (b_1, b_2, b_3, b_4) \). The fuzzy sets addition operation as: \( \tilde{a} \oplus \tilde{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4) \) [17].

Definition 2: Let \( x, y \in X \). If \( x \leq (\geq) y \), then \( x \) is called the minimum (maximum) of \( X \) with respect to the relation \( \leq (\geq) \). The \( \min(\max) \) operator gets the minimum (maximum) \( x \) of \( X \).

Definition 3: The distribution of possibility before and after \( \tilde{a} \) are the fuzzy sets \( \tilde{a}^b = (-\infty, a_2, a_3, a_4) \) and \( \tilde{a}^a = (a_1, a_2, a_3, +\infty) \), respectively; they are defined in [1] as a function \( \alpha_{-\infty,a_1}(\tau) = \sup(\tau) \) and \( \alpha_{a_1,+\infty}(\tau) = \sup(\tau) \), respectively.

B. Petri Nets

Definition 4: An ordinary PN structure \( G \) is a bipartite digraph represented by the 4-tuple \( G = (P, T, I, O) \) where
A. Basic Operators

The symbol \( t_i(t_f) \) denotes the set of all places \( p_i \) such that \( I(p_i, t_j) \neq 0 \) (or \( O(p_i, t_j) \neq 0 \)). Analogously, \( p_i(p_o) \) denotes the set of all transitions \( t_j \) such that \( O(p_i, t_j) \neq 0 \) (or \( I(p_i, t_j) \neq 0 \)).

The pre-incidence matrix of \( G \) is \( C^- = [c^-_{ij}] \) where \( c^-_{ij} = I(p_i, t_j) \); the post-incidence matrix of \( G \) is \( C^+ = [c^+_{ij}] \) where \( c^+_{ij} = O(p_i, t_j) \); the incidence matrix of \( G \) is \( C = C^- \cdot C^+ \).

A marking function \( M : P \rightarrow \mathbb{Z}^+ \) represents the number of tokens (depicted as dots) residing inside each place. The marking of a PN is usually expressed as an n-entry vector.

Definition 15: A fuzzy timed Petri net structure is a 3-tuple \( FTPN = (N, \Gamma, \xi) \); where \( N = (G, M_0) \) is a PN structure and \( M_0 \) is an initial token distribution.

In a PN system, a transition \( t_j \) is enabled at marking \( M_k \) if \( \forall p_i \in P, \ M_k(p_i) \geq I(p_i, t_j) \); an enabled transition \( t_j \) can be fired reaching a new marking \( M_{k+1} \) which can be computed as,

\[
M_{k+1} = M_k + C^+ v_k - C^- v_k
\]

where \( v_k(i) = 0, \ i \neq j, v_k(j) = 1 \), this equation is called the PN state equation.

The reachability set of a PN is the set of all possible reachable marking from \( M_0 \) firing only enabled transitions; this set is denoted by \( R(G, M_0) \).

A structural conflict is a PN sub-structure in which two or more transitions share one or more input places; such transitions are simultaneously enabled and the firing of one of them may disable the others (Fig.2(b)).

Definition 10: Let \( Y \) be the set of minimal \( p \)-invariants [11] of a PN, then \( Z \) is called the invariants base. The cardinality of \( Y \) is represented as \( |Y| \).

III. FUZZY TIMED PETRI NETS

A. Basic Operators

We introduce first some useful operators.

Definition 11: The \( f_{\min} \) and \( f_{\max} \) denote the minimum and maximum operators over fuzzy sets and they obtains the maximum common and maximum, respectively, \( \alpha(\tau) \) among fuzzy sets i.e.: \( f_{\min}(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n) = \max(\alpha(\tau)) \) where \( \alpha(\tau) \in \tilde{\alpha}_j; i = 1, \ldots, n \); \( f_{\max}(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n) = \max(\alpha(\tau)) \) where \( \alpha(\tau) \in \tilde{\alpha}_i; i = 1, \ldots, n \).

Definition 12: The intersection and union of fuzzy sets are defined in terms of \( f_{\min} \) and \( f_{\max} \) as, \( \tilde{\alpha} \cap \tilde{\beta} = f_{\min}(\tilde{\alpha}, \tilde{\beta}) \) and \( \tilde{\alpha} \cup \tilde{\beta} = f_{\max}(\tilde{\alpha}, \tilde{\beta}) \), respectively.

Definition 13: In order to get the fuzzy set between \( \tilde{\alpha} \) and \( \tilde{\beta} \) borders, the \( l_{\max} \) function is defined as,

\[
l_{\max}(\tilde{\alpha}, \tilde{\beta}) = f_{\min}(\tilde{\alpha}^b, \tilde{\beta}^b) \tag{2}
\]

Definition 14: The latest(earliest) operation picks the latest(earliest) fuzzy set among \( n \) fuzzy sets; they are calculated as:

\[
l_{\max}(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n) = f_{\max}\{f_{\max}\{f_{\max}\{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n\}, f_{\max}\{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n\}, \ldots, f_{\max}\{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n\}\}\} \tag{3}
\]

\[
l_{\min}(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n) = f_{\min}\{f_{\min}\{f_{\min}\{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n\}, f_{\min}\{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n\}, \ldots, f_{\min}\{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n\}\}\} \tag{4}
\]

Definition 15: The fuzzy conjugation operator is defined as, \( arg1 \cdot op \cdot arg2 \) where \( arg1, arg2 \) are arguments that can be matrices of fuzzy sets. The \( \cdot \) is a fuzzy and operation and \( op \) is any operation referred as, \(+, -, \cdot, >=, \cdot, \cdot, \cdot, \cdot\). For some file \( i = 1, \ldots, m \) and some column \( j = 1, \ldots, n \) the products and \( (\tilde{\alpha}_{ik}, \tilde{\beta}_{kj}) \) \( k = 1, \ldots, r \) are computed as \( op = (\tilde{\alpha}_{ik}, \tilde{\beta}_{kj}) \). For example:

\[
\begin{bmatrix}
\tilde{\alpha}_{11} & \cdots & \tilde{\alpha}_{1r} \\
\vdots & \ddots & \vdots \\
\tilde{\alpha}_{mr} & \cdots & \tilde{\alpha}_{rn}
\end{bmatrix}
\begin{bmatrix}
\tilde{\beta}_{11} & \cdots & \tilde{\beta}_{1n} \\
\vdots & \ddots & \vdots \\
\tilde{\beta}_{rn}
\end{bmatrix}
= 
\begin{bmatrix}
\sum_{k=1}^{r} \tilde{\alpha}_{ik}\tilde{\beta}_{kj} & \cdots & \sum_{k=1}^{r} \tilde{\alpha}_{ik}\tilde{\beta}_{kn} \\
\vdots & \ddots & \vdots \\
\sum_{k=1}^{r} \tilde{\alpha}_{mk}\tilde{\beta}_{kj} & \cdots & \sum_{k=1}^{r} \tilde{\alpha}_{mk}\tilde{\beta}_{kn}
\end{bmatrix}
\]

B. FTPN Definition

Definition 16: A fuzzy timed Petri net structure is a 3-tuple \( FTPN = (N, \Gamma, \xi) \); where \( N = (G, M_0) \) is a PN, \( \Gamma = \{\partial_1, \partial_2, \ldots, \partial_n\} \) is a collection of fuzzy sets, \( \xi : P \rightarrow \Gamma \) is a function that associates a fuzzy set \( \tilde{\alpha}_i \) to each place \( p_i \in P; i = 1, \ldots, |P| \).

Fuzzy timing of places.

The fuzzy set \( \tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) \) Fig.1(b) represents the static possibility distribution \( \alpha(\tau_0) \in [0, 1] \) of the instant at which a token leaves a place \( p \in P \), starting from the instant when \( p \) is marked. This set does not change during the FTPN execution.

Fuzzy timing of tokens.

The fuzzy set \( \tilde{b} = (\tilde{b}_1, \tilde{b}_2, \tilde{b}_3, \tilde{b}_4) \) Fig.1(c) represents the dynamic possibility distribution \( \beta(\tau_0) \in [0, 1] \) associated to a token residing within a \( p \in P \); it also represents the instant \( \tau_0 \) at which such a token leaves the place, starting from the instant when \( p \) is marked. \( \tilde{b} \) is computed from \( \tilde{a} \) every time the place is marked during the marking evolution of the FTPN.

A token begins to be available for enabling output transitions at \( \beta(\tilde{b}_1) \). Thus \( \tilde{b}^{*} = (\tilde{b}_1, \tilde{b}_2, \tilde{b}_3, +\infty) \) represents the possibility distribution of available tokens.
The fuzzy set \( \tilde{c} = (c_1, c_2, c_3, c_4) \), known as fuzzy timestamp, Fig. 1(d) is a dynamic possibility distribution \( \zeta(\tau_i) \in [0, 1] \) that represents the duration of a token within a place \( p \in P \).

### C. Enabling and firing of transitions.

#### Fuzzy enabling date.

The fuzzy enabling time \( e_h(\tau) \) of a transition \( t_k \) is a possibility distribution of the latest leaving instant among the leaving instants \( \tilde{b}_{p_i} \) of all tokens within the \( p_i \in \tilde{M} \). Fig.2(a).

\[
e_h(\tau) = \text{latest} \{ \tilde{b}_{p_i} \mid p_i \in \tilde{M} \}
\]

The \( \text{latest} \) operation obtains the latest date in which the input places \( p_i \) to \( t_k \) have a mark.

#### Fuzzy firing date.

The firing transition date \( o_h(\tau) \) of a transition \( t_k \) is determined with respect to the set of transitions \( \{t_j\} \) simultaneously enabled, Fig.2(b). This date, expressed as a possibility distribution, is computed as follows

\[
o_h(\tau) = \text{fmin} \{ e_h(\tau), \text{earliest} \{ e_j(\tau) \} \mid t_k \in \tilde{M} \}
\]

The \( \text{earliest} \) operation obtains the earliest date in which the transitions in a structural conflict are enabled.

#### Fuzzy timestamp.

For a given place \( p_s \), the possibility distribution \( \tilde{b}_{p_s} \) may be computed from \( \tilde{a}_{p_s} \) and the firing dates \( o_j(\tau) \) of a \( t_j \in \tilde{M} \), using the following expression: (see Fig.2(c)(d))

\[
\tilde{b}_{p_s} = \text{lmax} \{ o_j(\tau) \oplus \tilde{a}_{p_s} \mid t_j \in \tilde{M} \}
\]

The tokens do not disappear of \( \tilde{M} \) and appear in \( \tilde{M} \) instantaneously. The fuzzy timestamp \( \tilde{c}_{p_s} \) is the time elapse possibility that a token is in a place \( p_s \in \tilde{M} \). The possibility distribution \( \tilde{c}_{p_s} \) is computed from the occurrence dates of both \( \tilde{p}_s \) and \( \tilde{p}_s' \), Fig.2(e):

\[
\tilde{c}_{p_s} = \text{lmax} \{ \text{earliest} \{ o_h(\tau) \}, \text{latest} \{ o_j(\tau) \} \mid t_k \in \tilde{M}, t_j \in \tilde{M} \}
\]

### D. Marking matrix formulation

Now, we present the generalized equations that they are easy manipulated. The available tokens equation (7) is expressed as

\[
\tilde{B} = \left( C^T \text{lmax} \cdot \tilde{O} \right) \oplus \tilde{A}
\]

where, first the \( \tilde{O} \) sets are computed by \( \text{lmax} \) operation and after the \( \tilde{A} \) sets are added.

The fuzzy enabling transition date, equation (5), is generalized as follows

\[
\tilde{E} = \left( C^T \text{fmin} \cdot \tilde{E} \right)
\]

Then the matrix equation of fuzzy firing transition date derived from (6) is presented

\[
\tilde{Q} = \left( C^T \text{lmax} \cdot \tilde{Q} \right)
\]

The fuzzy enabling times sets of output transitions from each place are computed by \( \text{earliest} \) operation and the fuzzy enabling time of transitions in structural conflict are computed by \( \text{min} \) operator.

The equation (8) will be generalized in three steps. First, the earliest date of \( \tilde{Q} \) are computed from transitions to places input \( \left( C^T \text{earliest} \cdot \tilde{Q} \right) \). Second, the latest date of \( \tilde{O} \) are computed from places to transitions output \( \left( C^T \text{latest} \cdot \tilde{O} \right) \). Finally, these solutions are computed by \( \text{lmax} \) operation.

\[
\tilde{C} = \text{lmax} \left( C^T \text{earliest} \cdot \tilde{Q}, C^T \text{latest} \cdot \tilde{O} \right)
\]
E. Modeling example

Example 17: Consider the system shown in Fig. 3(a); it consists of two cars, car1 and car2, which move along independent sequences executing the set of activities \( Op = \{ \text{Right Car1, RightCar2, ChargeCar1, LeftCar12, DischargeCar12} \} \). The operation of the system is automated following the sequence described in the PN of Fig. 3(b) in which the activities are associated to places \( p_2, p_3, p_4, p_5, p_1 \) respectively. The ending time possibility \( \tilde{a}_p \) for every activity is given in the model.

Considering that there are not sensors detecting the activities in the system, the behavior is then analyzed through the estimated state.

a) Initial conditions: Initially \( M_0 = \{ p_1 \} \), therefore, the enabling date \( e_p(t) \) of transition \( t_1 \) is immediate i.e. \( (0,0,0,0) \). Since \( t_1 = 1 \) then \( a_{t_1} = e_{t_1}(t) \). The pre/post-incidence matrices are given below

\[
\begin{align*}
C^+ & = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1
\end{bmatrix} \\
C^- & = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1
\end{bmatrix}
\end{align*}
\]

The zeros are replaced by points with the purpose to do notorious the values.

b) The generalized equations: For the obtained \( \tilde{B} \), we solve the equation (9) as follow

\[
\tilde{B} = (C^+ \bullet O) \oplus \tilde{A}
\]

The enabling fuzzy time generalized equation is show,

\[
\tilde{E} = \left[ C^T \right]^{T \text{ latest}} \bullet \tilde{B}
\]

We notes that \( t_1 \) is an attribution transition and therefore it is computed as \( \text{latest} (\tilde{b}_{p_3}, \tilde{b}_{p_4}) \). The \( \tilde{O} \) is obtained with the equation 11 as follow,

\[
\tilde{O} = \left[ C^T \right]^{T \text{ min}} \left( C^- \bullet \tilde{E} \right)
\]

here, we observed that \( \tilde{O} \) coincides with \( \tilde{E} \). Finally, the fuzzy timestamp are computed by,

\[
\tilde{C} = \text{I_{max}} \left( C^+ \bullet \tilde{O}, C^- \bullet \tilde{O} \right)
\]

c) Firing \( t_1 \): When \( t_1 \) is fired, the token is removed from \( p_1 \); \( p_2 \) and \( p_3 \) get one token each one. The possibility sets \( \tilde{b}_{p_2}, \tilde{b}_{p_3} \) represent the end of activities \( \text{RightCar1} \) and \( \text{RightCar2} \), respectively, and they coincide with \( \tilde{a}_{p_2} \) and \( \tilde{a}_{p_3} \) respectively; therefore:

\[
\tilde{B} = \begin{bmatrix}
. & (0.9,1,1,1,1) & (0.8,1,1,1,2) \end{bmatrix}^T
\]

d) Firing \( t_2 \): When \( \text{RightCar1} \) is finishing, \( t_2 \) is being enabled. \( e_{t_2}(\tau) \) is a possible date for that \( \text{car1} \) is able to perform \( \text{ChargeCar1} \). The \text{enabling fuzzy time} and the \text{occurrence fuzzy time} are computed by (14) and (15) equations respectively.

\[
\tilde{E}, \tilde{E} = \begin{bmatrix}
. & (0.9,1,1,1,1) \end{bmatrix}
\]

The set \( \tilde{c}_{p_2} \) is the possibility distribution of the time at which \( \text{RightCar1} \) is executing. So, we computed the equation (16).

\[
\tilde{C} = \begin{bmatrix}
. & (0,0,1,1,1)
\end{bmatrix}
\]

The set \( \tilde{c}_{p_4} \) is the possibility distribution of the instant at which \( \text{car1} \) finishes \( \text{ChargeCar1} \) and it can be calculated by equation (13).

\[
\tilde{B} = \begin{bmatrix}
. & . & (2.6,3,3,3,4) \end{bmatrix}
\]

e) Firing \( t_3 \): When \( \text{ChargeCar1} \) and \( \text{RightCar2} \) are finishing, the transition \( t_3 \) is being enabled. \( e_{t_3}(\tau) \) is a possible date for that \( \text{car1} \) and \( \text{car2} \) may execute \( \text{LeftCar12} \).

\[
\tilde{O}, \tilde{E} = \begin{bmatrix}
. & . & (2.6,3,3,3,4)
\end{bmatrix}
\]

The execution of \( \text{ChargeCar1} \), is described by \( \tilde{c}_{p_3} \) and \( \tilde{c}_{p_1} \) describes the execution of \( \text{RightCar2} \); they are computed as

\[
\tilde{C} = \begin{bmatrix}
. & (0,0,3,3,4) & (0.9,1,3,3,4)
\end{bmatrix}
\]

Fig. 3. (a) Two cars system. (b) Fuzzy Petri net.
The set $\hat{b}_{ps}$ is the possibility distribution of the time at which car1 and car2 finish LeftCar12; it can be obtained by:

$$\hat{B} = \begin{bmatrix} 4.1, 5, 5, 9 \end{bmatrix}^T$$

f) Firing $t_4$: When LeftCar12 is finishing, the transition $t_4$ is being enabled. $\hat{\epsilon}_{ps}(t)$ is a possible date for that car1 and car2 are able to perform Discharge Car12. The set $\hat{b}_{ps}$ is the possibility distribution of the time at which LeftCar12 is executing. The set $\hat{b}_{ps}$ is the possibility distribution of the time at which car1 and car2 finish DischargeCar12.

$$\hat{O}, \hat{E} = \begin{bmatrix} 4.1, 5, 5, 9 \end{bmatrix}^T$$

$$\hat{C} = \begin{bmatrix} (4.1, 5, 5, 9) \end{bmatrix}^T$$

$$\hat{B} = \begin{bmatrix} 5.8, 7, 7, 8.2 \end{bmatrix}^T$$

g) Firing $t_5$: When Discharge Car12 is finishing, $t_5$ is being enabled. $\hat{\epsilon}_{ps}$ is the possibility distribution of the time at which Discharge Car12 is executing.

$$\hat{O}, \hat{E} = \begin{bmatrix} 5.8, 7, 7, 8.2 \end{bmatrix}^T$$

$$\hat{C} = \begin{bmatrix} (4.1, 5, 5, 9) \end{bmatrix}^T$$

The Fig.4 present the marking evolution of one cycle and some steps.

IV. FUZZY STATE EQUATION OF MARKED GRAPHS

We analyzed the equation (1) in order to obtain the fuzzy marking equation. The $C^+ V_k$ provides information about the places that get tokens. Also, we must consider that in FTPN the transition firing possibility evolves continuously. The variation of $\hat{O}(\hat{\tau}_d)$ during $\tau \in \hat{\tau}_d$ modifies the possibility of tokens residing in the output places of the firing transitions; thus the corresponding term to $V_k$ in FTPN is rather such a variation denoted by $C^+ O(\hat{\tau}_d)$, thus the marking variation is $C^+ V_k$. By a similar reasoning on the term $C^V_k$ corresponds to $C^- O(\hat{\tau}_d)$ in FTPN. The operation $C^+ V_k$ and $C^- O(\hat{\tau}_d)$ represent the possible marking change. Considering the marking $\Delta \tau$ instants before we obtain

$$M(\tau) = M(\tau - \Delta \tau) + C^+ V_k - C^- O(\hat{\tau}_d)$$

(17)

Here $\Delta O(\hat{\tau}_d) = \hat{O}(\tau) - \hat{O}(\tau - \Delta \tau)$ for $\tau \in \hat{\tau}_d$ and $\Delta O(\hat{\tau}_d) = \hat{O}(\tau - \Delta \tau) - \hat{O}(\tau)$ for $\tau \in \hat{\tau}_d$. The marking possibility obtained in (17) can be greater than 1; then since FTPN are safe, we use the $\min$ function to obtain $M(\tau) \leq 1$.

The new marking is denoted by $\hat{M}(\tau)$, i.e.,

$$\hat{M}(\tau) = \min \left\{ M(\tau - \Delta \tau) + C^+ V_k - C^- O(\hat{\tau}_d), 1 \right\}$$

(18)

At the beginning $M(0) = M_0$. If $\tau \neq 0$ then equation (18) is solved in three steps in order to illustrate the procedure.

- $\hat{M}(\tau) = C^+ \Delta O(\hat{\tau}_d) - C^- \Delta O(\hat{\tau}_d)$
- $M(\tau) = M(\tau - \Delta \tau) + \hat{M}(\tau)$
- $\hat{M}(\tau) = \min \left\{ M(\tau), 1 \right\}$

Remark 18: If $\Delta O(\hat{\tau}_d) = \Delta O(\hat{\tau}_d) = 0, 1$ then we have the ordinary timed marked Graph.

Example 19: For the system shown in Fig.3, we obtained the marking in some instants. The initially marking $M(0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}$. The transitions $t_1$ is firing at $\tau = 0^+$ therefore $M(0^+) = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \end{bmatrix}$. For $\tau \in (0^+, 0.9)$ the marking don’t change. For $\tau = 1$ the detail computed is shown,

$$\hat{M}(1) = \begin{bmatrix} C^+ \Delta O(\hat{\tau}_d) - C^- \Delta O(\hat{\tau}_d) \\ C^+ \Delta O(\hat{\tau}_d) - C^- \Delta O(\hat{\tau}_d) \\ C^+ \Delta O(\hat{\tau}_d) - C^- \Delta O(\hat{\tau}_d) \end{bmatrix}$$

$$M(1) = M(0) + \hat{M}(1)$$

$$\hat{M}(1) = \min \left\{ M(1), 1 \right\}$$

The change in the marking evolution is shown at some relevant instants in the next table:

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<th>$\tau$</th>
<th>0</th>
<th>0.9</th>
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<th>2.8</th>
<th>3</th>
<th>3.2</th>
<th>4</th>
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<td>1</td>
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The dashed line in Fig.4 represent the fuzzy marking evolution $\hat{M}(\tau)$. The Fig.5 show the firing possibility of transitions $\hat{O}$.

V. STATE ESTIMATION OF THE FTPN

A. Marking estimation

Definition 20: The marking estimation $\Xi$ in the instant $\tau$ is described by the function $\psi_i(\tau) \in [0, 1]$ which recognize the possible marked place $p_i \in \|Y_i\| \in \{1, ..., |Y|\}$, among
other possible places \( p_v \in ||Y|| \) \( v \neq u \). The function \( \psi_l(Y) \) is a value that indicates the minimal difference that exist among the bigger possibility that the token is in a place \( (\tilde{M}_p(\tau)) \) and the possibility that token is in any other place \( (\tilde{M}_l(\tau)) \). The function \( \psi_l(\tau) \) is calculated as,

\[
\psi_l(\tau) = \min \{ ||\tilde{M}_p(\tau) - \tilde{M}_l(\tau)|| \} \quad \forall\{p, p_1\} \in ||Y|| : v \neq u; Y \in Y
\]

In the previous definition \( \tilde{f}_{p}(\tau) - \tilde{g}_{p}(\tau) \neq 0 \) for any time, since it is always possible to find a token in some place.

Example 21: The FTPN in Fig.3(b) has two \( p \)-

invariants with \( ||Y_1|| = \{p_1, p_2, p_3, p_5\} \) and \( ||Y_2|| = \{p_1, p_3, p_5\} \). The Fig.6 shows the fuzzy sets \( C \) obtained from evolution of the marking in the \( p \)-invariant corresponding to \( Y_1 \). This evolution shows the first cycle and some next steps of other cycle. In order to obtain the activity estimation that the \( carl \) is executing, we need obtain the marking estimation \( (\psi_l(\tau)) \). During the time elapse \( \tau \in [0,0.9] \), it is observed that \( \psi_l(\tau) = \tilde{f}_{p}(\tau) \), because there exists not another \( \tilde{g}_{p}(\tau) \neq 0 \) indicating that the token exists in another place; in this case, \( p_2 \) is certainly marked. For \( \tau \in (0.9,1) \) the possibility that the place \( p_2 \) is marked is one. However, the possibility that the place \( p_2 \) is marked is increased; therefore when \( \tilde{f}_{p}(\tau) \) is increased, then \( \psi_l(\tau) \) is reduced. In \( \tau = 1 \), there exist absolute possibility that the token is in \( p_2 \) and \( p_4 \). In this case it is not possible to know where is the token, therefore \( \psi_l(\tau) = 0 \). When \( \tau \in [1,1.1] \), \( \tilde{f}_{p}(\tau) = 1 \) and \( \tilde{g}_{p}(\tau) \) is reduced, we obtain that \( \psi_l(\tau) \) is increased. Finally, we will see that for \( \tau \in [1.1,2.6] \), \( \psi_l(\tau) = \tilde{g}_{p}(\tau) \); it is absolutely possible that the \( p_3 \) is marked.

VI. CONCLUSION.

This paper addressed the state estimation problem of \( DES \) whose the duration of activities is ill known; fuzzy sets represent the uncertainty of the ending of activities. Several novel notions have been introduced in the FTPN definition, and a new matrix formulation for computing the fuzzy marking of marked graphs has been proposed. The extreme situation in which any activity of a system cannot be detected by sensors has been dealt for illustrating the degradation of the marking estimation when a cyclic execution is performed. The inclusion of sensors in the FTPN recovers the uncertainty to zero for a given path within the model; current research addresses the optimal placement of sensors in the system in order to keep bounded the uncertainty of the marking for any evolution of the system.

REFERENCES