A note on quasi-copulas and signed measures

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Abstract

In this note we provide two alternative proofs to that given in [15] of the fact that the best possible lower bound for the set of \( n \)-quasi-copulas does not induce a stochastic measure on \([0,1]^n\) for \( n \geq 3 \): firstly, by reviewing its mass distribution, and secondly, by using concepts of self-affinity.

Keywords: Copula; Quasi-copula; Self-affinity; Stochastic signed measure.

1 Preliminaries

A signed measure \( \mu \) on a measurable space \((S,A)\) is an extended real valued, countably additive set function on the \( \sigma \)-algebra \( A \) such that \( \mu(\emptyset) = 0 \) and \( \mu \) assumes at most one of the values \( \infty \) and \( -\infty \) [3, 9]. For a signed measure \( \mu \) on \((\Omega,\mathcal{F})\), there exist two sets \( \Omega^+, \Omega^- \in \mathcal{F} \) such that \((\Omega^+ \cup \Omega^-) = \Omega\), \((\Omega^+ \cap \Omega^-) = \emptyset\) and \( \mu(A \cap \Omega^+) \geq 0 \), \( \mu(A \cap \Omega^-) \leq 0 \) and \( \mu(A) = \mu(A \cap \Omega^+) + \mu(A \cap \Omega^-) \) for all \( A \in \mathcal{F} \). The measures \( \mu^+(A) = \mu(A \cap \Omega^+) \) and \( \mu^-(A) = -\mu(A \cap \Omega^-) \) satisfy \( \mu = \mu^+ - \mu^- \), which is called a Jordan decomposition and it is unique.

Let \( [0,1] \) denote the interval \([0,1] \). A stochastic measure \( \nu \) is a measure such that \( \nu_C(\mathbb{I}^{i-1} \times A \times \mathbb{I}^{k-i}) = \lambda(A) \), for every \( i = 1,2,\ldots,k \) and every Borel measurable set \( A \) in \( \mathbb{I} \)—the Borel \( \sigma \)-algebra for \( \mathbb{I}^k \) will be denoted by \( \mathcal{B}_{\mathbb{I}} \), where \( \lambda \) denotes the Lebesgue measure in \((\mathbb{R},\mathcal{B}_{\mathbb{R}})\) (see [14]).

As for positive measures, a signed measure \( \mu \) on \( S \) is said to be stochastic if \( \mu_C(\mathbb{I}^{i-1} \times A \times \mathbb{I}^{k-i}) = \lambda(A) \), for every \( i = 1,2,\ldots,k \) and every Borel measurable set \( A \in \mathcal{B}_{\mathbb{I}} \). A stochastic signed measure \( \mu \) on \( S \) satisfies \( \mu([0,1]^n) = 1 \) and is bounded.

The term copula, coined by A. Sklar [19], is now common in the statistical literature. Let \( n \) be a natural number such that \( n \geq 2 \). An \( n \)-copula is a function from \( \mathbb{I}^n \) to \( \mathbb{I} \) that link joint distribution functions to their one-dimensional margins (see [14] for a detailed introduction to copulas, and [4, 11, 17, 18] for some of their applications). It is known that every \( n \)-copula \( C \) induces a stochastic measure defined on \( \mathcal{B}_{\mathbb{I}} \).

An \( n \)-dimensional quasi-copula (briefly \( n \)-quasi-copula) is a function \( Q \) from \( \mathbb{I}^n \) to \( \mathbb{I} \) satisfying the following conditions (see [7] for the bivariate case and [5] for the \( n \)-dimensional case): (i) For every \( \mathbf{u} = (u_1,u_2,\ldots,u_n) \in \mathbb{I}^n \), \( Q(\mathbf{u}) = 0 \) if at least one coordinate of \( \mathbf{u} \) is equal to 0, and \( Q(\mathbf{u}) = u_k \)
whenever all coordinates of \( u \) are equal to 1 except maybe \( u_k \); (ii) \( Q \) is nondecreasing in each variable; and (iii) for every \( u, v \in \mathbb{I}^n \), it holds that \( |Q(u) - Q(v)| \leq \sum_{i=1}^n |u_i - v_i| \). Quasi-copulas were introduced in the field of probability (see [1, 16]), but they are also used in aggregation processes because they ensure that the aggregation is stable, in the sense that small error inputs correspond to small error outputs (for an overview, see [2, 8]).

Every \( n \)-copula is an \( n \)-quasi-copula, and a proper \( n \)-quasi-copula is an \( n \)-quasi-copula which is not an \( n \)-copula. Moreover, every \( n \)-quasi-copula \( Q \) satisfies the following inequalities:

\[
W^n(u) = \max \left( \sum_{i=1}^n u_i - n + 1, 0 \right) \leq Q(u) \leq \min(u_1, u_2, \ldots, u_n) = M^n(u)
\]

for all \( u \in \mathbb{I}^n \) (a superscript on the name of a copula or quasi-copula denotes dimension). It is known that \( M^n \) is an \( n \)-copula for every \( n \geq 2 \), \( W^2 \) is a 2-copula, and \( W^n \) is a proper \( n \)-quasi-copula for every \( n \geq 3 \). Note that for \( 2 \leq k < n \), the \( k \)-dimensional margin of \( W^n \) is \( W^k \).

When \( n = 2 \), the construction of proper \( 2 \)-quasi-copulas from the so-called proper quasi-transformation square matrices shows that these functions do not induce a stochastic signed measure on \( \mathbb{I}^2 \) [6], and the study of the mass distribution of \( W^n \), for \( n \geq 3 \), shows the result for the case of proper \( n \)-quasi-copulas (see [15]).

Finally, we will use some notation. We denote by \( T_1 \) the triangle in \( \mathbb{I}^3 \) whose vertices are \( A = (1, 1, 0) \), \( B = (1, 0, 1) \) and \( C = (0, 1, 1) \).

Our goal in this note is to provide two short proofs to the fact that \( W^n \) does not induce a stochastic signed measure on \( \mathbb{I}^n \), as a consequence of: firstly, by looking at the mass distribution of \( W^3 \), and secondly, by using the self-affinity of \( W^3 \).

2 The main result

We provide the main result of this note, whose alternative proofs are given in the next subsections.

**Theorem 2.1.** The 3-quasi-copula \( W^3 \) does not induce a stochastic signed measure on \( \mathbb{I}^3 \).

As a consequence of Theorem 2.1, we have:

**Corollary 2.1.** For every \( n \geq 3 \), the \( n \)-quasi-copula \( W^n \) does not induce a stochastic signed measure on \( \mathbb{I}^n \).

**Proof.** Suppose, if possible, that the \( n \)-quasi-copula \( W^n \) induces a stochastic signed measure on \( \mathbb{I}^n \) for every \( n \geq 4 \). Then, each of its \( \binom{n}{3} \) 3-variate margins, which are \( W^3 \), induces a stochastic signed measure on \( \mathbb{I}^3 \), which contradicts Theorem 2.1. \( \square \)

2.1 First alternative proof of Theorem 2.1

In the following, when we refer to “mass” of an \( n \)-quasi-copula \( Q \) —associated with a signed measure—on a set, we mean the value \( \mu_Q \) for that set, i.e., \( Q(D) = \mu_Q(D) \), with \( D \in \mathcal{B}_{\mathbb{I}^n} \).
Proving Theorem 2.1. Suppose, if possible, that $W^3$ induces a stochastic signed measure $\mu_{W^3}$ on $\mathbb{I}^3$. It is known that the mass of $W^3$ is spread on the triangle $T_1$ defined in Section 1 (see [13, 14] for details). Let $T_2$ be the triangle whose vertices are $A' = (1, 1, 1)$, $B$ and $C$, and consider the projection $\pi: T_1 \rightarrow T_2$ given by $\pi(u_1, u_2, u_3) = (u_1, u_2, 1)$. Since $W^3(u_1, u_2, 1) = W^2(u_1, u_2)$, we have that $\pi$ is an isomorphism between measurable spaces. Thus, we have $\mu_{W^3}(S) = \mu_{W^2}(\pi(S))$ for every Borel set $S$ such that $S \subset T_1$. Since $W^2$ is a copula whose mass is spread along the segment $CB$ which joins the points $C$ and $B$, and $\pi(CB) = CB$, then the mass of $W^3$ is distributed along the segment $CB$, and this implies $W^3 (u_1, u_2, u_3) = 0$ when $u_3 < 1$, which is absurd. Therefore, $W^3$ does not induce a stochastic signed measure on $\mathbb{I}^3$.

\[ \Box \]

2.2 Second alternative proof of Theorem 2.1

Before providing our second proof, we need some concepts on self-affinity (for more details, see [10, 12]).

Let $(X, d)$ be a metric space and $d(x, y)$ the distance between $x$ and $y$ in $(X, d)$. A map $S: (X, d) \rightarrow (X, d)$ is called a similarity with ratio $c$ if there exists a number $c > 0$ such that $d(S(x), S(y)) = c \cdot d(x, y)$ for all $x, y \in (X, d)$. Two sets are similar if one is the image of the other under a similarity transformation. A set $E$ is said to be self-similar if it can be expressed as a union of $m$ similar images of itself, that is, $E = \bigcup_{k=1}^{m} S_k(E)$. Standard examples of self-similar sets are the Cantor ternary set and the von Koch curve.

In $\mathbb{R}^3$, a concept related to self-similarity is self-affinity. A self-affine set is defined in the same way as a self-similar set with affine transformations instead of similarities. An affine transformation consists of a linear transformation and a translation and may contract with different ratios in different directions. Self-similarity is thus a particular case of self-affinity.

Let $T_{1,1}, T_{1,3}, T_{1,4}, T_{1,5}$ be the triangles in $\mathbb{I}^3$ with respective vertices $\{(1, 0, 1), (\frac{1}{2}, \frac{1}{2}, 1), (1, \frac{1}{2}, \frac{1}{2})\}$, $\{(0, 1, 1), (\frac{1}{2}, \frac{1}{2}, 1), (\frac{1}{2}, 1, \frac{1}{2})\}$, $\{(\frac{1}{2}, 1, 0), (\frac{1}{2}, \frac{1}{2}, 1), (1, \frac{1}{2}, \frac{1}{2})\}$ and $\{(1, 1, 0), (1, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, 1, \frac{1}{2})\}$. Following this notation, we also define—without confusion—the functions on $\mathbb{I}^3$ by

\[
T_{1,1}(u_1, u_2, u_3) = \left( \frac{u_1 + 1}{2}, \frac{u_2 + 1}{2}, \frac{u_3 + 1}{2} \right),
\]

\[
T_{1,3}(u_1, u_2, u_3) = \left( \frac{u_1}{2}, \frac{u_2 + 1}{2}, \frac{u_3 + 1}{2} \right),
\]

\[
T_{1,4}(u_1, u_2, u_3) = \left( \frac{1 - u_1}{2}, \frac{1 - u_2}{2}, \frac{1 - u_3}{2} \right),
\]

\[
T_{1,5}(u_1, u_2, u_3) = \left( \frac{u_1 + 1}{2}, \frac{u_2 + 1}{2}, \frac{u_3}{2} \right).
\]

Before proving Theorem 2.1 by using these ideas, we need a technical lemma, where $\mathcal{B}_{T_1}$ denotes the Borel $\sigma$-algebra for the triangle $T_1$ defined in Section 1.

**Lemma 2.1.** There does not exist a signed measure $\mu \neq 0$ on $(T_1, \mathcal{B}_{T_1})$ such that

\[
\mu(S) = (-1)^{j+1} 2\mu(T_{1,j}(S)), \quad j = 1, 3, 4, 5,
\]

for all $S \in \mathcal{B}_{T_1}$.
Proof. Suppose, if possible, that there exists a measure $\mu \neq 0$ such that condition (2.1) is satisfied. Then $\mu$ is a signed measure for which $\Omega^+$ and $\Omega^-$ are the two sets of its Jordan decomposition. Thus, we have

$$(\Omega^+ \cap T_{1,j}) = T_{1,j}(\Omega^+) \text{ and } (\Omega^- \cap T_{1,j}) = T_{1,j}(\Omega^-), \ j = 1, 3, 5$$

and

$$(\Omega^+ \cap T_{1,4}) = T_{1,4}(\Omega^-) \text{ and } (\Omega^- \cap T_{1,4}) = T_{1,4}(\Omega^+).$$

Therefore,

$$\mu(\Omega^+) = \mu(T_{1,1}(\Omega^+)) + \mu(T_{1,3}(\Omega^+)) + \mu(T_{1,5}(\Omega^+)) + \mu(T_{1,4}(\Omega^-))$$

$$= \frac{3\mu(\Omega^+)}{2} + \frac{\mu(\Omega^-)}{2},$$

i.e., $\mu(\Omega^+) = -\mu(\Omega^-)$, which implies $\mu(\Omega^+) = \mu(\Omega^-) = 0$, but this is a contradiction. \qed

We are now in position to provide the second of our proofs.

Proof of Theorem 2.1. We first note that

$$W^3(u_1, u_2, u_3) = 2W^3(T_{1,1}(u_1, u_2, u_3)) = 2W^3(T_{1,3}(u_1, u_2, u_3)) = 2W^3(T_{1,5}(u_1, u_2, u_3))$$

$$= 2W^3(T_{1,4}(u_1, u_2, u_3)) + u_1 + u_2 + u_3 - 2$$

for all $(u, v, w) \in \mathbb{R}^3$. Suppose, if possible, that $W^3$ induces a stochastic signed measure $\mu_{W^3}$ on $\mathbb{R}^3$. Then the identities in (2.2) show that the measure is self-similar, i.e., $\mu_{W^3}(S) = (-1)^{j+1}2\mu_{W^3}(T_{1,j}(S))$ for all $S \in \mathcal{B}^3$ and $j = 1, 3, 4, 5$. Since the mass of $W^3$ is spread on the triangle $T_1$ and $(T_{1,j}(\mathbb{R}^3) \cap T_1) = T_{1,j}$ for $j = 1, 3, 4, 5$, then we have that $\mu_{W^3}$ is a self-affine signed measure such that $\mu(S) = (-1)^{j+1}2\mu(T_{1,j}(S))$ for all $S \in \mathcal{B}_{T_1}$; however, in view of Lemma 2.1, this is a contradiction. \qed

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