A new family of fuzzy implication operators is introduced. The proposed class is based on the conditional version of a copula function. Properties of these operators are studied and several examples illustrate our results.

MSC2000: Primary 60E05, secondary 62H20.

Keywords: Conditional probability; copula; fuzzy implication.

1 Introduction

In fuzzy logic and approximate reasoning, the most usual method for managing the conditional statements “If A, then B” is done through functions $I : [0, 1]^2 \rightarrow [0, 1]$ in such a way that the result value of the conditional is functionally stated from the truth values of the fuzzy statements A and B. These functions $I$ are the so-called fuzzy implications (or simply, implications), and they play an important role in many fields where fuzzy logic applies (see, for example, [2]). Many authors have focused their interest in the theoretical study of implications and methods of constructing such functions (see [1, 3, 4, 15]). All these studies have led to the necessity of using many different models to perform fuzzy implications. The main reason is due to the fact that the choice of the implication cannot be made independently of the inference rule that is going to be applied. Implications are usually derived from $t$-norms and $t$-conorms (see [13]) via the so-called R-implications, (S,N)-implications, QL-implications and D-implications [14, 17].

In this paper, following the recent work by Grzegorzewski [10, 11], we propose a family of fuzzy implication operators based on the conditional version of a copula function (a special type of $t$-norm). The proposed family have a probabilistic interpretation and combines fuzzy implications with randomness [19]. After recalling some basic concepts of fuzzy implications and copulas (Section 2), in Section 3 we discuss the proposed family and provide several examples.

2 Preliminaries

In this section we recall some concepts and results which will be used in the rest of the paper. First, we give the definition of (fuzzy) implication [2, 8].
Definition 1. A binary operator $I : [0, 1]^2 \to [0, 1]$ is called a (fuzzy) implication if, for all $x, y \in [0, 1]$, it satisfies the following conditions:

(i) $I(0, 0) = I(1, 1) = 1$,
(ii) $I(1, 0) = 0$,
(iii) $I(x, y)$ is nondecreasing in the second variable,
(iv) $I(x, y)$ is nonincreasing in the first variable.

Since our methods involve the concept of a copula, we recall the definition and some of its basic properties. For a more detailed study, we refer to [12, 18].

Definition 2. A (bivariate) copula is a binary operation $C : [0, 1]^2 \to [0, 1]$ which satisfies: (i) for every $u \in [0, 1]$, $C(u, 0) = C(0, u) = 0$ and $C(u, 1) = C(1, u) = u$, and (ii) $C(u_2, v_2) - C(u_1, v_2) - C(u_2, v_1) + C(u_1, v_1) \geq 0$ for every $u_1, v_1, v_2, v_2 \in [0, 1]$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$.

The importance of copulas in probability and statistics comes from Sklar’s theorem [22], which shows that the joint distribution of a random vector and the corresponding one-dimensional marginal distributions are linked by these functions.

Let $\Pi$ denote the copula of independent random variables, i.e., $\Pi(u, v) = uv$ for all $(u, v) \in [0, 1]^2$, and let $M$ and $W$ denote the Fréchet-Hoeffding upper and lower bound copulas, respectively, which, for any copula $C$, satisfy: max($u + v - 1, 0) = W(u, v) \leq C(u, v) \leq M(u, v) = \min(u, v)$ for every $(u, v) \in [0, 1]^2$.

A special class of copulas are the so-called Archimedean copulas, which are given by

$$C_\phi(u, v) = \phi^{-1}(\phi(u) + \phi(v)),$$

where $\phi$ (the generator of $C$) is a continuous, strictly decreasing and convex function from $[0, 1]$ to $[0, \infty]$ such that $\phi(1) = 0$, and $\phi^{-1}$ denotes the pseudo-inverse of $\phi$. When $\phi^{-1} = \phi^{-1}$, it is said that the generator (or $C$) is strict. See [18] for a detailed study.

3 Construction of the new family of implications

Intuitively, when we say that an implication “If B, then A” is true, we mean that whenever B is true, we expect A is true as well. However, in practice, we have some confidence in B, but we are not completely sure that B is true. We can only estimate the probability that the implication holds. A natural way to interpret the probability of an implication is the conditional probability $P(B|A)$, defined by

$$P(B|A) = \frac{P(A \cap B)}{P(A)},$$

where $P(A) > 0$. Recently, Grzegorzewski [10, 11] has proposed an implication operator (called probabilistic implication) based on the conditional distribution of $V$ given $U \leq u$, where $(U, V)$ is a pair of uniform $[0, 1]$ random variables with associated copula $C$ (which we will suppose in the sequel). Since $P(V \leq v|U \leq u) = \frac{C(u, v)}{u}$ (for $u > 0$), the probabilistic implication $I_C$ (based on the copula $C$) is given by

$$I_C(u, v) = \begin{cases} 1, & \text{if } u = 0 \\ \frac{C(u, v)}{u}, & \text{if } u > 0. \end{cases}$$

(1)

It is shown that $I_C$ is an implication if, and only if, $V$ is left tail decreasing in $U$ [7], i.e., $\frac{C(u, v)}{u}$ is nonincreasing in $u$ for $v \in [0, 1]$ ([18], Theorem 5.2.5).
3.1 Main results

Following the ideas given in [10, 11] and described above, we construct a new family of implications. For that, consider the conditional distribution of $V$ given $U = u$ (instead of $U ≤ u$). It is clear that

$$P(Y ≤ y|X = x) = \lim_{\Delta u \to 0} \frac{C(u + \Delta u, v) - C(u, v)}{\Delta u} = \frac{\partial}{\partial u} C(u, v).$$

The function $v \mapsto \frac{\partial}{\partial u} C(u, v)$ is defined and nondecreasing almost everywhere (denoted a.e.) with respect to the Lebesgue measure $\lambda$ on $[0, 1]$ (see [18, 20]). Furthermore, since $C(u, v_0)$ is nondecreasing in the first variable, we have that the partial derivative $\frac{\partial}{\partial u} C(u, v_0)$ exists for a.e. $u \in [0, 1]$; and for such $v_0$ and $u$, $0 ≤ \frac{\partial}{\partial u} C(u, v_0) ≤ 1$. The partial derivative of a copula plays an important role, for instance, in credit derivative pricing and evaluating risks [5].

For a given copula $C$, our main purpose is to define an implication in terms of the partial derivative of $C$. For that, we need some preliminary considerations.

First, we extend $\frac{\partial}{\partial u} C(u, v_0)$ to every point in $[0, 1]$. Denoting by $D_{v_0} = \{ u \in [0, 1] : \frac{\partial}{\partial u} C(u, v_0) \text{ exists} \}$, then the extension is defined as

$$\frac{\partial^*}{\partial u} C(u, v_0) = \begin{cases} \frac{\partial}{\partial u} C(u, v_0), & \text{if } u \in D_{v_0} \\ \inf \left\{ \frac{\partial}{\partial u} C(u_1, v_0) : u_1 \in (0, u) \cap D_{v_0} \right\}, & \text{if } u \notin D_{v_0}. \end{cases}$$

Let $I_C^*: [0, 1]^2 \to [0, 1]$ be a binary operator defined by:

$$I_C^*(u, v) = \begin{cases} 1, & \text{if } u = 0 \\ \frac{\partial^*}{\partial u} C(u, v), & \text{if } u > 0. \end{cases} \tag{2}$$

Clearly, the function $I_C^*$ given by (2) satisfies requirements (i)-(iii) in Definition 1. Thus, we call this function as the conditional implication (based on the copula $C$) in the case that it is, in fact, an implication.

We have the following result, which provides sufficient conditions for the function given by (2) to be an implication.

**Proposition 1.** If $\frac{\partial}{\partial u} C(u, v_0)$ is nonincreasing in $D_{v_0}$, then the function given by (2) is an implication.

**Proof.** Since $\frac{\partial}{\partial u} C(u, v_0)$ is monotone in $D_{v_0}$ and $\lambda(D_{v_0}) = 1$ for all $v_0 ∈ (0, 1)$, then $\frac{\partial^*}{\partial u} C(u, v_0)$ is left-continuous and nonincreasing in the first variable. On the other hand, let $v_0, v_1$ be in $[0, 1]$ such that $v_0 < v_1$. Since $\lambda((0, u_0] \cap D_{v_0} \cap D_{v_1}) = u_0$, there exists a sequence $\{u_n\}$ — whose elements are in $D_{v_0} \cap D_{v_1}$ — which converges to $u_0$. Thus, we have $\frac{\partial}{\partial u} C(u_n, v_0) ≤ \frac{\partial}{\partial u} C(u_n, v_1)$ (see [18], Theorem 2.2.7), which implies that $\frac{\partial}{\partial u} C(u_0, v_0) ≤ \frac{\partial}{\partial u} C(u_0, v_1)$, i.e., $\frac{\partial}{\partial u} C(u, v)$ is nondecreasing in the second variable, whence the result follows. \[\square\]

The next result provides a characterization of the function given by (2) in terms of the concavity of the copula $C$ to be an implication.

**Theorem 2.** For any copula $C$, the function $I_C^*$ given by (2) is an implication if, and only if, $C$ is concave in the first variable.
Proof. Assume first \( I_1^* \) is an implication, and let \( u_0, u_1, v_0, \alpha \) be in \([0, 1]\) such that \( u_0 < u_1 \). Since \( C(u, v_0) \)—the horizontal section of \( C \)—is absolutely continuous \([18]\), we have

\[
\alpha C(u_0, v_0) + (1 - \alpha)C(u_1, v_0) = \alpha \int_0^{u_0} \frac{\partial}{\partial u} C(u, v_0) \, du + (1 - \alpha) \int_0^{u_1} \frac{\partial}{\partial u} C(u, v_0) \, du
\]

\[
= \alpha \int_0^{u_0} \frac{\partial}{\partial u} C(u, v_0) \, du + (1 - \alpha) \int_0^{u_1} \frac{\partial}{\partial u} C(u, v_0) \, du + (1 - \alpha) \int_0^{u_1 - u_0} \frac{\partial}{\partial u} C(u, v_0) \, du
\]

\[
= \int_0^{u_0} \frac{\partial}{\partial u} C(u, v_0) \, du + (1 - \alpha)(u_1 - u_0) \int_0^{1} \frac{\partial}{\partial u} C(u_0 + (1 - \alpha)(u_1 - u_0)t, v_0) \, dt.
\]

On the other hand,

\[
C(\alpha u_0 + (1 - \alpha)u_1, v_0) = \int_0^{\alpha u_0 + (1 - \alpha)u_1} \frac{\partial}{\partial u} C(u, v_0) \, du
\]

\[
= \int_0^{u_0} \frac{\partial}{\partial u} C(u, v_0) \, du + \int_{u_0}^{u_0 + (1 - \alpha)(u_1 - u_0)} \frac{\partial}{\partial u} C(u, v_0) \, du
\]

\[
= \int_0^{u_0} \frac{\partial}{\partial u} C(u, v_0) \, du + (1 - \alpha)(u_1 - u_0) \int_0^{1} \frac{\partial}{\partial u} C(u_0 + (1 - \alpha)(u_1 - u_0)t, v_0) \, dt.
\]

Since \( \frac{\partial}{\partial u} C(u_0 + (u_1 - u_0)t, v_0) \leq \frac{\partial}{\partial u} C(u_0 + (1 - \alpha)(u_1 - u_0)t, v_0) \), when both derivatives exist, then we have

\[
\int_0^{1} \frac{\partial}{\partial u} C(u_0 + (u_1 - u_0)t, v_0) \, dt \leq \int_0^{1} \frac{\partial}{\partial u} C(u_0 + (1 - \alpha)(u_1 - u_0)t, v_0) \, dt,
\]

and hence, \( \alpha C(u_0, v_0) + (1 - \alpha)C(u_1, v_0) \leq C(\alpha u_0 + (1 - \alpha)u_1, v_0) \), i.e., \( C \) is concave.

Conversely, if \( C \) is concave in the first variable, we have that \( \frac{\partial}{\partial u} C(u_0, v_0) \) is nonincreasing almost everywhere \( u_0 \in [0, 1] \), whence \( \frac{\partial}{\partial u} C(u_0, v_0) \) is nonincreasing in the first variable, and following the same steps for the rest of the conditions as in the proof of Proposition 1, the result follows.

The concavity of the copula \( C \) means that \( P(V \leq v|U = u_1) \geq P(V \leq v|U = u_2) \) for all \( v \) and \( u_1, u_2 \in [0, 1] \) such that \( u_1 \leq u_2 \). In this case, it is said that \( V \) is stochastically increasing in \( U \) \([18, 21]\).

### 3.2 Relationship between probabilistic and conditional implication

In what follows we provide some results concerning the relationship between the implications \( I_C \) and \( I_1^* \) defined by (1) and (2), respectively. As a consequence of the fact that for a pair \((U, V)\) of uniform \((0, 1)\) random variables, if \( V \) is stochastically increasing in \( U \) then \( V \) is left tail decreasing in \( U \) \([5.2.12],[18]\), Theorem 5.2.12), and that if \( V \) is left tail decreasing in \( U \) is equivalent to \( \frac{\partial}{\partial u} C(u, v) \leq \frac{C(u,v)}{u} \) for a.e. \( u \in [0,1] \) \([18]\), Corollary 5.2.6), we have the following result.

**Proposition 3.** If \( I_1^* \) is an implication then \( I_C \) is also an implication. Moreover, in such a case, \( I_1^*(u,v) \leq I_C(u,v) \) for all \( u, v \in [0,1] \).
We note that the converse of the above result is not true. For example, the copula
\[
C(u, v) = \begin{cases} 
\frac{3uv - u + v - 1}{2}, & \text{if } \frac{1}{3} \leq v \leq 1 - u \leq \frac{2}{3} \\
\frac{3uv}{2}, & \text{if } \frac{1}{3} \leq 1 - u \leq v \leq \frac{2}{3} \\
M(u, v), & \text{otherwise}
\end{cases}
\]
has the left tail decreasing property, but the stochastically increasing property fails ([18], Exercise 5.32).

A natural question now arises: For a given copula \(C\), when does the equality \(I^*_C = I_C\) hold? The following result provides the answer to this question.

**Proposition 4.** Let \(I_C\) and \(I^*_C\) be the implications defined by (1) and (2), respectively, based on a copula \(C\). Then \(I_C = I^*_C\) if, and only if, \(C = \Pi\).

**Proof.** Assume \(I_C = I^*_C\). If \(u = 0\) the result is obvious, so suppose \(u \in (0, 1]\). For a fixed \(v_0 \in [0, 1]\), since \(C(u, v_0)\) is absolutely continuous, we have
\[
\frac{\partial}{\partial u} C(u, v_0) = \frac{1}{u} \int_0^u \frac{\partial}{\partial u} C(t, v_0) \, dt,
\]
whence \(\frac{\partial}{\partial u} C(u, v_0)\) exists for all \(u \in (0, 1]\). From Equality (3), we have that \(C(u, v_0) \in C^\infty(0, 1]\), where \(C^\infty(0, 1]\) denotes the class of all continuous and infinitely differentiable functions (with continuous derivatives) in \((0, 1]\). Therefore,
\[
\frac{\partial}{\partial u} C(u, v_0) = \frac{C(u, v_0)}{u}
\]
is a differential equation whose solution is given by \(C(u, v_0) = f(v_0)u\), where \(f\) is a differentiable function in \((0, 1]\). Since \(C(1, v_0) = v_0\), we have \(f(v_0) = v_0\), and hence we obtain \(C = \Pi\).

Conversely, suppose \(C = \Pi\). Since \(\frac{\partial}{\partial u} \Pi(u, v_0) = v_0\), the result follows, which completes the proof. \(\square\)

The following result follows from the definitions of \(I_C\) and \(I^*_C\).

**Proposition 5.** For each pair of copulas \(C\) and \(D\) satisfying the conditions of Theorem 2, \(I^*_C(u, v) \leq I^*_D(u, v)\) implies \(I_C(u, v) \leq I_D(u, v)\) for every \(u, v \in [0, 1]\).

### 3.3 Examples

We provide examples concerning the conditional implication \(I^*_C\) defined by (2), and for which —via Theorem 2— we obtain several families of fuzzy implications.

**Example 1.** Consider the copula \(\Pi\). Then, the conditional implication based on \(\Pi\) is
\[
I^*_\Pi(u, v) = \begin{cases} 
1, & \text{if } u = 0 \\
v, & \text{if } u > 0,
\end{cases}
\]
which is the least \((S,N)\)-implication [2]. Observe also that, from Proposition 4, we have \(I^*_\Pi = I_\Pi\).

**Example 2.** Let us consider the copula \(M\). Then, the conditional implication based on \(M\) is
\[
I^*_M(u, v) = \begin{cases} 
1, & \text{if } u \leq v \\
0, & \text{if } u > v,
\end{cases}
\]
which is the Gaines-Rescher implication [2].
In Examples 1 and 2, we observe that \( \Pi \leq M \), but for the generated implications \( I^*_\Pi \) and \( I^*_M \) neither \( I^*_\Pi \leq I^*_M \) nor \( I^*_M \leq I^*_\Pi \) hold. Therefore, for two copulas \( C \) and \( D \) with \( C \leq D \), the respective generated implications \( I^*_C \) and \( I^*_D \) are not comparable in general.

**Example 3.** Let \( C_\theta \) be the Farlie-Gumbel-Morgenstern family of copulas (see [18] and the references therein) defined by \( C_\theta(u,v) = uv + \theta uv(1-u)(1-v) \) for all \( u, v \in [0,1] \) and \( \theta \in [-1,1] \). Note that this copula is concave if \( \theta \in [0,1] \). Thus, the function given by

\[
C^*_\theta(u,v) = \begin{cases} 
1, & \text{if } u = 0 \\
v[1 + \theta u(1-2u)(1-v)], & \text{if } u > 0 
\end{cases}
\]

is an implication if, and only if, \( \theta \in [0,1] \).

**Example 4.** Let \( C_{\alpha,\beta} \) be the Marshall-Olkin family of copulas [16] defined by

\[
C_{\alpha,\beta}(u,v) = \begin{cases} 
  u^{1-\alpha} v, & \text{if } u^\alpha \geq v^\beta \\
  u v^{1-\beta}, & \text{if } u^\alpha < v^\beta.
\end{cases}
\]

for every \((u,v) \in [0,1]^2\), with \( \alpha, \beta \in [0,1] \). Since \( C_{\alpha,\beta} \) is concave in each component, we have that \( I^*_C \) is an implication for all \( \alpha, \beta \in [0,1] \).

The next result provides a condition for strict Archimedean copulas with generator twice differentiable to generate conditional implications.

**Proposition 6.** Let \( C_\phi \) be an Archimedean copula with strict generator \( \phi \) twice differentiable. Then the conditional implication \( I^*_C \phi \) is a fuzzy implication if, and only if, \( \frac{1}{\phi'(t)} \) is concave for all \( t \in (0,1) \).

**Proof.** The copula \( C_\phi \) is concave in the first variable if, and only if, the function \( g : [0,1] \to [0,1] \) given by

\[
g(t) = \phi^{-1}\{\phi(t) + a}\),
\]

where \( a = \phi(v) \), is concave, which is equivalent to

\[
g''(t) = \frac{\phi''(t) (\phi'(\phi^{-1}(\phi(t) + a)))^2 - (\phi'(t))^2 \phi''(\phi^{-1}(\phi(t) + a))}{(\phi'(\phi^{-1}(\phi(t) + a)))^3} \leq 0
\]

for all \( t \in (0,1) \). Since \( \phi \) is strict, we have that \( \phi'(t) < 0 \) and \( \phi''(t) > 0 \) for every \( t \in (0,1) \). Thus, the inequality (4) turns out to be

\[
\frac{\phi''(t)}{(\phi'(t))^2} \geq \frac{\phi''(\phi^{-1}(\phi(t) + a))}{(\phi'(\phi^{-1}(\phi(t) + a)))^2}.
\]

Put \( s = \phi^{-1}(\phi(t) + a) \leq t \). Then inequality (5) amounts to

\[
\left( \frac{1}{\phi'(t)} \right)' \leq \left( \frac{1}{\phi'(s)} \right)'
\]

for all \( t, s \in (0,1) \) with \( t \geq s \), i.e., the derivative of the function \( \frac{1}{\phi'} \) is non-increasing, which implies the concavity of \( \frac{1}{\phi'} \).

In the next two examples, we provide fuzzy implications via families of Archimedean copulas.
Example 5. For every $\alpha > 0$, consider the Clayton family of copulas [6, 18] given by $C_\alpha(u, v) = \{\max(u^{-\alpha} + v^{-\alpha} - 1, 0)\}^{-1/\alpha}$, whose strict generator is $\phi(t) = (t^{-\alpha} - 1)/\alpha$ (the case $\alpha = 0$ corresponds to the product copula). Since $\left(\frac{1}{\phi(t)}\right)'' = -\alpha(\alpha + 1)t^{\alpha-1} < 0$ for all $\alpha > 0$, from Proposition 6 we have that $I_{C_\alpha}$ is an implication for every $\alpha \geq 0$.

Example 6. Consider the Frank family of copulas [9, 18] given by $C_\alpha(u, v) = \log_\alpha \{1 + (\alpha u^{-1})/(\alpha v^{-1})\}$ for all $(u, v) \in [0, 1]^2$ with $\alpha \in [0, +\infty) \setminus \{1\}$ (the case $\alpha = 1$ corresponds to the product copula). The strict generator is given by $\phi(t) = \ln(\frac{1-\alpha}{1-\alpha t})$, for $t \in (0, 1)$. Since $\left(\frac{1}{\phi(t)}\right)'' = \alpha^{-1} \ln \alpha < 0$ for $\alpha \in (0, 1)$, from Proposition 6 we have that $I_{C_\alpha}$ is an implication for every $\alpha \in (0, 1]$.

Acknowledgements

We first thank the comments by the referees in a previous version. The second author acknowledges the support of the Ministerio de Ciencia e Innovación (Spain), under research project MTM2011-22394. The third author acknowledges the support of the Ministerio de Ciencia e Innovación (Spain) and FEDER, under research project MTM2009-08724.

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