Global powerful $r$-alliances and total $k$-domination in graphs

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Abstract

Let $G = (V,E)$ denote a simple graph of order $n$ and size $m$. For a non-empty subset $S \subseteq V$, and a vertex $v \in V$, we denote by $N_S(v)$ the set of neighbors $v$ has in $S$. We denote the degree of $v$ in $S$ by $\deg_S(v) = |N_S(v)|$. The boundary of a set $S \subseteq V$ is defined as $\partial S = \bigcup_{v \in S} N_{\bar{S}}(v)$. $\bar{S}$ denotes the complement of $S$, i.e., $\bar{S} = V \setminus S$.

A set $D \subseteq V$ is called dominating if, for each $v \in \bar{D}$, $N_D(v) \neq \emptyset$. In this paper, we consider several variants of dominating sets.

Consider the following condition (∗): $\deg_S(v) \geq \deg_{\bar{S}}(v) + r$, which states that a vertex $v$ has at least $r$ more neighbors in $S$ than it has in $\bar{S}$. A non-empty set $S \subseteq V$ that satisfies Condition (∗) for every vertex $v \in S$ is called a defensive $r$-alliance; if $S \neq \emptyset$ satisfies Condition (∗) for every vertex $v \in \partial S$, then $S$ is called an offensive $r$-alliance. A set $S \subseteq V$ is a powerful $r$-alliance in $G$ if $S$ is both a defensive $r$-alliance and an offensive $(r+2)$-alliance in $G$. An alliance is called global if it is also a dominating set. The global powerful $r$-alliance number, denoted by $\gamma^*_r(G)$, is defined as the minimum cardinality of a global powerful $r$-alliance in $G$. We show that the natural decision problem associated to computing $\gamma^*_r(G)$ is NP-complete and we obtain tight bounds for this domination-related parameter.

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A set $S \subset V$ is a total $k$-dominating set if $\deg_S(v) \geq k$, $\forall v \in V$. The total $k$-domination number $\gamma_{kt}(G)$ is the minimum cardinality of a total $k$-dominating set. We show that the problem of computing $\gamma_{kt}(G)$ is NP-complete and we obtain a tight bound for this parameter, as well.

Finally, we exhibit close relationships between total $k$-dominating sets and global (defensive / offensive / powerful) $r$-alliances, with $r$ and $k$ tightly coupled.

1 Introduction

Since (defensive, offensive and powerful) alliances were first introduced by Kristiansen, Hedetniemi and Hedetniemi [20], several authors have studied their mathematical properties [1, 2, 4, 7, 15, 23, 24, 25, 26, 27, 30, 33, 35] as well as the complexity of computing the minimum cardinality of alliances [3, 9, 16, 17]. The minimum cardinality of a defensive (respectively, offensive or powerful) alliance in a graph $G$ is called the defensive (respectively, offensive or powerful) alliance number of $G$. The mathematical properties of defensive alliances were first studied in [20] where several bounds on the defensive alliance number were given. The particular case of global (strong) defensive alliances was investigated in [15] where several bounds on the global (strong) defensive alliance number were obtained. The powerful alliances were introduced and studied in [1, 2] as alliances that are both defensive and offensive. In [24] there were obtained several tight bounds on the defensive (offensive and powerful) alliance number. In particular, the relationships of the alliance numbers of a graph to several other graph parameters were investigated, e.g., to its algebraic connectivity, its spectral radius, and its Laplacian spectral radius. Alliances that are also dominating sets are called global alliances. The study of global defensive (offensive and powerful) alliances in planar graphs was initiated in [25] and the study of defensive alliances in the line graph of a simple graph was initiated in [33]. The particular case of global alliances in trees has been investigated in [4]. For many properties of offensive alliances, readers are referred to [7, 23, 34].

A generalization of (defensive and offensive) alliances called $r$-alliances was presented by Shafique and Dutton [28, 29, 30]. There, the study of $r$-alliance free sets and $r$-alliance cover sets was initiated. The aim of this work is to study mathematical properties of powerful $r$-alliances and total $k$-dominating sets. We begin by stating the terminology used. Throughout this article, $G = (V, E)$ denotes a simple graph of order $|V| = n$ and size $|E| = m$. We denote two adjacent vertices $u$ and $v$ by $u \sim v$. For a nonempty set $X \subseteq V$, and a vertex $v \in V$, $N_X(v)$ denotes the set of neighbors $v$ has in $X$: $N_X(v) := \{u \in X : u \sim v\}$, and the degree of $v$ in $X$ will be denoted by $\deg_X(v) = |N_X(v)|$. We denote the degree of a
A vertex \( v \in V \) by \( \text{deg}(v) \). The maximum and minimum degrees in \( G \) will be denoted by \( \Delta \) and \( \delta \), respectively. The subgraph induced by \( S \subset V \) will be denoted by \( G[S] \), i.e.,

\[
G[S] = (S, \{uv \in E : u, v \in S\})
\]

and the complement of the set \( S \in V \) will be denoted by \( \bar{S} \).

A nonempty set \( S \subset V \) is called **dominating** if, for each \( v \in \bar{S} \), \( N_S(v) \neq \emptyset \). A dominating set for \( G \) of minimal cardinality is also termed a **minimum dominating set**, and its cardinality is denoted by \( \gamma(G) \). In the next sections, we will employ the following derived decision problem that is well-known to be NP-complete:

**DOMINATING SET (DS)**

**INSTANCE:** A graph \( G \) and a bound \( \ell \in \mathbb{N} \)

**QUESTION:** Is \( \gamma(G) \leq \ell \)?

It is known that the DOMINATING SET problem, restricted to graphs of certain fixed minimum degree, is still NP-hard. For this fact and notions related to NP-completeness, we refer the reader to [12].

A nonempty set \( S \subseteq V \) is a **defensive** \( r \)-alliance in \( G = (V, E) \), with integer \( r \) in the range \( -\Delta \leq r \leq \Delta \), if

\[
\text{deg}_S(v) \geq \text{deg}_{\bar{S}}(v) + r, \text{ for every } v \in S. \tag{1}
\]

If \( r \geq 0 \), then this condition can be only satisfied for \( S \) where the minimum degree in \( G[S] \) is at least \( r \). Hence, in graphs \( G \) with \( \Delta(G) < r \), there is no defensive \( r \)-alliance.

A vertex \( v \in S \) is said to be **\( r \)-satisfied** with respect to \( S \) if (1) holds. Notice that (1) is equivalent to

\[
\text{deg}(v) \geq 2\text{deg}_{\bar{S}}(v) + r, \text{ for every } v \in S. \tag{2}
\]

A defensive \((-1)\)-alliance is a **defensive alliance** and a defensive \(0\)-alliance is a **strong defensive alliance** as defined in [20]. A defensive \(0\)-alliance is also known as a **cohesive set** [31].

A defensive \( r \)-alliance \( S \) is called **global** if it is a dominating set. The **global defensive \( r \)-alliance number** \( \gamma^d_r(G) \) is the minimum cardinality of any global defensive \( r \)-alliance in \( G \).

The **boundary** of a set \( S \subseteq V \) is defined as \( \partial S = \bigcup_{v \in S} N_{\bar{S}}(v) \). A nonempty set of vertices \( S \subseteq V \) is called **offensive \( r \)-alliance** in \( G \) if

\[
\text{deg}_S(v) \geq \text{deg}_{\bar{S}}(v) + r, \text{ for every } v \in \partial S \tag{3}
\]

where \( -\Delta + 2 < r \leq \Delta \). In particular, an offensive \( 1 \)-alliance is an “offensive alliance”, and an offensive \( 2 \)-alliance is a “strong offensive alliance” (as defined in [20]).
A non-empty set of vertices $S \subseteq V$ is a \textit{global offensive $r$-alliance} if
\[
\deg_S(v) \geq \deg_S(v) + r, \text{ for every } v \in \bar{S}.
\] (4)

The \textit{global offensive $r$-alliance number}, denoted by $\gamma^o_r(G)$, is defined as the minimum cardinality of a global offensive $r$-alliance in $G$.

An alliance is called \textit{powerful} if it is both defensive and offensive [20]. Hence, a \textit{global powerful alliance} is a global powerful $(-1)$-alliance, i.e., defensive $(-1)$-alliance and offensive 1-alliance, and a \textit{global strong powerful alliance} is a global powerful 0-alliance, i.e., $(0)$-defensive alliance and offensive 2-alliance. In general, a set $S \subseteq V$ is a powerful $r$-alliance in $G$ if $S$ is both defensive $r$-alliance and offensive $(r + 2)$-alliance in $G$. For powerful $r$-alliances the integer $r$ is in the range $1 - \Delta$ to $\Delta - 2$. The \textit{global powerful $r$-alliance number}, denoted by $\gamma^* r(G)$, is defined as the minimum cardinality of a global powerful $r$-alliance in $G$.

The global powerful alliance number $\gamma^*_{-1}$ is known for some families of graphs such as the complete graph $K_n$, the path graph $P_n$, the complete bipartite graph $K_{p,s}$, the cycle graph $C_n$ and the wheel graph of order $n+1$, $W_n$.

\textbf{Remark 1.} [32]

- $\gamma^*_{-1}(K_n) = \lceil \frac{n}{2} \rceil$.
- $\gamma^*_{-1}(P_n) = \lceil \frac{n^2}{3} \rceil$.
- $\gamma^*_{-1}(C_n) = \lceil \frac{n}{3} \rceil$.
- $p \leq s$, $\gamma^*_{-1}(K_{p,s}) = \min \{ \lceil \frac{p+1}{2} \rceil + \lceil \frac{s+1}{2} \rceil, p + \lceil \frac{s}{2} \rceil \}$.
- $\gamma^*_{-1}(W_n) = \lceil \frac{n+1}{2} \rceil$.

Fink and Jacobson [11] generalized the concept of domination and introduced the so called $k$-dominating sets. A set $S \subseteq V$ is a \textit{k-dominating set} if
\[
\deg_S(v) \geq k, \text{ for every } v \in \bar{S}.
\] (5)

Other generalization of the concept of domination is the concept of total domination, introduced by Cockayne, Dawes and Hedetniemi in [5]: a set $S \subseteq V$ is a \textit{total dominating set} if $\deg_S(v) \geq 1, \forall v \in V$. The \textit{total domination number} $\gamma_t(G)$ is the minimum cardinality of a total dominating set. So, a set $S \subseteq V$ is a \textit{total $k$-dominating set} if
\[
\deg_S(v) \geq k, \text{ for every } v \in V.
\] (6)

Therefore, every total k-dominating set is a k-dominating set, but the converse is not true. The \textit{total $k$-domination number} $\gamma_{kt}(G)$ is the minimum
cardinality of a total $k$-dominating set. Notice that the concept of total 2-domination is different from the concept of double domination introduced by Harary and Haynes in [14]. A set $S \subseteq V$ is a double dominating set if $\operatorname{deg}_S(v) \geq 2$, $\forall v \in S$ and $\operatorname{deg}_S(v) \geq 1$, $\forall v \in S$. A fortiori every double dominating set is a total 1-dominating set. Moreover, the concept of total $k$-dominating set coincides with the concept of $k$-tuple total dominating set introduced by Henning and Kazemi in [18].

The total 2-domination number $\gamma_2$ is easy to compute in the case of some families of graphs such as the complete graph $K_n$, the complete bipartite graph $K_{p,s}$ and the cycle graph $C_n$.

Remark 2.

- $\gamma_2(K_n) = 3$.
- $\gamma_2(C_n) = n$.
- $2 \leq p \leq s$, $\gamma_2(K_{p,s}) = 4$.

2 The global powerful $r$-alliance number

In this section, we address the complexity of the following problem, also called $r$-GPA for short: Given a graph $G$ and a bound $\ell$; determine if $\gamma_r^*(G) \leq \ell$. More precisely, for any fixed $r$, we consider the following decision problem:

GLOBAL POWERFUL $r$-ALLIANCE (r-GPA)

INSTANCE: A graph $G$ and a bound $\ell \in \mathbb{N}$

QUESTION: Is $\gamma_r^*(G) \leq \ell$?

Theorem 3. For all fixed $r$, the problem $r$-GPA is NP-complete.

We are using a construction based on ideas presented by Cami et al. [3] for the special case $r = -1$, although our construction is different concerning some details.

Proof. (A) Membership in NP is quite clear for each $r$: a nondeterministic Turing machine first guesses at most $\ell$ vertices and then verifies that this vertex set forms indeed a global powerful $r$-alliance. The latter property can be tested in polynomial time, on a deterministic Turing machine.

(B) Now, fix an $r \geq -1$. We consider graphs $G = (V,E)$ of minimum degree four as instances of DOMINATING SET, together with a bound $k$ on the dominating set size. We show how to construct an $r$-GPA instance $G' = (V',E')$ together with a bound $k'$, such that $G$ has a dominating set
of size at most $k$ if and only if $G'$ has a global powerful $r$-alliance of size at most $k'$.

The aforementioned graph $G'$ has $G$ as a subgraph. To each $v \in V$, $N_{V'}(v)$ contains $\deg_{V'}(v) + r$ new neighbors, which are collected in the vertex set $A(v)$. $G'[A(v)]$ forms a clique. Moreover, to $v$ we associate an independent set $B(v)$ of size $\deg_{V'}(v) - 2$. Every $x \in B(v)$ has $r + 2$ neighbors in $A(v)$ (and no other neighbors). Notice that, due to $r \geq -1$, the set of neighbors of each such $x$ is non-empty. We assume that $B(v)$ is “evenly distributed” among the $A(v)$, meaning that the number $b(x)$ and $b(y)$ of $B(v)$-neighbors of arbitrary $x \in A(v)$ and $y \in A(v)$ obey: $|b(x) - b(y)| \leq 1$. Hence, for the average $b = \frac{(r+2)(\deg_{V'}(v) - 2)}{\deg_{V'}(v) + r}$ we find $|b - b(y)| \leq 1$ for all $y \in A(v)$. We derive an estimate for $b$ in the following:

$$b = \frac{r\deg_{V'}(v) - 2r - 4}{\deg_{V'}(v) + r} = \frac{(\deg_{V'}(v) + r)\deg_{V'}(v) - \deg_{V'}^2(v) + \deg_{V'}(v) - 2r - 4}{\deg_{V'}(v) + r}$$

$$= \deg_{V'}(v) + \frac{-(\deg_{V'}(v) - 1)^2 - 2r - 3}{\deg_{V'}(v) + r}.$$

Observe that $\deg_{V'}(v) \geq 4$ implies that $(\deg_{V'}(v) - 1)^2 \geq 2\deg_{V'}(v)$, which shows $b < \deg_{V'}(v) - 2$. Hence, $b(x) \leq \deg_{V'}(v) - 2$ for each $x \in A(v)$.

We first show that if $D \subseteq V$ is a dominating set for $G$, then $S = D \cup \bigcup_{v \in V} A(v)$ is a solution to the $r$-GPA instance $G'$.

Consider some $u \in B(v)$. By construction, $u \in \partial S$, and $u$ has $r + 2$ neighbors in $A(v)$ and hence in $S$, but no neighbors in $\bar{S}$.

Now, discuss the case when $u \in A(v)$. On the one hand,

$$|N_{V'}(u) \cap S| \geq |A(v)| - 1 = \deg_{V'}(v) + r - 1,$$

since $A(v) \subseteq S$. On the other hand, $|N_{V'}(u) \cap \bar{S}| \leq (\deg_{V'}(v) - 2) + 1$. Hence, $|N_{V'}(u) \cap S| - |N_{V'}(u) \cap \bar{S}| \geq r$, as required.

Finally, consider some $v \in V$. There are two possible cases: (1) $v \in D$, i.e., $v \in S$. Then, $|N_{V'}(v) \cap S| \geq |A(v)| = \deg_{V'}(v) + r$, while $|N_{V'}(v) \cap \bar{S}| \leq \deg_{V'}(v)$, so that $|N_{V'}(v) \cap S| - |N_{V'}(v) \cap \bar{S}| \geq r$, as required by the defensive alliance condition. (2) $v \notin D$, i.e., $v \in \bar{S}$. Then, $|N_{V'} \cap S| = |A(v) + (D \cap N_{V'}(v))| = \deg_{V'}(v) + r + 2|D \cap N_{V'}(v)|$. Similarly, $|N_{V'} \cap \bar{S}| = |S \cap N_{V'}(v)| = \deg_{V'}(v) - |D \cap N_{V'}(v)|$. Therefore, since $D$ is a dominating set, we have $|N_{V'} \cap S| - |N_{V'} \cap \bar{S}| = r + 2|D \cap N_{V'}(v)| \geq r + 2$ as required. Notice that the converse is also true: if $S$ is constructed as described, namely as the union of $A(v)$-vertices and some vertices from $V$, then there must be a vertex from $V \cap S$ in the neighborhood of any vertex from $V \cap \bar{S}$, since otherwise $v \in \partial S$ (since $A(v) \subseteq S$) would not have more than $\deg_{V'}(v) + r$ neighbors in $S$ as required by the offensive alliance condition; hence, $S \cap V$ forms a dominating set in $G$.  

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Now, consider a set $S$ that forms a minimal powerful $r$-alliance in $G'$ of size at most $k + \sum_{v \in V} (\deg_{V}(v) + r) = k + 2|E| + r|V|$. If, for some $v \in V$, $B(v) \cap S \neq \emptyset$, then at least $r + 1$ of the $r + 2$ neighbors of $x \in B(v) \cap S$ must be in $S$, too. Since $x$ is dominating only its neighbors in $A(v)$, we cannot increase the size of $S$ by deleting $x$ from $S$ and putting all neighbors from $x$ into $S$, instead, and we would still find a powerful alliance. Continuing this way, we end up with some powerful alliance $S$ satisfying $A(v) \subseteq S$ and $B(v) \cap S = \emptyset$ for all $v \in V$. Our computations above show that such alliances have the property that $S \cap V$ forms a dominating set of size at most $k$ in $G$, as required.

(C) Observe that, for $r \leq -3$, on cubic graphs, each vertex set $S$ is a global defensive $r$-alliance if and only if $S$ is a dominating set. Moreover, for $r \leq -1$, on cubic graphs, each vertex set $S$ is a global offensive $r$-alliance if and only if $S$ is a dominating set. Since the Dominating Set problem is known to be NP-hard on cubic graphs, cf. [19], the claim easily follows for $r \leq -3$.

(D) The only remaining case is $r = -2$. In that case, we can provide similar observations as in (C) concerning the problem of total dominating set on cubic graphs, instead. This (restricted) problem was shown to be NP-complete in [21, Theorem 4.3.6].

The following result leads to several bounds on $\gamma^*_r(G)$.

**Theorem 4.** Let $G$ be a graph of order $n$, size $m$, minimum degree $\delta$ and maximum degree $\Delta$. If $S$ is a global powerful $r$-alliance in $G$, then the size of $G[S]$ is bounded below by $\frac{1}{4}(m + |S|(r - 1)) + \frac{1}{8}n(r + 2)$.

**Proof.** If $S \subset V$ is a global offensive $(r + 2)$-alliance, then
\[
\sum_{v \in S} \deg_{S}(v) \geq \sum_{v \in \bar{S}} \deg_{S}(v) + (n - |S|)(r + 2).
\]
(7)

Hence, as
\[
\sum_{v \in S} \deg_{S}(v) = \sum_{v \in \bar{S}} \deg_{S}(v),
\]
\[
\sum_{v \in S} \deg_{S}(v) \geq \left(2m - \sum_{v \in S} \deg_{S}(v) - 2 \sum_{v \in \bar{S}} \deg_{S}(v)\right) + (n - |S|)(r + 2).
\]
(8)

Thus,
\[
3 \sum_{v \in S} \deg_{S}(v) + \sum_{v \in S} \deg_{S}(v) \geq 2m + (n - |S|)(r + 2).
\]
(9)

On the other hand, if $S$ is a global defensive $r$-alliance in $G$,
\[
\sum_{v \in S} \deg_{S}(v) \geq \sum_{v \in S} \deg_{S}(v) + r|S|.
\]
(10)
Therefore, by (9) and (10) we have
\[
4 \sum_{v \in S} \deg_S(v) \geq 2m + n(r + 2) + 2|S|(r - 1). \tag{11}
\]
As the size of \(G[S]\) is \(\frac{1}{2} \sum_{v \in S} \deg_S(v)\), the result follows. \(\Box\)

**Corollary 5.** For any graph \(G\) of order \(n\) and size \(m\),
\[
\gamma_r^*(G) \geq \left\lceil \sqrt{8m + 4n(r + 2) + (r + 1)^2 + r + 1} \right\rceil.  
\]

**Proof.** If \(S \subseteq V\) is a global powerful \(r\)-alliance, by Theorem 4 and \(|S|(|S| - 1) \geq \sum_{v \in S} \deg_S(v)\), we obtain \(4|S|(|S| - 1) \geq 2m + n(r + 2) + 2|S|(r - 1)\). Hence, the lower bound follows. \(\Box\)

As we will show in Remark 7, the above bound is attained for the family of complete graphs of order \(n\).

**Theorem 6.** Let \(G\) be a graph of order \(n\) and minimum degree \(\delta \geq 1\). Let \(\alpha = \min\{\delta, \Delta - 2\}\). If \(r \in \{1 - \delta, \ldots, \alpha\}\), then \(\gamma_r^*(G) \leq n - \left\lceil \frac{\delta - r}{2} \right\rceil\).

**Proof.** Let \(x\) be a vertex of minimum degree in \(G = (V, E)\) and let \(Y \subseteq N_V(x)\) such that \(|Y| = \left\lceil \frac{\delta + r}{2} \right\rceil\). Let \(X = \{x\} \cup N_V(x) - Y\). As \(r + \delta \geq 1\), \(Y \neq \emptyset\), so \(x\) has at least one neighbor in \(X\). Moreover, if \(u \in X - \{x\}\), then \(u\) has at least one neighbor in \(X\), because otherwise \(\deg(u) < \delta\), a contradiction. Hence, \(X\) is a dominating set.

On the one hand, for all \(v \in X\) we have \(\deg_X(v) \geq \left\lceil \frac{\delta + r}{2} \right\rceil - 1 \geq \delta + r - \left\lceil \frac{\delta + r}{2} \right\rceil - 1 \geq \deg_X(v) + r - 2\). Thus, \(X\) is a global defensive \((r - 2)\)-alliance in \(G\).

On the other hand, since \(\deg_X(x) = \left\lceil \frac{\delta + r}{2} \right\rceil \geq \left\lceil \frac{\delta + r}{2} \right\rceil = \delta - \left\lfloor \frac{\delta + r}{2} \right\rfloor + r = \deg_X(x) + r\), we have \(\deg_X(u) \geq \deg_X(x) \geq \deg_X(x) + r \geq \deg_X(u) + r\), \(\forall u \in X\). Therefore, \(X\) is a global offensive \(r\)-alliance in \(G\). As a consequence, \(X\) is a global powerful \((r - 2)\)-alliance in \(G\). Hence, \(\gamma_r^{(r-2)}(G) \leq n - 1 - \delta + \left\lceil \frac{\delta + r + 2}{2} \right\rceil\). Thus, the bound follows:
\[
\gamma_r^*(G) \leq n - 1 - \delta + \left\lceil \frac{\delta + r + 2}{2} \right\rceil = n + \left\lfloor \frac{-2 - 2\delta + \delta + r + 2}{2} \right\rfloor = n - \left\lceil \frac{\delta - r}{2} \right\rceil.  
\]

From Theorem 6 and Corollary 5 we deduce the exact value of \(\gamma_r^*(G)\) for the complete graph \(G = K_n\).
Remark 7. $\gamma^*_r(K_n) = \left\lceil \frac{n + r + 1}{2} \right\rceil$.

We now relate the graph parameter $\gamma^*_r$ with another well-known parameter, the spectral radius, which is defined as the maximum eigenvalue of the adjacency matrix of the graph.

Theorem 8. For any graph $G$ of order $n$, size $m$, and spectral radius $\lambda$,

$$\gamma^*_r(G) \geq \left\lceil \frac{2m + (r + 2)n}{4\lambda - 2r + 2} \right\rceil.$$ 

Proof. Let $S$ be a global powerful $r$-alliance in $G$. Let $A$ be the adjacency matrix of $G$. Then

$$\lambda \geq \frac{(Aw, w)}{(w, w)}, \text{ for every } w \in \mathbb{R}^n \setminus \{0\},$$

(12)

where $(\cdot, \cdot)$ is the standard scalar product in the Euclidean space $\mathbb{R}^n$. Thus, taking $w \in \mathbb{R}^n$ defined as

$$w_i = \begin{cases} 1 & \text{if } v_i \in S; \\ 0 & \text{otherwise.} \end{cases}$$

we have

$$\lambda|S| \geq \sum_{v \in S} \deg_{S}(v).$$

(13)

Therefore, by (13) and Theorem 4 we obtain $4\lambda|S| \geq 2m + n(r + 2) + 2|S|(r - 1)$. So, the result follows.

The above bound is tight. For the left-hand side graph of Figure 1, Theorem 8 leads to $\gamma^*_1(G_1) \geq \left\lceil \frac{2 \cdot 9 + 10}{4\sqrt{6} + 4} \right\rceil = 3$ and, for the right-hand side graph of Figure 1, Theorem 8 leads to $\gamma^*_0(G_2) \geq \left\lceil \frac{2 \cdot 9 + 2 \cdot 6}{4(1 + \sqrt{5}) + 2} \right\rceil = 3$.

Figure 1: The spectral radius of the left-hand side graph $G_1$ is $\lambda = \sqrt{6}$ and the spectral radius of the right-hand side graph $G_2$ is $\lambda = 1 + \sqrt{5}$. 

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In the particular case of cubic graphs\footnote{A cubic graph is a 3-regular graph, i.e., each vertex has degree three.} we have:

\[
\frac{n}{4} \leq \gamma(G) = \gamma^*_3(G) \leq \gamma^*_2(G) = \gamma^*_1(G) \leq \gamma_0^*(G) = \gamma_1^*(G).
\]

So, in this case we only study \(\gamma^*_1(G)\) and \(\gamma_0^*(G)\). As a consequence of Corollary 5 we obtain that for any cubic graph of order \(n\),

\[
\gamma^*_1(G) \geq \sqrt{n}. \tag{14}
\]

\textbf{Theorem 9.} For any cubic graph \(G\) of order \(n\), \(\gamma_0^*(G) \geq \frac{3n}{4}\).

\textit{Proof.} As \(G = (V,E)\) is a cubic graph we have that if \(S \subset V\) is a global powerful 0-alliance in \(G\), then \(\bar{S}\) is an independent set. Therefore,

\[
3(n - |S|) \leq |S|. \tag{15}
\]

By solving (15) for \(|S|\) we obtain the result. \(\square\)

In the case of \(G = C_3 \times K_2\) Theorem 9 leads to \(\gamma_0^*(G) \geq 5\). Thus, the bound is tight.

\subsection{Powerful \(r\)-alliances in planar graphs}

\textbf{Theorem 10.} Let \(G\) be a graph of order \(n\) and size \(m\). Let \(S\) be a global powerful \(r\)-alliance in \(G\) such that \(G[S]\) is a planar graph.

(i) If \(n > 2(2 - r)\), then \(|S| \geq \left\lceil \frac{2(m+24)+(r+2)n}{2(13-r)} \right\rceil\).

(ii) If \(n > 2(2-r)\) and \(G[S]\) is a triangle free graph, then \(|S| \geq \left\lceil \frac{2(m+16)+(r+2)n}{2(9-r)} \right\rceil\).

\textit{Proof.}

(i) If \(|S| \leq 2\), for all \(v \in S\) we have \(\deg_S(v) \leq 1 - r\). Thus, \(n \leq 2(2 - r)\). It contradicts that \(n > 2(2 - r)\). Thus \(|S| > 2\).

As \(G[S]\) is planar and \(|S| > 2\), the size of \(G[S]\) is bounded by

\[
\frac{1}{2} \sum_{v \in S} \deg_S(v) \leq 3(|S| - 2). \tag{16}
\]

Hence, by Theorem 4 and (16) we obtain \(\frac{1}{4}(m+|S|)(r-1) + \frac{1}{5}n(r+2) \leq 3(|S| - 2)\), so the result follows.
(ii) If \( G[S] \) is a triangle free graph, then the size of \( G[S] \) is bounded by
\[
\frac{1}{2} \sum_{v \in S} \deg_S(v) \leq 2(|S| - 2). \tag{17}
\]
Moreover, by Theorem 4 and (17) we have \( \frac{1}{4}(m + |S|(r - 1)) + \frac{1}{8}n(r + 2) \leq 2(|S| - 2) \), so the result follows.

As a direct consequence of above theorem we obtain the following result.

**Corollary 11.** Let \( G \) be a planar graph of order \( n \) and size \( m \).

(i) if \( n > 2(2 - r) \), then \( \gamma^*_r(G) \geq \left\lceil \frac{2(m + 24) + (r + 2)n}{2(13 - r)} \right\rceil \).

(ii) If \( n > 2(2 - r) \) and \( G \) is a triangle free graph, then \( \gamma^*_r(G) \geq \left\lceil \frac{2(m + 16) + (r + 2)n}{2(9 - r)} \right\rceil \).

In the case of the right hand side graph of Figure 2, the set \( S = \{1, 2, 4\} \) is a global powerful \( r \)-alliance of minimum cardinality for \( r = -1 \) and 0, and (i) in Corollary 11 leads to \( \gamma^*_r(G) \geq 3 \). Moreover, the set \( S = \{1, 2, 3, 4\} \) is a global powerful 1-alliance of minimum cardinality and (i) leads to \( \gamma^*_r(G) \geq 4 \). On the other hand, if \( G = Q_3 \), (ii) in Corollary 11 leads to the exact value of \( \gamma^*_r \) in the following cases: \( 2 \leq \gamma^*_3(Q_3) \) and \( 4 \leq \gamma^*_{-2}(Q_3) = \gamma^*_{-1}(Q_3) \).

![Figure 2: Two planar graphs with \( \gamma^*_r(G) = 3 \) for \( r \in \{-1, 0\} \).](image)

**Theorem 12.** Let \( G \) be a graph of order \( n \). Let \( S \) be a global powerful \( r \)-alliance in \( G \) such that the subgraph \( G[S] \) is planar connected with \( f \) faces. Then,
\[
|S| \geq \left\lceil \frac{2(m - 4f + 8) + n(r + 2)}{2(5 - r)} \right\rceil.
\]
Proof. By Euler’s formula we have

\[ \sum_{v \in S} \deg_S(v) = 2(|S| + f - 2). \]  

(18)

Thus, by Theorem 4 and (18) we obtain

\[ |S| + f - 2 \geq \frac{1}{4}(m + |S|(r - 1)) + \frac{1}{8}n(r + 2), \]  

and the result immediately follows.

In the case of the left-hand side graph of Figure 2, the set \( S = \{2, 4, 6\} \) is a global powerful \( r \)-alliance of minimum cardinality for \( r \in \{-1, 0\} \) and \( G[S] \) is planar with two faces. In this case, Theorem 12 leads to \( \gamma^*_r(G) \geq 3 \).

**Theorem 13.** Let \( T \) be a tree of order \( n \). Let \( S \) be a global powerful \( r \)-alliance in \( T \) such that the subgraph \( G[S] \) has \( c \) connected components. Then,

\[ |S| \geq \left\lceil \frac{n(r + 4) + 8c - 2}{2(5 - r)} \right\rceil. \]

**Proof.** As the subgraph \( G[S] \) is a forest with \( c \) connected components,

\[ \sum_{v \in S} \deg_S(v) = 2(|S| - c). \]  

(19)

From Theorem 4 and (19) we have \( |S| - c \geq \frac{1}{4}(m + |S|(r - 1)) + \frac{1}{8}n(r + 2) \) Hence, the bound of \( |S| \) follows.

If \( S \) is a global strong powerful alliance in \( T \) and the subgraph induced by \( S \) has \( c \) connected components, then \( c \geq 2 \) (see Corollary 15 of [25]). Therefore, as a consequence of Theorem 13 we obtain that for any tree \( T \) of order \( n \),

\[ \gamma^*_r(T) \geq \left\lceil \frac{n + 7}{4} \right\rceil \quad \text{and} \quad \gamma^*_0(T) \geq \left\lceil \frac{2n + 7}{5} \right\rceil. \]

The above bounds were obtained in [25].

3 The total \( k \)-domination number

Given a graph \( G \), a \( k \)-total dominating set is a set \( D \) of vertices satisfying \( \deg_D(v) \geq k \) for all \( v \in V \). We consider the following related decidability problem **total \( k \)-dominating set** for each fixed integer \( k \geq 1 \):

**TOTAL \( k \)-DOMINATING SET (\( k \)-TDS)**

**INSTANCE:** A graph \( G \) and a bound \( \ell \in \mathbb{N} \)

**QUESTION:** Is there a \( k \)-dominating set \( D \) with \( |D| \leq \ell \)?
The smallest ℓ such that G together with ℓ forms a YES-instance of k-TDS is denoted as γ_kt(G); this parameter is called the total k-domination number of G. If k = 1, we write γ_t(G) for simplicity, according to the use in the literature.

**Theorem 14.** For all k ≥ 1, k-TDS is NP-complete.

**Proof.** This result is known for k = 1, where the problem becomes the classical total dominating set problem.

Membership in NP is quite clear: a nondeterministic Turing machine has to guess an ℓ-element vertex set D and then verify that, for all v ∈ V, \(\text{deg}_D(v) ≥ k\).

We still have to show NP-hardness for k ≥ 2. We show that 1-TDS can be solved in polynomial time if k-TDS can be solved in polynomial time. Consider a graph G = (V,E) and a parameter ℓ as an instance of 1-TDS. If G is trivial or if no feasible solution exists, we answer accordingly. Hence, from now on γ_t(G) > 1 and δ(G) ≥ 1.

Now, arbitrarily pick some v ∈ V; for all w ∈ N_V(v), we construct the graph \(G'_{v,w} = (V',E')\) as follows:

\[
V' = V \cup \{1, \ldots, k\}, \quad \text{where } V \cap \{1, \ldots, k\} = \emptyset,
\]

\[
E' = E \cup \{\{x,m\} \mid x \in V, 2 ≤ m ≤ k\}
\]

\[
\cup \{\{n,m\} \mid 1 ≤ n < m ≤ k\} \cup \{\{w,1\}\}.
\]

We claim that

\[
γ_t(G) = \min_{w \in N_V(v)} (γ_{kt}(G'_{v,w}) - (k - 1)).
\]

Namely, since \(\text{deg}(1) = k\), all neighbours of 1 must be in any total k-dominating set, in particular, all “new” vertices 2, ..., k plus one vertex from V.

Hence, a total k-dominating set \(D' \subseteq V'\) induces a total 1-dominating set on G as follows: If 1 \(\notin D'\), then \(D' \cap V\) is a total 1-dominating set for G. Now discuss 1 \(\in D'\). Since \(γ_t(G) > 1\), the only purpose of 1 within \(D'\) is to dominate w. Since \(N_V(w) \neq \emptyset\), we distinguish two further cases: If \(N_V(w) \cap D' \neq \emptyset\), then \((D' \setminus \{1\}\) is a total k-dominating set for \(G'_{v,w}\), as in particular any vertex from 2, ..., k is dominated from \((k - 1)\) vertices from 2, ..., k and by some vertices from \(N_V(w) \cap D'\), so we can proceed as in the previous case. Otherwise, \(D' = (D' \setminus \{1\}\) \cup \{v\}\) is a total k-dominating set, which again leads us to the case 1 \(\notin D'\) considered before.

Since one vertex among the \(w \in N_V(v)\) must be in any feasible solution for 1-TDS for G and since \(γ_t(G) > 1\), our algorithm (based on determining \(γ_{kt}(G'_{v,w})\)) would find \(γ_t(G)\) in polynomial time, contradicting the known NP-hardness of 1-TDS. □
Due to the simplicity of our construction, we can easily inherit W[2]-hardness and inapproximability results from what is known about 1-TD. The according definitions for the hardness notions of parameterized complexity, in particular the notion of W[2]-completeness, can be found in [6]. In short, W[2]-hardness of a parameterized problem means that it is not believed that a parameterized algorithm showing membership in the class FPT can be found, which would be, in our case, having a running time of $O(f(\ell)p(m))$, where $\ell$ is the integer parameter made explicit in the next statement, $f$ is some arbitrary function, $m$ is the size of the input graph, and $p$ is some polynomial.

More precisely, we can state:

**Corollary 15.** For each $k$, $k$-TDS, parameterized with an upperbound $\ell$ on the solution size, is W[2]-complete.

*Proof.** Hardness follows from the results on 1-TDS as stated in [6]. Membership in W[2] can be easiest seen by a multitape Turing machine construction, as displayed in [8] for a related domination-type problem. \qed

When talking about approximability, we mean the related minimization problem MINIMUM $k$-TDS, so the task is to find, given a graph $G = (V,E)$, a total $k$-dominating set $D$ whose size can be upper-bounded by a function $f$ depending on $n = |V|$ times $\gamma_{kt}(G)$, which is the size of an optimum solution.

**Corollary 16.** For each $k$, there is a constant $c_k > 0$ such that minimum $k$-TDS is not approximable within factor $c_k \times n \log(n)$ unless $\text{NP} \subseteq \text{DTIME}(n^{\log \log(n)})$, where $n$ denotes the number of vertices of the graph.

This follows directly from the corresponding fact for 1-TDS due to our construction, where this hardness assertion follows from [22] as for the classical MINIMUM DOMINATING SET problem, since both are deduced from the proven approximation hardness for the MINIMUM SET COVER problem.

Due to Theorem 14 and its corollaries, we can conclude that we cannot automate the process of obtaining nontrivial bounds for $\gamma_{kt}(G)$. Notice that if $S \subset V$ is a total $k$-dominating set of minimum cardinality $\gamma_{kt}(G)$, then $S - \{v\}$ is a total $(k - 1)$-dominating set, for all $v \in S$. Thus, if $G$ has a total $k$-dominating set, $k \geq 2$, then

$$\gamma_{kt}(G) \geq \gamma_{(k-1)t}(G) + 1.$$ \hspace{1cm} (20)

**Theorem 17.** For any graph $G$ of maximum degree $\Delta$ and order $n$,

$$\gamma_{kt}(G) \geq \left\lceil \frac{kn}{\Delta} \right\rceil.$$
Proof. If \( S \subset V \) is a total \( k \)-dominating set in \( G \),
\[
(n - |S|)k \leq \sum_{v \in S} \deg S(v) \leq \sum_{v \in S} (\deg(v) - k) \leq (\Delta - k)|S|.
\]
\(\blacksquare\)

The above bound is attained, for instance, in the case of the 3-cube graph \( G = Q_3 \). For \( k = 2 \) we have \( \gamma_{2t}(Q_3) = 6 \).

**Theorem 18.** For any simple graph of order \( n \) and size \( m \),
\[
\gamma_{kt}(G) \geq \left\lceil \frac{2(kn - m)}{k} \right\rceil.
\]

**Proof.** For \( S \subset V \) we have
\[
2m \geq 2 \sum_{v \in S} \deg S(v) + \sum_{v \in \bar{S}} \deg S(v). \tag{21}
\]

Moreover, if \( S \) is a total \( k \)-dominating set, then
\[
\sum_{v \in \bar{S}} \deg S(v) \geq k(n - |S|) \tag{22}
\]
and
\[
\sum_{v \in S} \deg S(v) \geq k|S|. \tag{23}
\]

Thus, by (21), (22) and (23) we deduce the result. \(\blacksquare\)

As we will see after Corollary 19, the above bound is tight.

### 3.1 Total \( k \)-domination in planar graphs

As the size of a planar graph is bounded by \( m \leq 3(n - 2) \) and the size of triangle free graph is bounded by \( m \leq 2(n - 2) \), by Theorem 18 we obtain the following result.

**Corollary 19.** For any planar graph of order \( n \),
\[
\gamma_{kt}(G) \geq \frac{2[n(k - 3) + 6]}{k}.
\]

Moreover, for any planar triangle-free graph of order \( n \),
\[
\gamma_{kt}(G) \geq \frac{2[n(k - 2) + 4]}{k}.
\]
Figure 3: The set $S = \{2, 4, 5, 6\}$ is a total 3-dominating set in the right hand side graph. In the case of the left-hand side graph, each set of four vertices composing a cycle is a total 2-dominating set.

The above bounds are tight. For instance, in the case of the right-hand side graph of Figure 3, Corollary 19 leads to $\gamma_3(G) \geq 4$. Moreover, in the case of the left-hand side graph of Figure 3, Corollary 19 leads to $\gamma_2(G) \geq 4$.

Theorem 20. For any planar graph of order $n$,

$$\gamma_k(G) \geq \left\lceil \frac{n(k - 2) + 16}{6} \right\rceil.$$ 

Moreover, for any planar triangle-free graph of order $n$,

$$\gamma_k(G) \geq \left\lceil \frac{n(k - 2) + 12}{4} \right\rceil.$$ 

Proof. Let $G = (V, E)$ be a planar graph and let $S \subset V$. Let $G' = (V, E')$ be the subgraph of $G$ whose edge set $E'$ is the set of all edges in $G$ with one endpoint in $S$ and one endpoint in $\overline{S}$. As $G'$ is a planar triangle-free graph,

$$2(n - 2) \geq |E'| = \sum_{v \in S} \deg_S(v). \quad (24)$$

If $S \subset V$ is a total $k$-dominating set, by (22) we have

$$2(n - 2) \geq k(n - |S|). \quad (25)$$

Moreover, the subgraph induced by $S$, $G[S] = (S, E'')$ is planar, so

$$6(|S| - 2) \geq 2|E''| = \sum_{v \in S} \deg_S(v) \geq k|S|. \quad (26)$$

Notice that (26) and (25) lead to the first bound.
Moreover, if $G$ is a triangle-free graph,

$$4(|S| - 2) \geq 2|E''| = \sum_{v \in S} \deg_S(v) \geq k|S|. \quad (27)$$

Thus, by (25) and (27) the second bound follows.

For $k = 2$ the first bound is attained in the case of the right hand side graph of Figure 1. The second bound is attained for $k = 3$ in the case of the right hand side graph of Figure 3.

4 Total $k$-domination and powerful $r$-alliances

Theorem 21.

1. Each total $k$-dominating set is a global defensive (offensive) $r$-alliance, where $-\Delta < r \leq 2k - \Delta$.

2. Each global powerful $r$-alliance, $r \geq 1$, is a total $r$-dominating set.

Proof.

1. If $S \subset V$ is a total $k$-dominating set in $G$ and $r \leq 2k - \Delta$, then

$$\deg_S(v) \geq k \geq r + \Delta - k \geq r + \deg(v) - k \geq r + \deg_S(v), \quad \forall v \in V.$$ 

Namely, $v$ as at least $k$ neighbours in $S$ (being total $k$-dominating) and therefore at most $k + \deg_S(v)$ neighbours in the whole graph. Therefore, $S$ is both defensive $r$-alliance and offensive $r$-alliance in $G$.

2. If $S \subset V$ is a global defensive $r$-alliance, then $\deg_S(v) \geq \deg_S(v) + r \geq r$, $\forall v \in S$. Moreover, if $S \subset V$ is a global offensive $(r + 2)$-alliance, then $\deg_S(v) \geq \deg_S(v) + r + 2 \geq r$, $\forall v \in S$. Therefore, $\deg_S(v) \geq r$, $\forall v \in V$.

$\square$

Corollary 22. Each total $k$-dominating set is a global powerful $r$-alliance, where $-\Delta < r \leq 2(k - 1) - \Delta$.

Corollary 23.

- For $-\Delta < r \leq 2k - \Delta$, $\gamma_{kt}(G) \geq \gamma^d_r(G)$ and $\gamma_{kt}(G) \geq \gamma^o_r(G)$.
- For $-\Delta < r \leq 2(k - 1) - \Delta$, $\gamma_{kt}(G) \geq \gamma^*_r(G)$.
• For $k \geq 1$, $\gamma_k^*(G) \geq \gamma_{kt}(G)$.

By Corollary 23, we have that lower bounds for $\gamma_t^d(G)$, $\gamma_o^r(G)$ and $\gamma_r^*(G)$ lead to lower bounds for $\gamma_{kt}(G)$. Moreover, upper bounds for $\gamma_{kt}(G)$ lead to upper bounds for $\gamma_t^d(G)$, $\gamma_o^r(G)$ and $\gamma_r^*(G)$.

**Theorem 24.** [10] Let $\varphi$ be the Laplacian spectral radius of $G$. Then $\gamma_r^o(G) \geq \left\lceil \frac{n}{\varphi} \left\lfloor \frac{4+r}{2} \right\rfloor \right\rfloor$.

**Corollary 25.** Let $\varphi$ be the Laplacian spectral radius of $G$. If $\frac{r+\Delta}{2} \leq k$, then $\gamma_{kt}(G) \geq \left\lceil \frac{n}{\varphi} \left\lfloor \frac{4+r}{2} \right\rfloor \right\rfloor$.

## 5 Conclusions

We introduced and discussed the concept of global powerful $r$-alliances from the viewpoint of complexity as well as from the viewpoint of combinatorics. The latter study is well-motivated by the derived NP-hardness results. Our study opens up quite natural lines of research, some of which we mention in the following:

1. Does the mentioned NP-hardness result transfer to other, more restricted graph classes, e.g., to planar graphs? (The cases when $r \leq -2$ surely do so in Theorem 1, due to what is known about (total) domination on planar cubic graph, see [21].) A similar question can be asked regarding $k$-total domination.

2. Conversely, are there interesting graph classes for which the mentioned decision problems can be solved in polynomial time?

3. Similar questions can be stated in terms of approximability. (Notice that the parameterized tractability results from [9] transfer to the alliance problems as discussed in this paper; however, approximability issues were not discussed as of yet.)

Also, many combinatorial questions remain. For instance, quite a number of results are known for the graph parameter $\gamma_t$, while far less is known for $\gamma_{kt}$ for $k > 1$. To give some concrete research direction: Gravier [13] considered total domination on grid graphs. So, can we compute $\gamma_{kt}(P_m \Box P_n)$? More generally speaking, upper bounds for $\gamma_t(G)$ should provide a generalized bound for $\gamma_{kt}(G)$. 

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