A note on the notion of singular copula

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Abstract

We clarify the link between the notion of singular copula and the concept of support of the measure induced by a copula.

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1. Introduction

In this note we wish to clarify the confusion, which is sometimes encountered in the literature, about the notion of singular (and, correspondingly, absolutely continuous) copula. In particular, we clarify the link between this notion and the concept of support of the measure induced by a copula.

2. Absolutely continuous and singular copulas

The reader is assumed to know the definition and the basic properties of copulas, which may be found, for instance in [6, 8].

Let \( d \geq 2 \). Let \( C \) be a \( d \)-dimensional copula. Any copula \( C \) induces on \( \mathcal{B}(\mathbb{I}^d) \), the class of Borel sets of \( \mathbb{I}^d \), a probability measure \( \mu_C \). Moreover, for every rectangle \( [\mathbf{u}, \mathbf{v}] \in \mathbb{I}^d \), one has

\[
\mu_C([\mathbf{u}, \mathbf{v}]) = V_C([\mathbf{u}, \mathbf{v}]),
\]

where

\[
V_C([\mathbf{u}, \mathbf{v}]) = \sum (-1)^{s(\mathbf{c})} C(\mathbf{c}),
\]

and the sum is taken over all the vertices \( \mathbf{c} \) of \( [\mathbf{u}, \mathbf{v}] \) (i.e., each \( c_k \) is equal to either \( u_k \) or \( v_k \)) and \( s(\mathbf{c}) \) is the number of indices \( k \)'s such that \( c_k = u_k \).

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The support of $\mu_C$ is the complement of the union of all open subsets of $\mathbb{I}^d$ with $\mu_C$-measure zero. In other words, it is the smallest closed set on which $\mu_C$ is concentrated.

In view of the Lebesgue decomposition theorem (see, e.g. [1, Theorem 2.2.6]), one has

$$\mu_C = \mu_C^{ac} + \mu_C^s,$$

where:

- $\mu_C^{ac}$ is a measure on $\mathcal{B}(\mathbb{I}^d)$ that is absolutely continuous with respect to the $d$-dimensional Lebesgue measure $\lambda_d$, i.e. for every $B \in \mathcal{B}(\mathbb{I}^d)$ $\lambda_d(B) = 0$ implies $\mu_C^{ac}(B) = 0$;

- $\mu_C^s$ is a measure on $\mathcal{B}(\mathbb{I}^d)$ that is singular with respect to the Lebesgue measure on $\mathbb{I}^d$, i.e. the probability measure is concentrated on a set $B$ such that $\lambda_d(B) = 0$.

Therefore, for all $u \in \mathbb{I}^d$, one can write

$$C(u) = C_{ac}(u) + C_s(u),$$

where $C_{ac}(u) = \mu_C^{ac}([0, u])$ and $C_s(u) = \mu_C^s([0, u])$ are called, respectively, absolutely continuous and singular component of $C$. In view of these facts, the following definition may be given.

**Definition 1.** A copula $C$ is said to be absolutely continuous (respectively, singular) when $C_s = 0$ (respectively, $C_{ac} = 0$).

If a copula $C$ is absolutely continuous, then it can be written in the form

$$C(u) = \int_{[0,u]^d} f(s)ds$$

where $f$ is a suitable function called density of $C$. In particular, for almost all $u \in \mathbb{I}^d$ one has

$$f(u) = \frac{\partial^d C(u)}{\partial u_1 \cdots \partial u_d}$$

As stressed in [7], eq. (1) is far from obvious. In fact, there are some facts that are implicitly used: first, the mixed partial derivatives of order $d$ of $C$ exist and are equal almost everywhere on $\mathbb{I}^d$; secondly, each mixed partial derivative is actually almost everywhere equal to the density $f$. The reader may refer to [2, section 4.1] and the references therein (in particular, [3], and [9, page 115]).

In general, given any copula $C$, in view of the Besicovitch derivation theorem (see, e.g. [1, Theorem 2.38]) one has

$$C(u) = \int_{[0,u]^d} f(s)ds + C_s(u),$$

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where $f$, which is the density of the absolutely continuous component of $C$, coincides almost everywhere with the Radon-Nikodym derivative of $\mu_C^a$ with respect to $\lambda_d$.

Following [1, Theorem 2.38], the function $f$ of eq. (2) coincides almost everywhere on $\mathbb{I}^d$ with the derivative of $\mu_C$, given, for almost all $u \in \mathbb{I}^d$ by:

$$D_{\mu_C}(u) = \lim_{r \to 0} \frac{\mu_C(B_r)}{\lambda_d(B_r)},$$

where $B_r$ ranges over all the open cubes of diameter less than $r$ that contain $u$. Notice that the existence a.e. of $D_{\mu_C}$ is guaranteed in view of [1, Theorem 2.38].

The following result easily follows.

**Proposition 1.** A copula $C$ is singular if, and only if, $D_{\mu_C}(u) = 0$ for almost all (with respect to $\lambda_d$) $u$ in $\mathbb{I}^d$.

When the mixed partial derivatives of order $d$ of $C$ exist for almost all $u \in \mathbb{I}^d$ and are all equal, it has been showed in [4] that

$$\frac{\partial^d C(u)}{\partial u_1 \cdots \partial u_d} = \lim_{h_d \to 0} \lim_{h_{d-1} \to 0} \lim_{h_{d-2} \to 0} \frac{V_C([u, u + h])}{\prod_{i=1}^d h_i},$$

i.e. $\partial_{1 \cdots d}^d C(u) = D_{\mu_C}(u)$ for almost all $u \in \mathbb{I}^d$.

Now, suppose that $C$ is a copula such that the following property holds:

(a) The support of $\mu_C$ has Lebesgue measure equal to 0, $\lambda_d(\text{supp}(\mu_C)) = 0$.

It can be easily derived from the Lebesgue decomposition of $\mu_C$ that such a $C$ is singular. However, if a copula is singular, then it may not satisfy property (a) (a confusion that we have sometimes encountered in the literature). In fact, we present an example of a singular bivariate copula whose support is $\mathbb{I}^2$.

**Example 1.** Let $C_n$ be the (bivariate) shuffle of Min obtained in the following way (see the proof of [8, Theorem 3.2.2]): consider a partition of the unit interval $\mathbb{I}$ into $n^2$ subintervals of length $1/n^2$ and the permutation $\pi$ of $\{1, 2 \ldots, n^2\}$ defined by $\pi(n^2(j-1)+k) = n^2(k-1)+j$ where $j, k = 1, 2, \ldots, n^2$. The copula $C_n$ approximates the independence copula $\Pi_2$ in the sense that

$$\sup_{(u,v) \in \mathbb{I}^2} |C_n(u,v) - \Pi_2(u,v)| < \epsilon,$$

for every $n \geq 4/\epsilon$. Consider the copula

$$C = \sum_{n \in \mathbb{N}} \frac{1}{2^n} C_n.$$

Let $T_n$ be the support of $\mu(C_n)$ and set $T := \cup_{n \in \mathbb{N}} T_n$: then $\mu_C(T) = 1$, viz., the probability mass of $\mu_C$ is concentrated on $T$. On the other hand, one has $\lambda_2(T_n) = 0$, so that $\lambda_2(T) = 0$. This implies that $C$ is singular.

Since the closure of $T$ is $\mathbb{I}^2$, the support of $\mu_C$ is $\mathbb{I}^2$, so that $C$ does not satisfy (a).
Another example of bivariate singular copula whose support is $\mathbb{I}^2$ appeared in [5, Example 3.2].

A said, both the examples show that, given a copula $C$, the support of $C$ being of measure zero is only a sufficient condition for $C$ being singular.

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