State Uncertainty Analysis of Fuzzy Timed DES

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Abstract—This paper addresses state estimation of discrete event systems (DES) exhibiting variations on the duration of activities; the system state is approximated by the marking of Fuzzy Timed Petri Net (FTPN) models. For this purpose a definition of FTPN is presented in which fuzzy sets associated to places represent the ending time uncertainty of activities. A fuzzy marking matrix equation is proposed, allowing computing the marking evolution of cyclic FTPN. Finally, the uncertainty of the estimated marking is evaluated as a measure of quality on the approximation.

I. INTRODUCTION

State estimation of discrete event systems (DES) has been addressed using a sensor based approach [1], [2] in which the marking of a Petri net (PN) model describing a DES is progressively computed after a finite number of event occurrences.

The state of a system can be also inferred from the knowledge of the duration of activities. However, this task becomes complex when, besides the absence of sensors, the duration of the operations is variable. In this situation the observer obtains and revises a belief that approximates the current system state. Consequently, this approach of state monitoring is useful for non critical applications in which an approximated computation of the state is sufficient.

The uncertainty of activities duration in DES can be handled using fuzzy PN (FPN) [4], [5], [6]; this PN extension has been applied to knowledge modeling [7], [8], planning [9], reasoning [10], and controller design [11].

In these works the proposed techniques include the computation of imprecise markings. However, the classes of models dealt do not include PN structures describing repetitive behavior nor decisions (structural conflicts). Also the quality of the marking estimation has not been analyzed.

In this paper we address the problem of state estimation of DES that exhibit variations on the duration of activities, by approximating the marking of a cyclic FTPN model. We consider that none of activities can be measured by sensors. A FTPN definition is presented in which the ending time uncertainty of activities is expressed with fuzzy sets associated to places. Several novel notions and operations are introduced to develop a fuzzy marking matrix equation, which allows the computation of the marking evolution of cyclic FTPN whose structure may include synchronizations or decisions. Previous results devoted to marked graphs [12] and state machines [13] are embedded into a more general state estimation technique based on a new matrix equation; additionally, the degradation of the marking estimation is analyzed and computed.

The remainder of this paper is structured as follows. In the next section, theories of fuzzy sets and Petri nets are overviewed. In section III the FTPN definition is introduced, and a new matrix formulation of dynamic parameters concerning enabling and firing of transitions is proposed. Section IV presents the procedure for state estimation in cyclic FTPN whose structure includes synchronizations and/or decisions. Finally, in section V the uncertainty of the estimated marking is analyzed and evaluated.

II. BACKGROUND

A. Possibility Theory

In possibility theory, a fuzzy set \( \tilde{a} \) is used to delimit ill-known values or to represent values characterized by linguistic variable expressions. The fuzzy set \( \tilde{a} \) in \( \mathfrak{F} \) is characterized by a membership function \( \alpha_\tilde{a}(\tau) \) which associates to each point \( \tau \) in \( \mathfrak{F} \) a real number in the interval [0,1]; the value \( \alpha_\tilde{a}(\tau) \) represents the "degree of membership" of \( \tau \) in \( \tilde{a} \) [14].

A fuzzy set can be defined in a trapezoidal form; thus a fuzzy set \( \tilde{a} \) can be characterized as \( \tilde{a} = (a_1,a_2,a_3,a_4) \) such that \( a_1,a_2,a_3,a_4 \in \mathbb{R} \), where \( (a_2,a_3) \) and \( (a_1,a_4) \) are the core and support of \( \tilde{a} \), respectively. \( \tilde{a}^\uparrow = (a_1,a_2) \) denotes a subset of \( \tilde{a} \) where the values \( \alpha_\tilde{a}(\tau) \) grow towards 1; similarly \( \tilde{a}^\downarrow = (a_3,a_4) \subseteq \tilde{a} \) denotes the decreasing values of \( \alpha_\tilde{a}(\tau) \). Figure 1(a) illustrates these notions.

\begin{definition}
Let \( \tilde{a} = (a_1,a_2,a_3,a_4) \) and \( \tilde{b} = (b_1,b_2,b_3,b_4) \) be two trapezoidal fuzzy sets. The fuzzy sets addition operation is: \( \tilde{a} \oplus \tilde{b} = (a_1+b_1,a_2+b_2,a_3+b_3,a_4+b_4) \) [15].
\end{definition}

\begin{definition}
The intersection and union of fuzzy sets are defined in terms of \( \min \) and \( \max \). \( \tilde{a} \cap \tilde{b} = \min(\tilde{a},\tilde{b}) = \min(\alpha_\tilde{a}(\tau),\alpha_\tilde{b}(\tau)) \) such that \( \tau \in \text{support of } \tilde{a} \cap \tilde{b} \) and \( \tilde{a} \cup \tilde{b} = \max(\tilde{a},\tilde{b}) = \max(\alpha_\tilde{a}(\tau),\alpha_\tilde{b}(\tau)) \) such that \( \tau \) belongs to the support of \( \tilde{a} \cup \tilde{b} \), where the \( \min(\max) \) operator gets the minimum (maximum) \( \tau \) of \( \mathfrak{F} \). We use these intersection and union operators as a t-norm and a s-norm, respectively. The null element for \( \min(\max) \) operation is 0 [13].
\end{definition}

\begin{definition}
The distribution of possibility before and after \( \tilde{a} \) are the fuzzy sets \( \tilde{a}^\uparrow = (-\infty,a_2,a_3,a_4) \) and \( \tilde{a}^\downarrow = (a_1,a_2,a_3,\infty) \), respectively. They are defined in [3] as a function \( \alpha_{(-\infty,a]}(\tau) = \sup_{\tau \leq \tau} \alpha(\tau) \) and \( \alpha_{[a,\infty]}(\tau) = \sup_{\tau \leq \tau} \alpha(\tau) \), respectively (see Fig.1(c),(d)).
\end{definition}
B. Petri Nets

Definition 4: An ordinary PN structure $G$ is a bipartite digraph represented by the 4-tuple $G = (P,T,I,O)$ where $P = \{p_1,p_2,...,p_m\}$ and $T = \{t_1,t_2,...,t_n\}$ are finite sets of vertices called respectively places and transitions, $I(O) : P \times T \to \{0,1\}$ is a function representing the arcs going from places to transitions (transitions to places).

Places are depicted as circles, transitions are represented by bars, and arcs are depicted as arrows. The symbol $\bullet t_j(p_i)$ denotes the set of all places $p_i$ such that $I(p_i,t_j) \neq 0$ \((O(p_i,t_j) \neq 0).\) Analogously, $\bullet p_i(t_j)$ denotes the set of transitions $t_j$ such that $O(p_i,t_j) \neq 0 \((I(p_i,t_j) \neq 0).\)

The pre-incidence matrix of $G$ is $C^- = [c^-_{ij}]$ where $c^-_{ij} = I(p_i,t_j);$ the post-incidence matrix of $G$ is $C^+ = [c^+_{ij}]$ where $c^+_{ij} = O(p_i,t_j);$ the incidence matrix of $G$ is $C = C^+ - C^-.$

A marking function $M : P \to \mathbb{Z}^+$ gives the number of tokens (depicted as dots) residing inside each place. The marking of a PN is usually expressed as an n-entry vector.

Definition 5: A PN system is the pair $N = (G,M_0),$ where $G$ is a PN structure and $M_0$ is an initial token distribution.

In a PN system, a transition $t_j$ is enabled at the marking $M_k$ if $\forall p_i \in P,M_k(p_i) \geq I(p_i,t_j);$ an enabled transition $t_j$ can be fired reaching a new marking $M_{k+1}$ which can be computed using the PN state equation:

$$M_{k+1} = M_k + C^+ v_k - C^- v_k \quad (1)$$

where $v_k(i) = 0, i \neq j, v_k(j) = 1.$

The reachability set of a PN is the set of all possible reachable markings from $M_0$ firing only enabled transitions; this set is denoted by $R(G,M_0).$

A structural conflict is a PN sub-structure in which two or more transitions share one or more input places; such transitions are simultaneously enabled and firing one of them disables the others.

Definition 6: A transition $t_k \in T$ is live, for a marking $M_0$, if $\forall M_k \in R(G,M_0), \exists M_n \in R(G,M_0)$ such that $t_k$ is enabled in $M_n.$

Definition 7: A PN is said to be l-bounded, or safe, for a marking $M_0,$ if $\forall p_i \in P$ and $\forall M_j \in R(G,M_0)$, it holds that $M_j(p_i) \leq 1.$

In this work we deal with live and safe PN.

Definition 8: A p-invariant $Y_i$ (t-invariant $X_i$) of a PN is a positive integer solution of the equation $Y_i^T C = 0 \mid Y_i \neq 0 \mid (X_i = 0 \mid X_i \neq 0).$ The support of the p-invariant $Y_i$ (t-invariant $X_i$) is the set $||Y_i|| = \{p_j \mid Y_i(p_j) \neq 0\}$ \((||X_i|| = \{t_j \mid X_i(t_j) \neq 0\}).\)

Definition 9: Let $Y_i$ be a p-invariant of a Petri net $(G,M_0)$, and let $||Y_i||$ be the support of $Y_i,$ then the subnet induced by $Y_i$ is $PY_i = \{p_j \mid Y_i(p_j) \neq 0\}.$ Let $I_{PY_i} \subseteq p_j, t_j \in p_j, \bullet t_j$ be the p-component, such that $I_{PY_i} = P \times T_i \cap I,$ and $O_{PY_i} = P \times T_i \cap O.$

Definition 10: Let $X_i$ be a t-invariant of a PN, and let $||X_i||$ be the support of $X_i,$ then the subnet induced by $X_i$ is $TX_i = \{p_j \mid p_j, t_j \in t_j \bullet t_j \in ||X_i||\}.$ The pre-component, such that $I_{TX_i} = X_j \times I_i, O_j = P \times T_i \cap O.$

III. FUZZY TIMED PETRI NETS

A. Basic Operators

We introduce first some useful operators.

Definition 11: Let $\tilde{a}$ and $\tilde{b}$ two fuzzy sets. The sum operation between $\tilde{a}$ and $\tilde{b}$ is computed as:

$$\text{sum} (\tilde{a}, \tilde{b}) = \alpha_0 (\tau) + \alpha_0 (\tau) \quad (2)$$

such that $\text{sum} (\tilde{a}, \tilde{b}) = 1$ if $\alpha_0 (\tau) + \alpha_0 (\tau) \geq 1.$

Definition 12: Let $\tilde{a}$ a fuzzy set and $w$ a real number. The product between $\tilde{a}$ and $w$ is defined as:

$$\text{prod} (\tilde{a}, w) = w \cdot \alpha_0 (\tau) \quad (3)$$

such that $\text{prod} (\tilde{a}, w) = 1$ if $w \cdot \alpha_0 (\tau) \geq 1.$

Definition 13: Let $\tilde{a}$ and $\tilde{b}$ two fuzzy sets such that $a_2 < b_3.$ The extended union operation between $\tilde{a}$ and $\tilde{b}$ is

$$\text{ext} (\tilde{a}, \tilde{b}) = \min (\tilde{a}^+, \tilde{b}^-) \quad (4)$$

this operation is illustrated in Fig.1(e).

Fig. 1. Fuzzy sets and operations

Definition 14: The latest (earliest) operation selects the latest (earliest) fuzzy set among $n$ fuzzy sets; they are calculated as follows:

$$\text{latest} (\tilde{a}_1, ..., \tilde{a}_n) = \min (\max (\tilde{a}_1^-, ..., \tilde{a}_n^-), \min (\tilde{a}_1^+, ..., \tilde{a}_n^+)) \quad (5)$$

$$\text{earliest} (\tilde{a}_1, ..., \tilde{a}_n) = \min (\max (\tilde{a}_1^+, ..., \tilde{a}_n^+), \max (\tilde{a}_1^-, ..., \tilde{a}_n^-)) \quad (6)$$

Definition 15: The conjugation operator is defined as $\text{arg}1 \bullet \text{arg}2$, where $\text{arg}1, \text{arg}2$ are arguments that can be matrices of fuzzy sets; $\bullet$ is the fuzzy and operation and $\text{oper}$ is any operation referred as $+, -, \text{latest}, \text{min},$ etc. For two fuzzy sets $\tilde{a} \text{ oper } \tilde{b} = \text{oper} (\text{arg}1, \text{arg}2).$ For matrices $A(m \times r)$ and $B(r \times n), A \text{ oper } B = \begin{bmatrix} \text{oper} (\text{arg}_k, \text{arg}_j) \end{bmatrix}_{k=1,...,r}$. i = 1,...,m.
and \( j = 1, \ldots, n \). For example: \( \tilde{A} \circ \tilde{B} = \)
\[
\begin{bmatrix}
\tilde{a}_{11} & \cdots & \tilde{a}_{1r} \\
\vdots & \ddots & \vdots \\
\tilde{a}_{m1} & \cdots & \tilde{a}_{mr} \\
\end{bmatrix} + \)
\[
\begin{bmatrix}
\tilde{b}_{11} & \cdots & \tilde{b}_{1n} \\
\vdots & \ddots & \vdots \\
\tilde{b}_{m1} & \cdots & \tilde{b}_{mn} \\
\end{bmatrix}
\]
Actually, ˜cpₜ represents the fuzzy marking in pₜ at instant τ.

The enabling times fuzzy sets of output transitions from each place are computed by the *earliest* operator; the fuzzy enabling times of transitions in structural conflicts are computed using the *min* operator.

### Step 4
Equation (10) is generalized in three steps. a) the earliest dates of ˜O are computed from transitions to input places (C⁺ *earliest* ˜O).  b) the latest dates of ˜O are computed from places to output transitions (C⁻ *latest* ˜O).  c) these solutions are integrated through the ext operator.

$\tilde{C} = \text{ext} \left( C^+ \bullet \tilde{O}, C^- \bullet \tilde{O} \right)$ (14)

where $\tilde{C} = [ \tilde{c}_{p_1}, \tilde{c}_{p_2}, \ldots \tilde{c}_{p_n} ]$.

### Step 5
Finally, $\tilde{O}' = \tilde{O}$.

**EndLoop.**

**Remark 21:** In this work, row and column vectors are represented as X.

### IV. FUZZY MARKING OF THE FTPN

In this section we develop a fuzzy state equation analogous to the discrete state equation (1). First we present the expressions for computing the evolution of the fuzzy marking in Marked Graphs; then we propose a technique for marking approximation in structural conflicts. Finally a general marking equation is developed.

#### A. State equation of Fuzzy Timed Marked Graphs

In an ordinary Petri Net, the terms C⁺ᵥₖ, C⁻ᵥₖ of the state equation (1) determine the variations on the marking. In a Fuzzy Timed Marked Graph (FTMG) such variations are determined by the variations on ˜O, more precisely ˜O₁, ˜O⁻¹. Then considering the marking after a time elapse Δτ we obtain:

$\tilde{M}(\tau) = \tilde{M}(\tau - \Delta\tau) + C^+ \triangle \tilde{O}^i - C^- \triangle \tilde{O}^i$ (15)

where $\triangle \tilde{O}^i = \tilde{O}^i(\tau) - \tilde{O}^i(\tau - \Delta\tau)$ and $\triangle \tilde{O}^i = \tilde{O}^i(\tau - \Delta\tau) - \tilde{O}^i(\tau)$.

The marking possibility obtained in (15) can be greater than 1 and since FTMG are *safe* then we use the min operator to bound $M(\tau) \leq 1$. The new marking is denoted by ˜M(τ), i.e.:

$\tilde{M}(\tau) = \min \left( M(\tau - \Delta\tau) + C^+ \triangle \tilde{O}^i - C^- \triangle \tilde{O}^i, 1 \right)$ (16)

where $\overline{1}$ is a n-entry vector having 1 in all its entries.

At the start $M(0) = M_0$.

**Remark 22:** If $\triangle \tilde{O}^i, \triangle \tilde{O}^i \in \{0, 1\}$ the behavior is that of a timed marked graph.

#### B. Marking approximation considering structural conflicts

If we would obtain the state equation for Fuzzy Timed State Machines (FTSM), then structural conflicts must be carefully analyzed. In order to deal with conflicts we need to declare an a priori knowledge about the firing of transitions in a conflict set; this knowledge states a relative possibility...
of firing among such transitions called branching rate, which is the certainty degree of the path that the token follows.

Definition 23: The branching rate \( b \) is a function \( w: T \rightarrow [0, 1] \). The \( w(t) \) value, denoted as \( w_i \), is a static value associated to a transition \( t \) representing the relative possibility to be fired. It fulfills the following constraint:

\[
\sum_{t \in \mathcal{P}} w_t = 1; \forall p \in \mathcal{P}
\]  

(17)

The branching rate \( w \) is defined for every transition; when \( p \bullet \{ t \} \), \( w_t = 1 \). In a conflict set, if \( w \) is not specified then \( w_i = w_j \) such that \( t_i, t_j \in p \bullet \). A diagonal matrix \( W \) represents the values \( w_t \) such that \( t \in T \).

\[
W = \text{diag} \left[ w_1, w_2, \ldots, w_{|T|} \right] \tag{18}
\]

The branching rate will affect the marking possibility of the output places, which will increase their marking value proportionally to both \( w_t \) and the marking of the input place. This notion is named the firing degree (FD) of a transition in conflict.

Definition 24: The firing degree is a dynamic value \( g_t \in [0, 1] \) associated to a transition \( t \), which represents the firing of \( t \) according to both the marking of the input place \( p \in \mathcal{P} \) and \( w_t: g_t = M_p(\tau) w_t \). It fulfills the following constraint:

\[
\sum_{t \in \mathcal{P}^*} g_t = M_p(\tau)
\]  

(19)

The diagonal matrix \( G \) represents the values \( g_{t_j} \), \( \forall t_j \in T \).

Now, consider the structural conflict depicted in Fig.7 where \( p_1 \) is marked. Initially, the \( FD \) of \( t_1 \) coincides with its branching rate, hence \( g_{t_1} = 0.4 \). The place \( p_2 \) will get "an amount" of marking proportionally to \( g_{t_1} \) whilst \( p_3 \) will get a marking at most of 0.6 by the firing of \( t_2 \). Next, the \( FD \) of \( t_3 \) is computed from \( w_{t_3} \) and the marking of \( p_2 \), that is \( g_{t_3} = M_{p_2} w_{t_3} \); then \( g_{t_3} \) will be at most 0.4. Similarly \( g_{t_4} = M_{p_2} w_{t_4} \).

The marking from 0 to 0.9 does not change (car1 is moving to right) i.e., \( M(0) = M(0.9) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \). At the instant \( \tau = 1 \), the transitions \( t_2 \) and \( t_3 \) are fired. For computing the marking, first the firing degree is obtained using (20).

\[
G(0.9) = \text{diag} (M(0.9) C^- W)
\]

\[
G(0.9) = \text{diag} \left[ \begin{bmatrix} 0 & 0.4 & 0.6 & 0 & 0 & 1 & 0 \end{bmatrix} \right]
\]

Now, we compute the marking at instant \( \tau = 1 \) using (21).

\[
M(1) = M(0.9) + C^+ G(0.9) - C^- G(0.9)
\]

\[
\begin{bmatrix}
0 \\
1 \\
1 \\
0 \\
0 \\
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\end{bmatrix}
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\begin{bmatrix}
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\end{bmatrix}
\]

Fig. 7. Structural conflict and branching rate

Now we can compute the firing degree of all the transitions at the instant \( \tau \) as a vector \( G(\tau) \) using \( M(\tau), W, \) and the structural information provided by \( C^- \):

\[
G(\tau) = M(\tau) C^- W
\]  

(20)

We can use the \( G(\tau) \) to weight the flow of marking by the firing of the involved transitions by introducing this matrix within the state equation given for Marked Graphs (16) as follows:

\[
M(\tau) = M(\tau - \Delta \tau) + C^+ G(\tau) \Delta \phi - C^- G(\tau) \Delta \phi 
\]  

(21)
The Fig. 10 shows the evolution of fuzzy marking. The uncertainty of the estimated marking grows during the evolution of the system. In this section we study the degradation of the estimated marking in order to provide a measure of the quality on the approximated marking.

**Definition 26:** The $p$-component marking deviation at the instant $\tau$ is described by the value $\psi'(\tau) \in [0, 1]$ which distinguishes the plausible (having the highest $\tilde{M}_{p_\tau}(\tau)$) marked place $p_\tau \in \|Y_r\|$ among other possible places $p_r \in \|Y_r\|$ such that $r \in \{1, \ldots, |Y_r|\}$, $v \neq u$. $\psi'(\tau)$ indicates the minimal difference between $\tilde{M}_{p_\tau}(\tau)$ and the marking of the rest of the places $p_v$ in the component $r$ ($\tilde{M}_{p_\tau}(\tau)$). The function $\psi'(\tau)$ is computed as:

$$\psi'(\tau) = \min \left( |\tilde{M}_{p_\tau}(\tau) - \tilde{M}_{p_\tau}(\tau)| \right)$$

where $\forall p_\tau, p_r \in \|Y_r\|; v \neq u, Y_r \in Y$ (22)

**Example 27:** Consider again the system of Fig. 8 handling a single type of material; a simpler sequence description yields the model in fig. 11. This model has two p-invariants with supports $\|Y_1\| = \{p_1, p_2, p_4, p_5\}$ and $\|Y_2\| = \{p_1, p_3, p_6\}$, as showed in Fig 12 where fuzzy sets $\tilde{C}$ are obtained from evolution of the marking in the p-component corresponding to $Y_1$. This evolution shows the first cycle and some few more steps of the next cycle. In order to obtain the activity estimation of $\text{car1}$, we need to obtain the marking deviation $\psi_1(\tau)$. During the time interval $\tau \in [0, 9, 1)$ the possibility that the place $p_2$ is marked is equal to one. However, the possibility that the place $p_1$ is marked is increased. Therefore, when $\tilde{M}_{p_1}(\tau)$ is increased, then $\psi_1(\tau)$ is reduced. In $\tau = 1$, the fuzzy marking indicates that the token is in $p_2$ and $p_4$. In this case it is not possible to know where the token is and therefore $\psi_1(\tau) = 0$. When $\tau \in (1, 1.1)$, $\tilde{M}_{p_1}(\tau) = 1$; $\tilde{M}_{p_2}(\tau)$ is reduced and $\psi_1(\tau)$ is increased. Finally, we can see that for $\tau \in [1.1, 2.6]$, $\psi_1(\tau) = \tilde{M}_{p_1}(\tau)$; it means that $p_4$ is certainly marked.

Definition 28: The p-component marking certainty degree $V'(\tau)$ is a measure of the marking deviation in a p-component $r$ along the time interval $[\tau_1, \tau_2]$. It is computed as:

$$V'(\tau) = \int_{\tau_1}^{\tau_2} \psi'(\tau) d(\tau) \left| \tau_2 > \tau_1 \right. \quad (23)$$

Definition 29: The global marking deviation at the instant $\tau$ is described by the function $\xi(\tau) \in [0, 1]$, which determines the possible state of the system among other possible states; it is calculated by:

$$\xi(\tau) = \min (\psi'(\tau)) \forall Y_r; \quad r \in \{1, \ldots, |Y_r|\} \quad (24)$$

Example 30: Following the previous example, in order to obtain an estimation about the activity the cars are executing, we need to obtain the state estimation. In this case $\xi(\tau) = \psi_1(\tau)$.
The extreme situation in which any activity of a system cannot be detected by sensors has been dealt to illustrate the degradation of the marking estimation when a cyclic execution is performed; the proposed technique for evaluating the uncertainty in the approximated marking provides a useful measure on the quality of the estimation.

REFERENCES