Bivariate quasi-copulas and
doubly stochastic signed measures

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Abstract

We show that there exist bivariate proper quasi-copulas that do not induce a doubly stochastic signed measure on $[0, 1]^2$. We construct these quasi-copulas from the so-called proper quasi-transformation square matrices.

Keywords: Copula; Quasi-copula; Signed measure; Doubly stochastic measure; Doubly stochastic signed measure.

1 Introduction

Bivariate copulas and quasi-copulas were introduced in the field of probability (see [18] and [1], respectively), where they have provided a great number of applications (see, for instance, [3, 14]). They are also used in aggregation processes because they ensure that the aggregation is stable, in the sense that small error inputs correspond to small error outputs (see [11]). In the few last years an increasing interest has been devoted to these functions by researchers in some topics of fuzzy sets theory, such as preference modeling, similarities and fuzzy logics: see, for instance, [4, 6, 7, 8, 12].

In the literature, several interesting similarities and differences between copulas and proper quasi-copulas have been shown (see, for instance, [5, 10, 14, 15, 17]). On that score, the following problem has been posed: it is known that every $n$-variate copula induces a multiply stochastic measure on $[0, 1]^n$; can this fact be extended to $n$-variate proper quasi-copulas? Specifically, does every $n$-variate quasi-copula induce a multiply stochastic signed measure on $[0, 1]^n$? The answer to this question is negative when $n \geq 3$ (see [16]). In this paper we prove that the answer is also negative for the bivariate case. Since we will only deal with bivariate copulas and quasi-copulas, we will usually remove—for brevity—the word “bivariate” before the words “copula” and “quasi-copula.”
The paper is structured as follows: In Section 2 we provide some preliminary concepts and results. In Section 3 we study a class of quasi-copulas constructed from a type of square matrices that we call quasi-transformation matrices. Finally, in Section 4 we construct proper bivariate quasi-copulas which do not induce doubly stochastic signed measures on $[0,1]^2$.

## 2 Preliminaries

A copula is a function $C: [0,1]^2 \rightarrow [0,1]$ that satisfies the following two conditions:

(C1) $C(0,t) = C(t,0) = 0$ and $C(1,t) = C(t,1) = t$, for every $t \in [0,1]$,

(C2) $V_C([u_1,u_2] \times [v_1,v_2]) = C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \geq 0$, for all $u_1, u_2, v_1, v_2$ in $[0,1]$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$.

$V_C(B)$ is usually called the $C$-volume of the box $B = [u_1,u_2] \times [v_1,v_2]$.

The original concept of a quasi-copula is not interesting for this paper; thus, we define this concept by one of its characterizations, the one which now is usually utilized as definition (see [10]). A quasi-copula is a function $Q: [0,1]^2 \rightarrow [0,1]$ that satisfies (C1) and, instead of (C2), the following two weaker conditions:

(Q1) $Q$ is increasing in each variable;

(Q2) $Q$ is 1-Lipschitz: $|Q(u_1,v_1) - Q(u_2,v_2)| \leq |u_1 - u_2| + |v_1 - v_2|$ for all $u_1, v_1, u_2, v_2 \in [0,1]$.

While every copula is a quasi-copula, there exist proper quasi-copulas, i.e., quasi-copulas that are not copulas. The following result (see [10]) provides a useful characterization of the concept of a quasi-copula.

**Theorem 2.1.** A function $Q: [0,1]^2 \rightarrow [0,1]$ is a quasi-copula if, and only if, it satisfies (a) $Q(0,t) = Q(t,0) = 0$ and $Q(t,1) = Q(1,t) = t$ for every $t \in [0,1]$, and (b) $V_Q(B) \geq 0$ for every box $B = [u_1,u_2] \times [v_1,v_2]$ in $[0,1]^2$ such that at least one of $u_1,u_2,v_1$ or $v_2$ is either equal to 0 or to 1.

Here $V_Q(B)$, the $Q$-volume of the box $B$, is defined as for copulas.

For each $n = 1,2,\ldots$, let $\mathcal{B}^n$ denote the Borel $\sigma$-algebra for $[0,1]^n$, and let $\mathcal{S}^n$ denote the measurable space $([0,1]^n, \mathcal{B}^n)$. It is known that every bivariate copula $C$ induces a probability measure $\mu_C$ on $\mathcal{S}^2$ such that $\mu_C([0,1] \times A) = \mu_C([A \times [0,1]) = \lambda(A)$ for every set $A$ in $\mathcal{B}^1$, where $\lambda$ denote the Lebesgue measure in $\mathbb{R}$; i.e., $\mu_C$ is a doubly stochastic measure (see, for instance, [14]). This measure is characterized by the fact that $\mu_C(B) = V_C(B)$ for every box $B = [u_1,u_2] \times [v_1,v_2]$ in $[0,1]^2$. It also satisfies $0 \leq \mu_C(D) \leq 1$ for every set $D$ in $\mathcal{B}^2$.

A signed measure $\mu$ on $\mathcal{S}^2$ is an extended real valued, countably additive set function on $\mathcal{B}^2$, such that $\mu(\emptyset) = 0$ and $\mu$ assumes at most one of the values $\infty$ and $-\infty$. Equivalently, $\mu$ is
the difference between two (positive) measures $\mu_1$ and $\mu_2$ on $S^2$, such that at least one of them is finite. For every signed measure $\mu$ on $S^2$, there exist two sets $D_+$ and $D_-$ in $\mathcal{B}^2$ such that (i) $D_+ \cap D_- = \emptyset$, (ii) $D_+ \cup D_- = [0, 1]^2$, and (iii) $\mu(E \cap D_+) \geq 0$ and $\mu(E \cap D_-) \leq 0$ for every $E$ in $\mathcal{B}^2$. The sets $D_+$ and $D_-$ are said to form a Hahn decomposition of $[0, 1]^2$ with respect to $\mu$. For more details see, for instance, [13].

As for positive measures, a signed measure $\mu$ on $S^2$ is said to be doubly stochastic if $\mu([0, 1] \times A) = \mu(A \times [0, 1]) = \lambda(A)$ for every set $A$ in $\mathcal{B}^1$. Thus, a doubly stochastic signed measure $\mu$ on $S^2$ satisfies $\mu([0, 1]^2) = 1$; and, for each Hahn decomposition $D_+$ and $D_-$ of $[0, 1]^2$ with respect to $\mu$, we have $-\infty < \mu(D_-) \leq \mu(E) \leq \mu(D_+) = 1 - \mu(D_-) < \infty$, for every set $E$ in $\mathcal{B}^2$; whence every doubly stochastic signed measure is bounded.

3 Quasi-copulas induced by quasi-transformation matrices

In this section we construct sequences of quasi-copulas from a type of real matrices that we call quasi-transformation matrices. We show several properties of such quasi-copulas and study the limits of those sequences, which will be useful in the next section in order to reach the main goal of this paper.

It is easy to check that the set of quasi-copulas is a closed set in the complete metric space of the continuous functions defined on $[0, 1]^2$, whence we have the following preliminary result:

**Theorem 3.1.** The set of quasi-copulas, endowed with the sup metric, is a complete metric space.

As a consequence, if $\{Q_l : l \in \mathbb{N}\}$ is a sequence of quasi-copulas which converges pointwise, then $\lim_{l \to \infty} Q_l$ is a quasi-copula, and the following property holds immediately:

$$\lim_{l \to \infty} V_{Q_l}(B) = V_{\lim_{l \to \infty} Q_l}(B) \quad \text{for every box } B \text{ in } [0, 1]^2. \quad (3.1)$$

It is known that the set of copulas, endowed with the sup metric, is also a complete metric space. However, the set of proper quasi-copulas is not a complete metric space with respect to the sup metric, since we can find sequences of proper quasi-copulas whose limit is not a proper quasi-copula (i.e., it is a copula). An example of a sequence fulfilling this property can be easily constructed from Theorem 4.1 in [15].

It is known that, for every quasi-copula $Q$, the $Q$-volume of any box in $[0, 1]^2$ must lie between $-1/3$ and 1 (see [15]). Taking into account this fact and part (b) of Theorem 2.1, we consider the following definition in order to (i) represent some quasi-copulas with the help of matrices, and (ii) define successive transformations of a given quasi-copula.

**Definition 3.1.** A **quasi-transformation matrix** is a square matrix $T = (t_{ij}) \ (i, j = 1, 2, \ldots, m)$, with the column index first and the rows ordered from bottom to top, satisfying the following
conditions: (a) every entry \( t_{ij} \) is in \([-1/3, 1]\); (b) the sum of all entries in \( T \) is 1; (c) the sum of the entries in any row or column of \( T \) is positive; and (d) the sum of the entries in any submatrix of \( T \), which contains at least one entry from the first or last row or column of \( T \), is nonnegative.

We note that this definition can be extended to the case of non-square matrices, which would let us to extend many results below in this paper. However, it suffices for our purposes to consider square matrices. Thus, for brevity, we will only deal with this case.

An example of a quasi-transformation matrix of order 4 is the following:

\[
T_1 = \begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0
\end{pmatrix} = \frac{1}{16} \begin{pmatrix}
0 & 2 & 2 & 0 \\
3 & -2 & -1 & 3 \\
3 & 0 & -1 & 3 \\
0 & 2 & 2 & 0
\end{pmatrix}.
\] (3.2)

A quasi-transformation matrix \( T \) is proper if some entry in \( T \) is negative; for instance, the matrix \( T_1 \) above is a proper quasi-transformation matrix. Otherwise, if all entries in a quasi-transformation matrix \( T \) are nonnegative, then \( T \) is a transformation matrix (see [9]).

Let \( p_i \) (respectively, \( q_j \)) denote the sum of the entries in the first \( i \) columns (respectively, first \( j \) rows) of \( T \), for every \( i \) (respectively, \( j \)) in \( \{0, 1, \ldots, m\} \). So \( p_0 = 0 \) and \( p_m = 1 \) (respectively, \( q_0 = 0 \) and \( q_m = 1 \)). For every \( i, j = 1, 2, \ldots, m \), let \( R_{ij} \) be the box in \([0, 1]^2\) given by \( R_{ij} = [p_{i-1}, p_i] \times [q_{j-1}, q_j] \). In what follows, we refer to these boxes as the rectangles associated with the quasi-transformation matrix \( T \). From the definition of quasi-transformation matrix, all these rectangles are not degenerated (they have positive area). The rectangles associated with the quasi-transformation matrix \( T_1 \) given by (3.2) are shown in Figure 1.

Now, for every quasi-transformation matrix \( T = (t_{ij}) \), with associated rectangles \( R_{ij} \) \((i, j = 1, 2, \ldots, m)\), and every quasi-copula \( Q \), the \( T \)-transformation of \( Q \), denoted by \( T(Q) \), is defined as the function on \([0, 1]^2\) given by

\[
T(Q)(u, v) = \sum_{i' < i, j' < j} t_{i'j'} + \frac{u - p_{i-1}}{p_i - p_{i-1}} \sum_{j' < j} t_{i'j} + \frac{v - q_{j-1}}{q_j - q_{j-1}} \sum_{i' < i} t_{i'j} + t_{ij} \cdot Q \left( \frac{u - p_{i-1}}{p_i - p_{i-1}}, \frac{v - q_{j-1}}{q_j - q_{j-1}} \right)
\]

for every \( i, j = 1, 2, \ldots, m \) and every \((u, v) \in R_{ij}\) (the empty sums are considered equal to zero). Roughly speaking, for every \( i, j = 1, 2, \ldots, m \), \( T(Q) \) spreads mass \( t_{ij} \) on \( R_{ij} \) in the same (but re-scaled) way in which \( Q \) spreads mass on \([0, 1]^2\).

The following result is a consequence of Theorem 2.1 and simple computations.

**Theorem 3.2.** Let \( T \) be a quasi-transformation matrix of order \( m \in \mathbb{N} \). Let \( R_{ij} = [p_{i-1}, p_i] \times [q_{j-1}, q_j] \) \((i, j = 1, 2, \ldots, m)\) be the rectangles associated with \( T \). Then, for every quasi-copula \( Q \), the \( T \)-transformation of \( Q \), \( T(Q) \), is a quasi-copula. Moreover, for every \( i, j = 1, 2, \ldots, m \) and every \((u, v) \in R_{ij}\), the following equality holds:

\[
V_{T(Q)}([p_{i-1}, u] \times [q_{j-1}, v]) = t_{ij} \cdot Q \left( \frac{u - p_{i-1}}{p_i - p_{i-1}}, \frac{v - q_{j-1}}{q_j - q_{j-1}} \right).
\]
In particular, we have

\[ V_{T(Q)}(R_{ij}) = t_{ij} \quad \text{for every } i, j = 1, 2, \ldots, m. \] (3.3)

From the first statement in this theorem, given a quasi-transformation matrix \( T \), we can define a mapping from the metric space of quasi-copulas to itself by \( Q \rightarrow T(Q) \). Without confusion, this mapping will be also denoted by \( T \).

The following result proves that the mapping \( T \) is contractive except for a trivial case.

**Theorem 3.3.** Let \( T = (t_{ij}) \) be a quasi-transformation matrix of order \( m \), and let

\[ \alpha = \max(|t_{ij}| : 1 \leq i, j \leq m). \] (3.4)

If \( d \) denotes the sup metric, then

\[ d(T(Q_1), T(Q_2)) = \alpha \cdot d(Q_1, Q_2) \] (3.5)

for any pair of quasi-copulas \( Q_1 \) and \( Q_2 \). As a consequence, \( T \) is a contraction mapping if, and only if, \( m \geq 2 \).

**Proof.** The proof of the equality (3.5) is simple, and analogous to that of part (b) of Proposition 4 in [9]. From the definition of a quasi-transformation matrix, we have \( 0 < \alpha \leq 1 \). Let \( R_{ij} \) be the rectangles associated with the quasi-transformation matrix \( T \). Observe that the only quasi-transformation matrix of order 1 is \( T = (1) \). In this case, the only \( R_{ij} \) is the unit square \([0,1]^2\), and the number \( \alpha \) defined by (3.4) is \( \alpha = 1 \) (the mapping \( T \) is an isometry). Suppose now that \( T \neq (1) \), i.e., \( T \) is a quasi-transformation matrix of order greater than 1. For every quasi-copula \( Q \) and a box \( B \), if \( V_Q(B) = 1 \) then \( B = [0,1]^2 \) (see [15]). Therefore, in this case \( V_Q(R_{ij}) < 1 \) for every \( i, j = 1, 2, \ldots, m \) and we have \( \alpha < 1 \), which completes the proof. \( \square \)
From every quasi-copula $Q$ and every quasi-transformation matrix $T$, we can obtain a sequence of quasi-copulas by successive compositions of the associated mapping $T$: $T^0(Q) = Q$ and $T^l(Q) = T(T^{l-1}(Q))$ for every $l \in \mathbb{N}$. Let $R_{ij}$ $(i, j = 1, 2, \ldots, m)$ be the rectangles associated with the matrix $T$, and let $Q_1$ and $Q_2$ be quasi-copulas. From the definition of $T^l$, observe that the properties (3.3) and (3.5) can be extended to the following:

$$V_{T^l(Q)}(R_{ij}) = t_{ij} \quad \text{for every } i, j = 1, 2, \ldots, m \text{ and every } l \in \mathbb{N},$$

and

$$d(T^l(Q_1), T^l(Q_2)) = \alpha^l \cdot d(Q_1, Q_2) \quad \text{for every } l \in \mathbb{N},$$

respectively.

The following result provides other properties satisfied by the quasi-copulas of the form $T^l(Q)$.

**Theorem 3.4.** Let $T = (t_{ij})$ be a quasi-transformation matrix of order $m \geq 2$, and let $l \in \mathbb{N}$.

1. If $T$ is proper, then $T^l(Q)$ is a proper quasi-copula for every quasi-copula $Q$.
2. If $T$ is a transformation matrix, then $T^l(Q)$ is a proper quasi-copula if, and only if, $Q$ is a proper quasi-copula.

**Proof.** The first part of the theorem is an immediate consequence of property (3.6). For the second part, it is sufficient to prove that $T(Q)$ is a proper quasi-copula if, and only if, $Q$ is a proper quasi-copula. First, suppose that $T(Q)$ is a proper quasi-copula. Then, $Q$ cannot be a copula since $T$ is a transformation matrix: if $Q$ is a copula then $T(Q)$ is also a copula (see [9]). Thus, $Q$ is a proper quasi-copula. Conversely, if $Q$ is a proper quasi-copula, then there exists a box $B = [u_1, u_2] \times [v_1, v_2]$ such that $V_Q(B) < 0$. Let $R_{ij} = [p_{i-1}, p_i] \times [q_{j-1}, q_j] \ (i, j = 1, 2, \ldots, m)$ be the rectangles associated with $T$. It is easy to check that, for each $i, j = 1, 2, \ldots, m$ such that $t_{ij} > 0$, the box $B' = [p_{i-1} + u_1(p_i - p_{i-1}), p_{i-1} + u_2(p_i - p_{i-1})] \times [q_{j-1} + v_1(q_j - q_{j-1}), q_{j-1} + v_2(q_j - q_{j-1})]$ is included in $R_{ij}$ and $V_{T(Q)}(B') = t_{ij} V_Q(B) < 0$, whence the proof follows. □

Theorems 3.1 and 3.3, and the Contraction-Mapping Theorem yield one of the main results in this paper.

**Theorem 3.5.** Let $T$ be a quasi-transformation matrix of order greater than 1. Then, there is a unique quasi-copula $Q_T$ for which $T(Q_T) = Q_T$. Moreover, for every quasi-copula $Q$, $Q_T = \lim_{l \to \infty} T^l(Q)$.

If the hypothesis in Theorem 3.5 does not hold, i.e., if $T = (1)$, then it is clear that every quasi-copula is a fixed point of $T$: $T(Q) = Q$ for every quasi-copula $Q$.

The following result extends Theorem 3.4 to the limit.
Theorem 3.6. Let $T = (t_{ij})$ be a quasi-transformation matrix of order $m \geq 2$. Let $Q_T$ be the fixed point of the mapping $T$. Then, $Q_T$ is a proper quasi-copula if, and only if, $T$ is a proper quasi-transformation matrix.

Proof. From Theorem 2 in [9], if $T$ is not proper then $Q_T$ is a copula. On the other hand, suppose that $t_{ij} < 0$ for some $i, j = 1, 2, \ldots, m$. Let $R_{ij}$ be the respective rectangle associated with $T$. From properties (3.1) and (3.6), we have $V_{Q_T}(R_{ij}) = V_{\lim_{l \to \infty} T^l(Q)}(R_{ij}) = \lim_{l \to \infty} V_{T^l(Q)}(R_{ij}) = t_{ij} < 0$, whence $Q_T$ is a proper quasi-copula. □

Observe that, as a trivial consequence of Theorem 3.6, we have that $Q_T$ is a copula if, and only if, $T$ is a transformation matrix.

4 Bivariate quasi-copulas which do not induce doubly stochastic signed measures

Let $Q$ be a quasi-copula, and let $\mathcal{B}$ denote the algebra generated by all the boxes in $[0, 1]^2$. There exists a unique finitely additive (signed when $Q$ is proper) measure $m_Q$ such that $m_Q(B) = V_Q(B)$ for every box $B$ in $[0, 1]^2$. Observe that, as a consequence, $m_Q([0, 1] \times [a, b]) = m_Q([a, b] \times [0, 1]) = \lambda([a, b]) = b - a$ for every closed interval $[a, b]$ in $[0, 1]$. For a detailed study on finitely additive signed measures, also called charges, see [2].

At a first sight, it seems that every proper quasi-copula $Q$ appearing in the literature induces a doubly stochastic signed measure $\mu_Q$ on $S^2 = ([0, 1]^2, \mathcal{B}^2)$. An example of this fact has been shown in [16]. In this section we show that this fact is not general, i.e., there exist proper quasi-copulas $Q$ such that the finitely additive signed measure $m_Q$ associated with $Q$ cannot be extended to a doubly stochastic signed measure $\mu_Q$ on $S^2$. We will find these quasi-copulas among the quasi-copulas $Q_T$ introduced in Section 3.

Let $T = (t_{ij})$ be a quasi-transformation matrix of order $m \geq 2$, and let $l \in \mathbb{N}$. Let $R_{ij} = [p_{i-1}, p_i] \times [q_{j-1}, q_j]$ $(i, j = 1, 2, \ldots, m)$ be the rectangles associated with $T$, and let $\Delta p_r = p_r - p_{r-1}$ and $\Delta q_r = q_r - q_{r-1}$ for every $r = 1, 2, \ldots, m$. From the definition of $T^l$, given any quasi-copula $Q$, the definition of the quasi-copula $T^l(Q)$ is associated with $m^{2l}$ rectangles $R_{ir} = [p'_{i-1}, p'_i] \times [\sigma_{r-1}^i, \sigma_r^i]$ in $[0, 1]^2$, with $i, \kappa = 1, 2, \ldots, m^l$ (for $l = 1$ we obtain again the rectangles $R_{ij}$). The indices $\iota$ and $\kappa$ can be obtained from other indices $i_1, i_2, \ldots, i_l = 1, 2, \ldots, m$ and $j_1, j_2, \ldots, j_l = 1, 2, \ldots, m$ as follows:

$$\iota(i_1, i_2, \ldots, i_l) = 1 + \sum_{k=1}^{l} (i_k - 1)m^{l-k} \quad \text{and} \quad \kappa(j_1, j_2, \ldots, j_l) = 1 + \sum_{k=1}^{l} (j_k - 1)m^{l-k}$$

(observe that $\iota(i_1, i_2, \ldots, i_l) < \iota(i'_1, i'_2, \ldots, i'_l)$ if, and only if, the first $k = 1, 2, \ldots, l$ such that $i_k \neq i'_k$ satisfies $i_k < i'_k$; and an analogous observation can be made for the indices $\kappa(j_1, j_2, \ldots, j_l)$);
and then, the numbers $\rho^t_i$ and $\sigma^t_i$ are given by $\rho^0_i = \sigma^0_i = 0$ and

$$
\rho^t_i(i_1, i_2, \ldots, i_l) = p_{i_1-1} + \Delta p_{i_1} (p_{i_2-1} + \Delta p_{i_2}(p_{i_3-1} + \Delta p_{i_3} \ldots (p_{i_{l-2}} + \Delta p_{i_{l-2}}(p_{i_{l-1}} + \Delta p_{i_{l-1}} p_{i_l})) \ldots)),
$$

$$
\sigma^t_i(l_j, j_1, j_2, \ldots, j_l) = q_{j_1-1} + \Delta q_{j_1} (q_{j_2-1} + \Delta q_{j_2} (q_{j_3-1} + \Delta q_{j_3} \ldots (q_{j_{l-2}} + \Delta q_{j_{l-2}}(q_{j_{l-1}} + \Delta q_{j_{l-1}} q_{j_l})) \ldots)),
$$

for every $i_1, i_2, \ldots, i_l, j_1, j_2, \ldots, j_l = 1, 2, \ldots, m$. Then, it is clear that

$$
V^t_{T(Q)}(R^t_{i_1, i_2, \ldots, i_l} l_{j_1, j_2, \ldots, j_l}) = \prod_{k=1}^l t_{i_k j_k}, \quad \text{for all } i_1, i_2, \ldots, i_l, j_1, j_2, \ldots, j_l = 1, 2, \ldots, m, \quad (4.1)
$$

which is a new generalization of property (3.3).

Given a proper quasi-transformation matrix $T$, a quasi-copula $Q$ and a natural number $l$, let $\mathcal{R}^l_+$ (respectively, $\mathcal{R}^l_-$) be the set of boxes $R^l_{i_1, i_2, \ldots, i_l} l_{j_1, j_2, \ldots, j_l}$ such that their $T^l(Q)$-volumes are positive (respectively, negative). In order to prove the next result, we need to compute the sum of all these positive (respectively, negative) $T^l(Q)$-volumes, sum which we denote by $V^t_+$ (respectively, $V^t_-$):

$$
V^t_+ = \sum_{R^l_{i_1, i_2, \ldots, i_l} l_{j_1, j_2, \ldots, j_l} \in \mathcal{R}^l_+} V^t_{T(Q)}(R^l_{i_1, i_2, \ldots, i_l} l_{j_1, j_2, \ldots, j_l}),
$$

$$
V^t_- = \sum_{R^l_{i_1, i_2, \ldots, i_l} l_{j_1, j_2, \ldots, j_l} \in \mathcal{R}^l_-} V^t_{T(Q)}(R^l_{i_1, i_2, \ldots, i_l} l_{j_1, j_2, \ldots, j_l}).
$$

Let $-\tau$ be the sum of all negative entries of the matrix $T$ ($\tau > 0$). Then, the sum of all positive entries of the matrix $T$ is $1 + \tau$, and it is clear that $V^t_+ = 1 + \tau$ and $V^t_- = -\tau$. From (4.1) we have that $V^2_+$ is the sum of all positive products of the form $t_{i_1 j_1} t_{i_2 j_2}$, and $V^2_-$ is the sum of all negative products, i.e.,

$$
V^2_+ = \sum_{t_{i_1 j_1} t_{i_2 j_2} > 0} t_{i_1 j_1} t_{i_2 j_2} + \sum_{t_{i_1 j_1} t_{i_2 j_2} < 0} t_{i_1 j_1} t_{i_2 j_2} = \sum_{t_{i_1 j_1} > 0} t_{i_1 j_1} \sum_{t_{i_2 j_2} > 0} t_{i_2 j_2} = (V^1_+)^2 + (V^1_-)^2.
$$

$$
V^2_- = \sum_{t_{i_1 j_1} t_{i_2 j_2} > 0} t_{i_1 j_1} t_{i_2 j_2} + \sum_{t_{i_1 j_1} t_{i_2 j_2} < 0} t_{i_1 j_1} t_{i_2 j_2} = V^1_+ V^1_- + V^1_- V^1_+ = 2V^1_+ V^1_-.
$$

Similar reasonings yield the extension of the last two equalities to the following:

$$
V^k_+ = V^{k-1}_+ V^1_+ + V^{k-1}_- V^1_+ \quad \text{and} \quad V^k_- = V^{k-1}_+ V^1_- + V^{k-1}_- V^1_- \quad \text{for every } k = 2, 3, \ldots
$$

Then, by induction, we can easily obtain

$$
V^t_+ = \frac{1 + (1 + 2\tau)^l}{2} \quad \text{and} \quad V^t_- = \frac{1 - (1 + 2\tau)^l}{2} \quad \text{for every } l \in \mathbb{N}. \quad (4.3)
$$

The following example illustrates some of the properties studied in this section.
Example 4.1. Let $T$ be the quasi-transformation matrix given by

$$T = \begin{pmatrix} 0 & 1/3 & 0 \\ 1/3 & -1/3 & 1/3 \\ 0 & 1/3 & 0 \end{pmatrix},$$

and let $Q$ be a quasi-copula. The sum of the negative entries of $T$ is $-1/3$, i.e., $\tau = 1/3$. The rectangles associated with $T$ are $R_{ij} = [\frac{i-1}{3}, \frac{i}{3}) \times [\frac{j-1}{3}, \frac{j}{3})$, with $i, j = 1, 2, 3$. And, for every $l \in \mathbb{N}$, the rectangles associated with $T^l(Q)$ are $R_{i\kappa}^l = [\frac{i-1}{3^l}, \frac{i}{3^l}) \times [\frac{\kappa-1}{3^l}, \frac{\kappa}{3^l})$, with $i, \kappa = 1, 2, \ldots 3^l$. Observe that $V_{T^l(Q)}(R_{i\kappa}^l)$ can be either positive (specifically, $3^{-l}$), or negative (equal to $-3^{-l}$), or zero. Moreover, from (4.3), we have $V_+^l = \frac{1}{2} \left[ 1 + \left( \frac{3}{2} \right)^l \right]$ and $V_-^l = \frac{1}{2} \left[ 1 - \left( \frac{3}{2} \right)^l \right]$.

Before introducing the main result of this paper, we need a further result about $T^l(Q)$-volumes. From the definition of $T$-transformation of a quasi-copula, since $T^{l+1}(Q) = T(T^l(Q))$, it is clear that $V_{T^{l+1}(Q)}(R_{i_1j_1,\ldots,i_lj_l}) = V_{T^l(Q)}(R_{i_1j_1,\ldots,i_lj_l}) = \prod_{k=1}^l t_{i_kj_k}$ for all $i_1, i_2, \ldots, i_l, j_1, j_2, \ldots, j_l = 1, 2, \ldots, m$. Hence, it is easy to conclude

$$V_{T^h(Q)}(R_{i_1j_1,\ldots,i_lj_l}) = V_{T^l(Q)}(R_{i_1j_1,\ldots,i_lj_l}) = \prod_{k=1}^l t_{i_kj_k} \quad (4.4)$$

for every $h \geq l$ and $i_1, i_2, \ldots, i_l, j_1, j_2, \ldots, j_l = 1, 2, \ldots, m$.

Finally, we can prove the following theorem, which gives an answer to the main question posed in this paper.

Theorem 4.1. For every proper quasi-transformation matrix $T$, the proper quasi-copula $Q_T$ does not induce a doubly stochastic signed measure on $[0, 1]^2$.

Proof. Let $T$ be a proper quasi-transformation matrix. Let $\tau$ be the positive real number such that the sum of the positive entries of $T$ is $1 + \tau$. From Theorems 3.5 and 3.6, we know that $Q_T$ is a proper quasi-copula that can be obtain as a limit: $Q_T = \lim_{l \to \infty} T^l(Q)$ for any quasi-copula $Q$. Suppose $Q_T$ induces a doubly stochastic signed measure $\mu_{Q_T}$ on $[0, 1]^2$. For every $l \in \mathbb{N}$, we consider the rectangles $R_{i_1j_1,\ldots,i_lj_l}^l (i_1, i_2, \ldots, i_l, j_1, j_2, \ldots, j_l = 1, 2, \ldots, m)$ associated with the quasi-copula $T^l(Q)$, as shown above in this section. From property (3.1), since equality (4.4) holds for every $h \geq l$ and $i_1, i_2, \ldots, i_l, j_1, j_2, \ldots, j_l = 1, 2, \ldots, m$, we have

$$V_{Q_T}(R_{i_1j_1,\ldots,i_lj_l}^l) = \lim_{h \to \infty} T^h(Q)(R_{i_1j_1,\ldots,i_lj_l}^l) = \lim_{h \to \infty} V_{T^h(Q)}(R_{i_1j_1,\ldots,i_lj_l}^l) = V_{T^l(Q)}(R_{i_1j_1,\ldots,i_lj_l}^l)$$

for every $i_1, i_2, \ldots, i_l, j_1, j_2, \ldots, j_l = 1, 2, \ldots, m$. From (4.2) and (4.3), if

$$U_+^l = \bigcup_{R_{i_1j_1,\ldots,i_lj_l}^l \in R_+^l} R_{i_1j_1,\ldots,i_lj_l}^l,$$

then $\mu_{Q_T}(U_+^l) = \frac{1 + (1 + 2\tau^l)}{2}$. Therefore, $\lim_{l \to \infty} \mu_{Q_T}(U_+^l) = \infty$, i.e., $\mu_{Q_T}$ is not bounded, whence $\mu_{Q_T}$ is not a doubly stochastic signed measure on $[0, 1]^2$, against our hypothesis.
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References


