ON THE CLASSES OF COPULAS AND QUASI-COPULAS WITH A
GIVEN DIAGONAL SECTION

FABRIZIO DURANTE
Department of Knowledge-Based Mathematical Systems
Johannes Kepler University, Linz, Austria
School of Economics and Management
Free University of Bozen-Bolzano, Bolzano, Italy
e-mail: fabrizio.durante@unibz.it

JUAN FERNÁNDEZ-SÁNCHEZ
Grupo de Investigación de Análisis Matemático
Universidad de Almería, La Cañada de San Urbano, Almería, Spain
e-mail: juanfernandez@ual.es

Received 12 February 2010
Revised August 24, 2010

Let $C_δ$ and $Q_δ$ be, respectively, the classes of all copulas and quasi–copulas whose diagonal section is $δ$. We determine under which conditions on $δ$ we have: (a) both $C_δ$ and $Q_δ$ consist of a singleton; (b) $C_δ = Q_δ$. Moreover, a simple construction of copulas with a given convex diagonal section is introduced.

Keywords: Copula; Section of a copula; Tail dependence.

1. Introduction

A two–dimensional copula $^1^,$ $^2$ (a copula, for short) is a function $C: I^2 \to I$ ($I = [0, 1]$) that satisfies the following properties:

(C1) $C(x, 1) = C(1, x) = x$ for all $x \in I$;
(C2) $C$ is increasing in each variable;
(C3) for all $x_1, x_2, y_1, y_2$ in $I$ with $x_1 \leq x_2$ and $y_1 \leq y_2$,

$$V_C ([x_1, x_2] \times [y_1, y_2]) = C(x_2, y_2) - C(x_1, y_2) - C(x_2, y_1) + C(x_1, y_1) \geq 0.$$

Condition (C3) is called the 2–increasing property of $C$, and $V_C ([x_1, x_2] \times [y_1, y_2])$ is called the $C$–volume of the rectangle $[x_1, x_2] \times [y_1, y_2]$.

According to Sklar’s Theorem, copulas are functions that express the rank–invariant dependence between a pair of continuous random variables and, as such, they have been largely used in statistical applications $^1$, $^3$. In particular, examples of copulas are given by $M(x, y) = \min\{x, y\}$, $\Pi(x, y) = xy$ and $W(x, y) =$
\[ \max\{x + y - 1, 0\} \] expressing, respectively, comonotonicity, independence and countemonotonicity between random variables.

Today, copulas (and their generalizations) are also of interest in many other fields, requiring the aggregation of incoming data, like multi–criteria decision making and fuzzy set theory \(^4,5\). In particular, in view of its stability with respect to the aggregation of different inputs into a single output, the concept of quasi–copula, which generalizes the notion of copula, has been used \(^6,7,8\). We recall that a quasi–copula \(Q: \mathbb{I}^2 \rightarrow \mathbb{I}\) is a function that satisfies (C1), (C2), and the 1–Lipschitz condition, i.e., for all \(x_1, x_2, y_1\) and \(y_2\) in \(\mathbb{I}\),

\[ |Q(x_1, y_1) - Q(x_2, y_2)| \leq |x_1 - x_2| + |y_1 - y_2|. \]

The diagonal section of a quasi–copula \(Q\) is the function \(\delta_Q : \mathbb{I} \rightarrow \mathbb{I}\) defined by \(\delta_Q(t) = Q(t, t)\). When \(Q\) is a copula, its diagonal section provides some information about the tail dependence \(^9\) of a bivariate random vector whose associated copula is \(Q\).

During recent years, special attention has been devoted to the construction of copulas (and quasi–copulas) with a given diagonal section. Specifically, given a suitable \(\delta: \mathbb{I} \rightarrow \mathbb{I}\), (hereafter called diagonal), the class of (quasi–)copulas having diagonal section equal to \(\delta\) has been the object of several investigations \(^10,11,12,13,14,15,16\). Although a number of investigations has been conducted in this direction, it is worth of further consideration to clarify some aspects about the classes \(C_\delta\) and \(Q_\delta\) formed, respectively, by all copulas and quasi–copulas whose diagonal section is \(\delta\).

As known \(^1\), \(C_\delta\) and \(Q_\delta\) are nonempty sets for any diagonal \(\delta\); moreover, \(C_\delta \subseteq Q_\delta\). In particular, if \(\delta = \delta_M\), the diagonal section of the copula \(M\), then \(C_{\delta_M} = Q_{\delta_M} = \{M\}\) (see, for instance, Proposition 2.7 in the paper \(^17\)). However, as stressed by Nelsen et al. \(^16\), two problems are still open in this context:

- determine which diagonals \(\delta\) ensure that \(C_\delta\) (or \(Q_\delta\)) is a singleton.
- determine under which conditions on a diagonal \(\delta\), \(C_\delta = Q_\delta\).

In the following, by exploiting recent developments in copula theory, we provide an answer to these questions (Section 3). Preliminarily, we recall some known facts on copulas with given diagonal section and present another simple construction method (Section 2). Section 4 concludes.

### 2. Constructions of copulas with given diagonal section

We call diagonal every function \(\delta: \mathbb{I} \rightarrow \mathbb{I}\) that satisfies

- (D1) \(\delta(1) = 1\);
- (D2) \(\delta(t) \leq t\) for all \(t \in \mathbb{I}\);
- (D3) \(\delta\) is increasing;
- (D4) \(|\delta(v) - \delta(u)| \leq 2|v - u|\) for all \(u, v \in \mathbb{I}\).

The diagonal section \(\delta_C\) of every copula \(C\) is a diagonal.
As already mentioned, given a diagonal $\delta$, one is interested in determining $C_\delta$. To the best of our knowledge, first examples of copulas in $C_\delta$ are represented by the so-called diagonal copulas $^{18,19}$, defined by the expression

$$K_\delta(x, y) = \min \left\{ x, y, \frac{\delta(x) + \delta(y)}{2} \right\}. \quad (1)$$

However, it should be also mentioned that a copula belonging to $C_\delta$ could be also derived from previous work by Bertino $^{20,21}$. Based on some recent results $^{14,22}$, diagonal copulas have a nice probabilistic interpretation (which can serve also as a tool to prove that they are actually copulas). In fact, note that, for every $(x, y) \in \mathbb{I}^2$, one has

$$K_\delta(x, y) = \min \left\{ \frac{2x - \delta(x)}{2}, \delta(y) \right\} + \min \left\{ \frac{2y - \delta(y)}{2}, \delta(x) \right\}.$$ 

Now, the functions $F$ and $G$ from $\mathbb{I}^2$ to $\mathbb{I}$ given by

$$F(x, y) = \min \{2x - \delta(x), \delta(y)\} \quad \text{ and } \quad G(x, y) = \min \{2y - \delta(y), \delta(x)\}$$

are 2-increasing. In fact, they are bivariate distribution functions obtained by composing the copula $M(x, y) = \min \{x, y\}$ with the univariate distribution functions $t \mapsto \delta(t)$, $t \mapsto (2t - \delta(t))$ (such mappings are increasing as a consequence of (D3) and (D4)). Thus, any diagonal copula is a convex combination of two bivariate distribution functions with the same copula, namely $M$, but different marginals.

Under additional assumption about a diagonal $\delta$, several other copulas in $C_\delta$ can be constructed $^{23,10,24,25,26,27}$. Here we would like to propose another construction of such a type.

**Proposition 1.** For every convex diagonal $\delta$, the function $C_\delta: \mathbb{I}^2 \to \mathbb{I}$ given by

$$C_\delta(x, y) = \min \left\{ x, y, \delta \left( \frac{x + y}{2} \right) \right\}. \quad (2)$$

is a copula.

Before proving this result, we need some preliminary results. The first lemma is contained in Proposition 4.B.2 of $^{28}$ The second one is presented in Theorem 7.1 of $^{29}$.

**Lemma 1.** Let $A$ be an interval of $\mathbb{R}$ and let $f: A \to \mathbb{R}$. If $f$ is convex and increasing, then, for every $a_1, a_2, a_3, a_4 \in A$ such that

$$a_1 \leq \min\{a_2, a_3\} \leq \max\{a_2, a_3\} \leq a_4,$$

and $a_1 + a_4 \geq a_2 + a_3$, one has

$$f(a_1) + f(a_4) \geq f(a_2) + f(a_3).$$
Lemma 2. For every mapping \( A : \mathbb{I}^2 \rightarrow \mathbb{I} \) that is \( 2 \)–increasing and \( 1 \)–Lipschitz in each argument with \( A(0,0) = 0 \) and \( A(1,1) = 1 \), the function

\[
C(x,y) = \min\{x, y, A(x,y)\}
\]

is a copula.

Proof. [Proof of Proposition 1] Let \( A : \mathbb{I}^2 \rightarrow \mathbb{I} \) be defined by

\[
A(x,y) = \delta \left( \frac{x+y}{2} \right).
\]

In order to prove that \( C_\delta \) is a copula, we have just to verify that \( A \) satisfies the assumptions of Lemma 2. Easy calculations show that \( A(0,0) = 0 \) and \( A(1,1) = 1 \). Moreover, since \( \delta \) satisfies (D4), for every \( x, y, x', y' \in [0,1] \), we have that

\[
|A(x,y) - A(x',y')| = \left| \delta \left( \frac{x+y}{2} \right) - \delta \left( \frac{x'+y'}{2} \right) \right| \leq |x - x'| + |y - y'|,
\]

i.e. \( A \) is 1–Lipschitz in each argument.

Finally, for every \( x, y, x', y' \in [0,1] \) with \( x \leq x' \) and \( y \leq y' \), we have that

\[
A(x,y) + A(x',y') - A(x,y') - A(x',y) = \delta \left( \frac{x+y}{2} \right) + \delta \left( \frac{x'+y'}{2} \right) - \delta \left( \frac{x'+y}{2} \right) - \delta \left( \frac{x+y'}{2} \right),
\]

and this expression is non-negative since \( \delta \) is convex and, hence, Lemma 1 can be used.

3. On the structure of the classes \( \mathcal{C}_\delta \) and \( \mathcal{Q}_\delta \)

Now, we discuss the two problems that we have formulated in Section 1. First of all, we are interested in those diagonals \( \delta \) such that \( \mathcal{C}_\delta \) is a singleton. The following result characterizes such diagonals.

Proposition 2. Let \( \delta \) be a diagonal. The following statements are equivalent:

(a) there exists a copula \( C \) such that \( \mathcal{C}_\delta = \{C\} \);
(b) \( \delta = \delta_M \), or, in other words, \( C = M \).

In particular, since \( \mathcal{C}_\delta \subseteq \mathcal{Q}_\delta \), one can easily obtain that \( \mathcal{Q}_\delta \) is a singleton only when \( \delta = \delta_M \).

Here, we would like to present two proofs of Proposition 2, since each of them underlines different aspects of the structure of the class of copulas and is grounded on different techniques. Notice that, in order to prove Proposition 2, it is enough to show that (a) implies (b), since the other implication is true as a consequence of Proposition 2.7 in \(^1\)\(^7\).
We introduce some notations. Given a function $H : [a_1, a_2] \times [b_1, b_2] \to [c_1, c_2]$, the margins of $H$ are the functions $h_{b_1}^H, h_{b_2}^H, v_{a_1}^H$, and $v_{a_2}^H$ defined by

\[
\begin{align*}
    h_{b_1}^H &: [a_1, a_2] \times [b_1, b_2] \to [c_1, c_2], & h_{b_2}^H &= H(x, b_1), \\
    h_{b_2}^H &: [a_1, a_2] \times [b_1, b_2] \to [c_1, c_2], & h_{b_2}^H &= H(x, b_2), \\
    v_{a_1}^H &: [a_1, a_2] \times [b_1, b_2] \to [c_1, c_2], & v_{a_1}^H &= H(a_1, y), \\
    v_{a_2}^H &: [a_1, a_2] \times [b_1, b_2] \to [c_1, c_2], & v_{a_2}^H &= H(a_2, y).
\end{align*}
\]

**Proof.** [First proof of Proposition 2] Suppose that $C_{\delta}$ has the copula $C$ as its unique element. Suppose, ab absurdo, that $\delta(t_0) \neq t_0$ for some $t_0 \in [0, 1]$. Set $R = [0, t_0] \times [t_0, 1]$. As a consequence of property (C1), one has $V_C(R) = t_0 - \delta(t_0) = \lambda_R > 0$. Let $C_1$ and $C_2$ be two different copulas, $C_1 \neq C_2$, and consider, for $i = 1, 2$, the functions:

\[
\tilde{C}_i(x, y) = \begin{cases} 
    \lambda_R C_i \left( \frac{V_C([0, x] \times [t_0, 1])}{\lambda_R}, \frac{V_C([0, t_0] \times [t_0, y])}{\lambda_R} \right) + h_{C_i}^r(x), & (x, y) \in R, \\
    C(x, y), & \text{otherwise}.
\end{cases}
\]

In view of Theorem 2.2 in $\S 30$, $\tilde{C}_1$ and $\tilde{C}_2$ are copulas, $\tilde{C}_1 \neq \tilde{C}_2$ (notice that, possibly, $\tilde{C}_1$, respectively $\tilde{C}_2$, may be equal to $C$). Moreover, one can easily check that $\tilde{C}_1$ and $\tilde{C}_2$ share the same diagonal section that is equal to $\delta$. As a consequence, $C_{\delta}$ consists of at least two distinct elements, which is a contradiction. It follows that $\delta$ must be equal to $\delta_M$. \(\square\)

**Proof.** [Second proof of Proposition 2] Suppose that the set $C_{\delta}$ has the copula $C$ as its unique element. Suppose, ab absurdo, that $\delta(t_0) \neq t_0$ for some $t_0 \in [0, 1]$. Then, $C$ is not absolutely continuous. In fact, one can associate to the diagonal $\delta$ the singular copula $K_{\delta}$ given by (1). Therefore, if $C$ were absolutely continuous, then $C_{\delta}$ would contain at least two elements.

Now, it is known from Theorem 2.3 in $\S 10$ that $C_{\delta}$ contains an absolutely continuous copula when $\delta(t) < t$ for all $t \in [0, 1]$. Therefore, since $\delta$ is a Lipschitz function, there exist a non-empty interval $[t_1, t_2] \neq \emptyset$ such that $\delta(t_1) = t_1$, $\delta(t_2) = t_2$ and $\delta(t) < t$ on $[t_1, t_2]$. Moreover, Theorem 3.2.1 in $\S 1$ ensures that $C$ is an ordinal sum, and, hence, $C(x, y) = \min\{x, y\}$ on the boundary of $[t_1, t_2]^2$. It can be easily checked that the function $C_1 : \mathbb{I}^2 \to \mathbb{I}$ given by

\[
C_1(x, y) = \frac{C((t_2 - t_1)x + t_1, (t_2 - t_1)y + t_1) - t_1}{t_2 - t_1},
\]

is a copula whose diagonal section is

\[
\delta_1(t) = \frac{\delta((t_2 - t_1)t + t_1) - t_1}{t_2 - t_1}.
\]

Notice that $\delta_1(t) < t$ for all $t \in [0, 1]$. Let $C_2$ be another copula whose diagonal section is $\delta_1$ and $C_2 \neq C_1$. Such a $C_2$ may be either an absolutely continuous copula...
whose diagonal is $\delta_1$ (whose existence is guaranteed by Theorem 2.3 in 10) or a singular copula of type (1). Now, let us define a copula $\tilde{C}$ in the following way:

$$
\tilde{C}(x, y) = \begin{cases} 
  t_1 + (t_2 - t_1)C_2 \left( \frac{x-t_1}{t_2-t_1}, \frac{y-t_1}{t_2-t_1} \right), & (x, y) \in \left[ t_1, t_2 \right] \times \left[ t_1, t_2 \right], \\
  C(x, y), & \text{otherwise}.
\end{cases}
$$

Then, $\tilde{C} \in \mathcal{C}_\delta$ and $\tilde{C} \neq C$. As a consequence, $\mathcal{C}_\delta$ contains two elements, which is a contradiction. It follows that $\delta$ must be equal to $\delta_M$.

The latter version of the proof of Proposition 2 can be used in order to prove a $d$–dimensional analogous ($d \geq 3$) of this result, as stated in the following corollary.

**Corollary 1.** Let $\delta$ be a $d$–diagonal, i.e. $\delta$ satisfies (D1)–(D3) and

$$(D4) \ |\delta(v) - \delta(u)| \leq d|v - u| \text{ for all } u, v \in I.$$  

The following statements are equivalent:

(a) the class of all $d$–dimensional copulas whose diagonal section is $\delta$ consists of a singleton;

(b) $\delta(t) = t$ on $I$.

The proof of Corollary 1 can be obtained by mimicking the second version of the proof of Proposition 2 and by using the results presented in 14 concerning the class of $d$–dimensional copulas with given diagonal section.

**Remark 1.** Notice that, if the class $\mathcal{C}_\delta$ contains two elements, then it contains an infinitude of elements. In fact, it can be trivially checked that, if $C_1, C_2 \in \mathcal{C}_\delta$ and $C_1 \neq C_2$, then $\alpha C_1 + (1 - \alpha) C_2 \in \mathcal{C}_\delta$ for every $\alpha \in I$.

Now, let us consider the second open problem of our interest. Let $\delta$ be a diagonal and consider the classes $\mathcal{C}_\delta$ and $\mathcal{Q}_\delta$. It is well known that $\mathcal{C}_\delta = \mathcal{Q}_\delta$, when $\delta = \delta_M$ (see, for instance, Proposition 2.7 in 17). In the following, we prove that, for any other diagonal $\delta \neq \delta_M$, $\mathcal{C}_\delta$ is strictly included in $\mathcal{Q}_\delta$.

**Proposition 3.** Let $\delta$ be a diagonal. Then $\mathcal{C}_\delta = \mathcal{Q}_\delta$ if, and only if, $\delta = \delta_M$.

**Proof.** Let us suppose that $\mathcal{C}_\delta = \mathcal{Q}_\delta$ and suppose, ab absurdo, that $\delta(t_0) \neq t_0$ for some $t_0 \in [0, 1]$. As in the first proof of Proposition 2, set $R = [0, t_0] \times [t_0, 1]$ and $V_C(R) = \lambda_R > 0$. Let $Q$ be a proper quasi–copula, i.e. a quasi–copula that is not a copula (its existence is guaranteed 7). Consider the function

$$
\tilde{C}(x, y) = \begin{cases} 
  D(x, y), & (x, y) \in R, \\
  C(x, y), & \text{otherwise},
\end{cases}
$$

(4)
where \( D: R \rightarrow I \), \( D(x, y) = \lambda_R Q (F(x), G(y)) + h_\lambda^C(x) \), while \( F: [0, t_0] \rightarrow I \) and \( G: [t_0, 1] \rightarrow I \) are defined, respectively, by

\[
F(x) = \frac{V_C([0, x] \times [t_0, 1])}{\lambda_R} = \frac{x - h_\lambda^C(x)}{\lambda_R}, \\
G(y) = \frac{V_C([0, t_0] \times [t_0, y])}{\lambda_R} = \frac{v^C_\lambda(y) - C(t_0, t_0)}{\lambda_R}.
\]

Thanks to Propositions 4, 5 and 6 in \(^{31}\), in order to prove that \( \tilde{C} \) is a quasi–copula, one has to prove that \( C = D \) on the boundaries of \( R \). \( D \) is increasing in each variable, and the mappings \( t \mapsto D(t, y_0) \) and \( t \mapsto D(x_0, t) \) are 1–Lipschitz for every \( x_0 \in [0, t_0] \) and \( y_0 \in [t_0, 1] \).

Since, the first two requirements for \( D \) are easily proved, it remains to prove that the horizontal and vertical sections of \( D \) are 1–Lipschitz. To this end, let \( t_1, t_2 \) be in \([0, t_0]\); without loss of generality, assume that \( t_1 < t_2 \). Let \( y_0 \in [t_0, 1] \). One has

\[
D(t_2, y_0) - D(t_1, y_0) = \lambda_R (Q (F(t_2), G(y_0)) - Q (F(t_1), G(y_0))) + (h_\lambda^C(t_2) - h_\lambda^C(t_1))
\leq (t_2 - t_1) - (h_\lambda^C(t_2) - h_\lambda^C(t_1)) + (h_\lambda^C(t_2) - h_\lambda^C(t_1)) = t_2 - t_1.
\]

Analogously, let \( t_1, t_2 \) be in \([t_0, 1]\); without loss of generality, assume that \( t_1 < t_2 \). Let \( x_0 \in [0, t_0] \). One has

\[
D(x_0, t_2) - D(x_0, t_1) = \lambda_R (Q (F(x_0), G(t_2)) - Q (F(x_0), G(t_1)))
\leq v^C_{\lambda}(t_2) - v^C_{\lambda}(t_1) \leq t_2 - t_1.
\]

We can conclude that \( t \mapsto D(t, y_0) \) and \( t \mapsto D(x_0, t) \) are 1–Lipschitz, and, hence, \( \tilde{C} \) is a quasi–copula. Moreover, it can be easily checked that \( \tilde{C} \in \mathcal{Q}_3 \).

Notice that, \( \tilde{C} \) is actually a proper quasi–copula. In fact, since \( Q \) is a proper quasi–copula, there exists a rectangle \( R^* = [a_1^*, a_2^*] \times [b_1^*, b_2^*] \subseteq I^2 \) such that \( V_Q (R^*) < 0 \). Now, let \( \overline{a}_1, \overline{a}_2, \overline{b}_1 \) and \( \overline{b}_2 \) be the points in \( R \) such that

\[
F(\overline{a}_1) = a_1^*, \ G(\overline{a}_2) = a_2^*, \ F(\overline{b}_1) = b_1^*, \ F(\overline{b}_2) = b_2^*.
\]

One obtains

\[
V_{\tilde{C}} ([\overline{a}_1, \overline{a}_2] \times [\overline{b}_1, \overline{b}_2]) = \lambda_R V_Q (R^*) < 0,
\]

that is \( \tilde{C} \) is not a copula.

It follows that \( Q \in \mathcal{Q}_3 \setminus \mathcal{C}_t \), which is a contradiction. Thus, \( \delta = \delta_M \). Since the converse implication in the statement is obviously true, the desired assertion has been proved.

\( \square \)

**Remark 2.** From the proof of Proposition 3, one can derive that every copula \( C \in \mathcal{C}_5 \), \( \delta \neq \delta_M \), can approximated by means of a proper quasi–copula \( Q \in \mathcal{Q}_3 \). In fact, given \( \varepsilon > 0 \), following the notations of Proposition 3, let consider a rectangle \( R = R^* \subseteq [0, t_0] \times [t_0, 1], V_C (R) > 0 \), and the quasi–copula \( Q \) of eq. (4). For \( R \) sufficiently small (but of positive \( C \)-volume), one can derive that \( |C(x, y) - Q(x, y)| < \varepsilon \) for all \( (x, y) \in I^2 \). Furthermore, by using Theorem 4.2 in \(^{32}\), one can assume that the
approximating quasi-copula $Q$ has so much negative mass as desired in some given subset of $R$.

Notice that the proper quasi–copula $\tilde{C}$ of eq. (4) is non-symmetric, in the sense that $\tilde{C}(x, y) \neq \tilde{C}(y, x)$ for some $(x, y) \in \mathbb{R}^2$. However, it is not difficult to construct a symmetric proper quasi–copula in $Q_\delta$. In fact, one can consider the quasi–copula $\tilde{C}^*(x, y) = \tilde{C}(\min\{x, y\}, \max\{x, y\})$ (see Remark 3 in [33]). As a consequence, denoted by $Q_\delta$ the class of symmetric quasi–copulas, we have the following result.

**Corollary 2.** Let $\delta$ be a diagonal. Then $C_\delta \cap Q_\delta = Q_\delta \cap Q_\delta$ if, and only if, $\delta = \delta_M$.

Interestingly, while for any diagonal $\delta \neq \delta_M$, $C_\delta \cap Q_\delta \subset Q_\delta \cap Q_\delta$, it is known that the pointwise lower and upper bounds of $C_\delta \cap Q_\delta$ and $Q_\delta \cap Q_\delta$ coincide [15,34]. Actually, another way of proving Corollary 2 consist of showing, with some computational effort, that upper and lower bounds in $Q_\delta \cap Q_\delta$ coincide if, and only if, $\delta = \delta_M$.

4. Conclusions

In this paper, we have solved two open problems concerning the relationships between the classes of copulas and quasi–copulas with the same diagonal section. Such solutions used some results originated recently in copula theory. The obtained results could help in clarifying better the differences among copulas and quasi–copulas. Moreover, we expect that the obtained results could be useful in order to understand general bounds for bivariate distribution functions, when the information about the diagonal section of the underlying copula is given (in the spirit of [34]).

Acknowledgements

Part of this work was realized when Fabrizio Durante was on research stay at the Dipartimento di Matematica “Ennio De Giorgi”, Università del Salento (Italy). He also thanks Prof. Carlo Sempi for some useful comments about a first version of this manuscript.

References


