Explicit Bounding Circles for IFS Fractals

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Abstract

The attractors of Iterated Function Systems in Euclidean space - IFS fractals - have been the subject of great interest for their ability to visually model a wide range of natural phenomena. Indeed computer-generated plants are often modeled using 3D IFS fractals, and thus their extent in virtual space is a fundamental question, whether for collision detection or ray tracing. A great variety of algorithms exist in the literature for finding bounding circles, polygons, or rectangles for these sets, usually tackling the easier question in 2D first, as a basis for the 3D bounding problem. The existing algorithms for finding bounding circles are mostly approximative, with significant computational and methodological complexity. We intend to hereby introduce explicit formulas for bounding circles in the plane, and some generalizations to space, thereby providing readily applicable bounding sets for IFS fractals.*

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Earlier efforts to provide bounding sets to IFS fractals, have been mostly algorithmic with significant computational complexity, while explicit bounding circles are yet to be introduced. Our circumcircles generalize the concept of the circumscribed circle of triangles in classical geometry to IFS fractals, hinting at a novel polygonal viewpoint of IFS fractals. Our general bounding circle resolves the bounding problem for any number of maps and dimensions. Our paper is intended for mathematicians and computer scientists alike. The latter may appreciate our discussion of the visualization of fractals and our ready bounding circle formulas, while the former, our attempt to take the bounding problem to a geometrical level. Indeed our primary goal is to reason the usefulness of bounding circles as a numerical panacea for any geometrical problem related to IFS fractals.

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The relevance of bounding sets lies in their property of “iterative containment” by which they provide improving approximations to the IFS fractal. The idea of “bounding” is important so that no points of the fractal are excluded, and thus we may devise numerical theorems and algorithms for virtually any geometrical question about an IFS fractal, such as its intersection with Euclidean sets, or even other fractals. The initial bounding set can be a circle, the convex hull, or any other, which may then be iterated for better approximations to the fractal.

Figure 1: A bounding circle iteratively approximating the fractal.

The challenge is therefore to find bounding circles which are as tight as possible, and preferably have a ready formula for their center and radius. In the current literature, such explicit bounding circles are scarce, while there are several defined via algorithms. Virtual plants as well as other objects in 3D, are often generated in an iterative way, essentially producing “embellished” IFS fractals. Their bounding can be relevant for multiple applied reasons. For morphing fractals, real-time bounding via explicit formulas is most ideal. Though research often begins by treating the 2D bounding problem first, the eventual goal is to generalize such results to 3D space. Our “general bounding circle” indeed also extends to 3D.

For the visualization of IFS attractors, or really for any object in virtual space, there are two primary questions arising from the problem of lighting the object or “ray tracing” it: the object’s spatial extent, and its intersection with various rays of light. With classical Euclidean objects these questions are much easier. As illustrated by Figure 2, ray tracing is composed of a series of steps for shading an object and its environment. Testing for whether a camera ray intersects the object involves: (1) sending out a test ray, (2) testing for intersection with the convex hull of the object, (3) if the ray intersects the hull, testing for intersection with the object itself. Determining the effect that a light source has on the object, involves similar steps. Paper [7] provides a full description of the rendering / visualization of 3D IFS fractals. It emphasizes the importance and difficulties of the ray / line intersection problem for a fractal. The paper also describes a bounding sphere, and utilizes it for the rendering process.
Other than ray tracing, the fundamental question of handling an IFS fractal in virtual space may arise. We must be able to calculate its interaction with other virtual objects. We must know its extent to be able to bump into it, or hold it in an avatar’s hands even. Thus we may observe that the bounding problem leads to a resolution of fundamental geometrical questions in computer graphics applications, related to IFS fractals.

We will hereby review some of the main cornerstones in bounding circle research. Essentially, the main intentions here are to give some sort of approximation to a fractal, in order to assist its display. Article [16] ends with a bounding circle for two-map IFS, similar to our general bounding circle for multi-map IFS. That method however does not optimize for a center, but always uses the origin. Paper [4] comes even closer to our definition of the general bounding circle, having a variable center. However in this form, it is unclear how the optimal center may be found in general, possibly using an optimization algorithm. By our alternative formulation however, it becomes clear that the minimization involved is equivalent to the Smallest Circle / Ball Problem of finitely many points, for which specialized efficient algorithms exist. Paper [2] discusses the connectivity of a special class of two-map IFS fractals. Though not its primary focus, the paper introduces a bounding circle for a certain class of two-map IFS, which is in fact a special case of our circumcircle. In [13] we are introduced to a method for multi-map IFS, similar to our circumcircle method. In fact, the paper even conjectures the existence of the circumcircle, as an ideal tangential solution to the thereby presented numerical method, in the two-map case. Again this method gives no explicit formula, but assumes the use of an efficient optimization algorithm at each iterative step, leading to a bounding circle. The method presented in paper [3] gives an algorithm for finding not a single bounding circle, but rather a finite set of circles whose union covers the fractal. An outer bounding circle to this set of circles can thus be calculated and will be a bounding circle of the fractal itself. Considering the figures, the method seems rather inefficient, and the size of the resulting bounding circle is generally quite large relative to the fractal. Both [5] and [14] assume an initial good center estimate which remains fixed throughout the algorithm, and makes no improvement upon it. In [11] a novel algorithm is presented which is supposedly able to arrive at the absolute tightest bounding sphere of the
fractal, via a series of subroutines. The sphere however is restricted to be centered at the centroid. The method is founded upon the theory of IFS moments and the simplex method from linear optimization, that has exponential complexity in the worst case. Thus the implementation of the method requires significant computational and methodological complexity. In our Remarks, we reason a much simpler method for finding the tightest bounding circle, by improving an existing one.

We shall now turn to introducing the basic mathematical concepts related to IFS fractals and bounding circles. For an introduction to fractals, see [10, 1, 6].

1 IFS Fractals and Bounding Circles

The attractors of Iterated Function Systems - IFS Fractals - are likely the most basic types of fractals possible, first defined in [8]. They are the attractors of a finite set of affine linear contractive maps - the “function system” - which when combined and iterated to infinity, converges to an attracting limit set, the IFS fractal itself. There is a fundamental motivation in studying linear contractive maps in the plane, as they are the simplest contractions possible, and give rise to many logarithmic spiral structures in Nature, such as snail shells, galaxies, or eddies. Their iterative interactions reveal models for analogous interactions in Nature. We begin by defining the generating maps of an IFS Fractal, and go on to stating its existence and uniqueness.

1.1 The Concept of IFS Fractals

Definition 1.1 Let a 2-dimensional affine contractive mapping (briefly: contraction or contraction map) $T : \mathbb{C} \rightarrow \mathbb{C}$ be defined for all $z \in \mathbb{C}$ as $T(z) := p + \varphi(z - p)$ where $p \in \mathbb{C}$ is the fixed point of $T$, and $\varphi = \lambda e^{\theta i} \in \mathbb{C}$ is the factor of $T$, with $\lambda \in (0, 1)$ the contraction factor of $T$, and $\theta \in (-\pi, \pi]$ the rotation angle of $T$.

In higher dimensions, an equivalent definition may be given using unitary rotation matrices $R^T R = I$, $R \in \mathbb{R}^{n \times n}$. Then the contraction maps take the form

$$T(z) = p + \lambda R(z - p), \ p, z \in \mathbb{R}^n, \ \lambda \in (0, 1)$$

This formulation shall be useful for our general bounding circle.

Definition 1.2 A 2-dimensional affine contractive $n$-map iterated function system (briefly: IFS or $n$-map IFS) is defined as a finite set of contractions, and denoted as $\mathcal{T} := \{T_1, \ldots, T_n\}$, $n \in \mathbb{N}$. We will denote the set of indices as $\mathcal{N} := \{1, \ldots, n\}$, the set of fixed points as $\mathcal{P} := \{p_1, \ldots, p_n\}$, and the set of factors as $\Phi := \{\varphi_1, \ldots, \varphi_n\}$.
Definition 1.3 Let \( \mathcal{T} = \{T_1, \ldots, T_n\}, n \in \mathbb{N} \) be an IFS. Define the Hutchinson operator \( H \) belonging to \( \mathcal{T} \) as

\[
H(S) := \bigcup_{k=1}^{n} T_k(S), \quad T_k(S) := \{T_k(z) : z \in S\}, \text{ for any } S \subset \mathbb{C}
\]

Theorem 1.1 For any IFS with Hutchinson operator \( H \), there exists a unique compact set \( F \subset \mathbb{C} \) such that \( H(F) = F \). Furthermore, for any compact \( S_0 \subset \mathbb{C} \), the recursive iteration \( S_{n+1} := H(S_n) \) converges to \( F \) in the Hausdorff metric.

Proof The proof follows from the Banach Fixed Point Theorem, after proving that \( H \) is contractive in the Hausdorff metric over compact sets [8]. \( \square \)

Figure 4: Generation of a fractal by iterating a square (created by Scott Draves).

Definition 1.4 Let the set \( F \) in the above theorem be called a fractal generated by an IFS with Hutchinson operator \( H \) (briefly: fractal, IFS fractal, attractor, or IFS attractor). Denote \( \langle T_1, \ldots, T_n \rangle = \langle \mathcal{T} \rangle := F \). We will say that \( F \) is a bifractal, trifractal, and polyfractal for the cases of \( n = 2, 3 \) and any \( n > 3 \) respectively. In the special case of \( \vartheta_k = 0, \ k \in \mathcal{N} \) we shall speak of Sierpinski fractals.

Note that since by the above theorem, an IFS fractal is the limit set of a sequence that can also be considered a dynamical system, it is often referred to as an attractor of the IFS (or of \( H \)), which is a more general term. Mostly such attractors are self-similar and have non-integer (sometimes fractional) Hausdorff dimension, hence the term “fractal”. There are cases when such an attractor is an ordinary Euclidean set. By introducing the last few terms, we wish to emphasize the “polygonal” nature of IFS fractals, especially apparent in the case of Sierpinski fractals.
1.2 Bounding Circles and Iterative Containment

The relevance of a bounding circle lies in the fact that we may approximate the fractal via iteration of the Hutchinson operator over it, as illustrated by Figure 1 - this is the idea of “iterative containment”. Therefore a bounding circle serves as a tool for answering geometrical questions numerically to some $\varepsilon > 0$ accuracy. We hereby give an intuitive definition of a bounding circle.

**Definition 1.5** A circle $C = (c, r) \in \mathbb{C} \times \mathbb{R}_+$ is called a bounding circle of the IFS fractal $F$, if it is contained in the disk determined by $C$, denoted as $B(c, r) := \{z \in \mathbb{C} : |z-c| \leq r\}$.

We shall refer to the iteration of sets numerous times throughout the text. This will mean the application of the Hutchinson operator $H$ recursively to any set - primarily bounding circles - as in Theorem 1.1. The motivation being that for any $B_1 \supset F$ we have $T_k(F) \subset T_k(B_1)$ ($k \in \mathbb{N}$) and

$$F = H(F) = \bigcup_{k=1}^{n} T_k(F) \subset \bigcup_{k=1}^{n} T_k(B_1) = H(B_1) =: B_2$$

Since $H$ is contractive in the Hausdorff metric over compact sets, then if $B_1$ was compact, then so will $B_2$ be, and it will be closer to $F$ in the metric. In practice, this will imply that taking any bounding set, such as a circle, we can improve it by one or several applications of $H$ to new tighter bounding sets.

**Theorem 1.2** (Containment Theorem) If $B \subset \mathbb{C}$ is compact and $H(B) \subset B$ then $F \subset B$.

**Proof** The proof follows directly from Theorem 1.1. $\square$

The above theorem provides an easily verifiable condition for whether a set bounds the fractal. This theorem is the key inspiration for our explicit bounding circle definitions.

2 The General Bounding Circle for Polyfractals

Let the radius function with respect to the IFS $\mathcal{T} = \{T_1, \ldots, T_n\}$ be denoted

$$\varrho(z) := \max_k |p_k - z| \ (z \in \mathbb{C})$$

This is the radius of the smallest circle centered at $z$ that contains all the fixed points $p_k \in \mathcal{P}$. It can be shown that $\varrho$ is a continuous convex function, so it is a natural inquiry to look for its minimum - the center $c_*$ which gives the lowest $\varrho$ value. The minimizing circle $(c_*, \varrho(c_*))$ is called the minimum covering circle or minimal bounding circle. Determining it is the Smallest Circle Problem for a finite set of points in the plane, first investigated in modern times by Sylvester [15] in 1857. The minimum covering circle is known to be unique. Several algorithms exist for finding it, and the fastest run in linear time. We note the algorithms of Megiddo [12] and Welzl [17] specifically, both having linear runtime.
Theorem 2.1 For any $c \in \mathbb{C}$ the circle 

$$(c, r(c)) \in \mathbb{C} \times \mathbb{R}_+, \quad r(c) := \frac{\mu_* \varrho(c)}{1 - \lambda_*}$$

$$\lambda_* := \max_k |\varphi_k|, \quad \mu_* := \max_k |1 - \varphi_k|$$

is a bounding circle of $F$. With the center $c_* = \text{argmin}_{z \in \mathbb{C}} \varrho(z)$ and radius $r_* := r(c_*)$ this circle is minimal, and we will call this the general bounding circle of the IFS fractal $F$.

Proof Considering the Containment Theorem, we see that $(c, r(c))$ must satisfy $T_k(B(c, r(c))) \subset B(c, r(c))$, $k \in N$. Let $k$ be fixed and arbitrary. Then we can see geometrically that this containment is equivalent to

$$|T_k(c) - c| + \lambda_k r(c) \leq r(c) \iff |1 - \varphi_k| \cdot |c - p_k| \leq (1 - \lambda_k)r(c)$$

Estimating the left-hand side by $\mu_* \varrho(c)$ from above, the right-hand side by $(1 - \lambda_*)r(c)$ from below, and the two estimates being equal, we have the desired inequality above. Therefore $H(B(c, r(c))) \subset B(c, r(c))$ so by the Containment Theorem we have that $F \subset B(c, r(c))$, which also holds for $(c_*, r_*)$. □

The above proof works in higher dimensional spaces for a corresponding IFS with maps of the form $T_k : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $T_k(z) = p_k + \lambda_k R_k(z - p_k), \ R_k^T R_k = I$ and $\mu_k := \|I - \lambda_k R_k\|$, where $\|\cdot\|$ is the matrix norm induced by the Euclidean norm, or whatever norm was used in the definition of $\varrho$ (easy-to-compute compatible matrix norms with the vector norm are also sufficient). Thus in 3D we may talk about a general bounding sphere $(c_*, r_*)$ of the above form, obtained from its corresponding minimal bounding sphere $(c_*, \varrho(c_*))$. Algorithms do exist for finding the minimal bounding sphere of a finite set of points as well [9].

If we wish to obtain a good estimate of $c_*$, we may work with the arithmetic or harmonically-weighted mean of the fixed points of the IFS.

$$c_A := \frac{\sum_k p_k}{n}, \quad c_H := \frac{\sum_k \frac{1}{\varrho(p_k)} p_k}{\sum_k \frac{1}{\varrho(p_k)}}$$

For these special centers, we may find by the convexity of $\varrho$ that their value will be less than or equal to the arithmetic and harmonic mean of the $\varrho$ values of the fixed points, thus proving these centers reasonable. Calculating the $\varrho$ of both means, we may find whichever gives a lower value and use that center, rather than attempting the algorithmic calculation of the optimal $c_*$. 

Philosophically speaking, we have reduced the problem of bounding an IFS fractal having an infinite number of points, to the bounding of a finite set of points, the fixed points of the IFS. Since we have pointed out specific efficient algorithms for finding $c_*$, as well as reasonable explicit centers $c_A$ and $c_H$, the definition of our general bounding circle can be considered an explicit one.
3 The Circumcircle for Trifractals

The bounding of polygons by circles is in general formulaically non-explicit, since the minimum covering circle of the vertices can only be determined algorithmically. When the polygon is a triangle however, such a circle - the circumcircle - may be determined explicitly. Since from a bounding perspective, Sierpinski polyfractals are essentially polygons of the fixed points of the IFS, their non-algorithmic explicit circular bounding seems to be a hopeless venture, for more than three maps. For the case of three or two maps however, such explicit circles may exist for any fractal. Since the Hutchinson of the circumcircle of a Sierpinski trifractal results in tangential iterates, it suggests a similar Apollonian definition for trifractals in general. This implies the following definition, illustrated by Figure 5.

Definition 3.1 Let the circumcircle of a trifractal $F = \langle T_1, T_2, T_3 \rangle$ be the smallest circle $C = (c, r) \in \mathbb{C} \times \mathbb{R}_+$ satisfying the conditions

$$|T_k(c) - c| + \lambda_k r = r, \ k = 1, 2, 3$$

![Figure 5: A Sierpinski trifractal under rotational perturbations.](image)

Definition 3.2 Denote the dot and cross product of $z_1, z_2 \in \mathbb{C}$ respectively by

$$z_1 \cdot z_2 := \text{Re}(z_1)\text{Re}(z_2) + \text{Im}(z_1)\text{Im}(z_2), \ z_1 \times z_2 := \text{Re}(z_1)\text{Im}(z_2) - \text{Im}(z_1)\text{Re}(z_2)$$

Theorem 3.1 Let us introduce the following auxiliary variables

$$\alpha_k := \frac{1 - |\varphi_k|}{|1 - \varphi_k|}, \ k = 1, 2, 3$$

$$A := (\alpha_3^2 - \alpha_2^2)p_1 + (\alpha_1^2 - \alpha_3^2)p_2 + (\alpha_2^2 - \alpha_1^2)p_3$$

$$B := (|p_2|^2 - |p_3|^2)p_1 + (|p_3|^2 - |p_1|^2)p_2 + (|p_1|^2 - |p_2|^2)p_3$$

$$C := 2(p_2 - p_1) \times (p_2 - p_3)$$

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\[ c_0 := \frac{B}{C_1}, \quad r_0 := |c_0 - p_1|, \quad c_1 := \frac{A}{C_1} \]
\[ D := c_0 \cdot c_1 + \frac{1}{C} (\alpha_1^2 p_2 \times p_3 + \alpha_2^2 p_3 \times p_1 + \alpha_3^2 p_1 \times p_2) \]

The circumcircle exists iff \( C \neq 0 \), meaning iff \( p_{1,2,3} \) are non-collinear.

If \( A \neq 0 \) it has the form
\[ c = c_0 + c_1 r^2, \quad r = \frac{1}{|c_1|} \sqrt{-D - \sqrt{D^2 - |c_1|^2 r_0^2}} \]

If \( A = 0 \) then \( c = c_0 \) and \( r = r_0/\alpha_1 \). (Note: \( r_0 \) can be defined above with any \( p_k \).)

**Proof** First we note that the circumcircle equations reduce to the conditions
\[ |p_k - c|^2 - \alpha_k^2 r^2 = 0, \quad k = 1, 2, 3 \]

Denoting \( p_{k1} := \Re(p_k), \quad p_{k2} := \Im(p_k), \quad x := \Re(c), \quad y := \Im(c) \) and expanding these conditions, then multiplying each by the factors \( p_{31} - p_{21}, \quad p_{11} - p_{31}, \quad p_{21} - p_{11} \) respectively, and lastly summing the three equations, several terms drop out in the resulting equation, which we denote as \( E_1 = 0 \). We may do similarly with the factors \( p_{32} - p_{22}, \quad p_{12} - p_{32}, \quad p_{22} - p_{12} \), resulting in an analogous equation \( E_2 = 0 \). Taking \( E_1 + E_2 i = 0 \) and collecting terms, we get that \( Ar^2 + B - C \cdot c = 0 \), which gives the formula for \( c \). To find \( r \), we expand the circumcircle condition say for \( k = 1 \), resulting in
\[ 0 = |c_1|^2 r^4 + 2 \left( -\frac{\alpha_1^2}{2} + c_1 \cdot (c_0 - p_1) \right) r^2 + r_0^2 \]

When \( c_1 = 0 \), we get the stated formula for \( r \). When \( c_1 \neq 0 \), we may solve this equation for \( r^2 \). Upon some calculation, we deduce that the coefficient of the \( r^2 \) term in the equation is \( 2D \). The quadratic formula gives the desired expression for \( r \), by choosing the negative sign in \( \pm \), since we have defined the circumcircle to be the smallest circle satisfying the conditions. \( \square \)

We remark that when \( \vartheta_k = 0 \) for some \( k \), then \( \alpha_k = 1 \), so by the corresponding circumcircle condition, we get that \( |p_k - c| = r \). This means that when \( \vartheta_k = 0 \), the fixed point \( p_k \) lies on the circumcircle. For Sierpinski trifractals, all three fixed points lie on the circumcircle, as expected.

### 4 The Circumcircle for Bifractals

Similarly to the case of three maps, we will define the two-map circumcircle in an Apollonian manner, to “optimally” satisfy the condition of the Containment Theorem. We derive it in quite a different manner however. Having any bounding circle to the fractal, we may get a potentially smaller one, by applying the Hutchinson operator to it, and then taking the
outer tangential circle as in Figure 6 below - this will be calculated using the map $M$. So
the two-map circumcircle will then be defined as the fixed point - or fixed circle - of $M$,
containing the fractal by its definition and the Containment Theorem.

If $(a, r_A)$ and $(b, r_B)$ represent the circles received from a bounding circle under the ac-
tion of $T_1$ and $T_2$ respectively, then finding their outer tangential circle $(c, r_C)$ will usually
result in a tighter bounding circle than the original.

Figure 6: The Circumcircle for Bifractals.

From Figure 6, the following relations can be seen
\[
c = \frac{1}{2}((a - r_A u_{AB}) + (b + r_B u_{AB})), \quad r_C = \frac{1}{2}(r_A + r_B + |b - a|), \quad u_{AB} := \frac{b - a}{|b - a|}
\]

These relations suggest the following transformation. Given a circle $C = (c, r)$ in the plane,
let $M := (m_1, m_2) : \mathbb{C} \times \mathbb{R} \to \mathbb{C} \times \mathbb{R}$ be defined as
\[
m_1(c, r) := \frac{T_1(c) + T_2(c)}{2} + r \frac{\lambda_2 - \lambda_1}{2} \frac{T_2(c) - T_1(c)}{|T_2(c) - T_1(c)|}
\]
\[
m_2(c, r) := \frac{\lambda_1 + \lambda_2}{2} r + \frac{|T_2(c) - T_1(c)|}{2}
\]

**Theorem 4.1** Let the circumcircle of a bifractal be defined as the unique fixed point of $M$,
which is given by the following $(c, r) \in \mathbb{C} \times \mathbb{R}_+$
\[
c = \frac{(1 - \nu)(1 - \varphi_1) p_1 + (1 + \nu)(1 - \varphi_2) p_2}{(1 - \nu)(1 - \varphi_1) + (1 + \nu)(1 - \varphi_2)}
\]
\[
r = \frac{|1 - \varphi_1| \cdot |1 - \varphi_2|}{(1 - \lambda) \cdot |(1 - \nu)(1 - \varphi_1) + (1 + \nu)(1 - \varphi_2)|} \cdot |p_2 - p_1|
\]
\[ \lambda := \frac{\lambda_1 + \lambda_2}{2}, \quad \nu := \frac{\lambda_2 - \lambda_1}{2(1 - \lambda)} \]

**Proof** The fixed point equation \( M(c, r) = (c, r) \) implies that \( m_2(c, r) = r \) which implies that \( r = \frac{1}{2} |T_2(c) - T_1(c)|/(1 - \lambda) \). By plugging this into the equation \( m_1(c, r) = c \) and some further manipulation, we get the above formula for \( c \). To get the formula for \( r \), according to the above equation it is sufficient to calculate \( T_2(c) - T_1(c) \) in terms of \( p_{1,2} \) and \( \varphi_{1,2} \). Since we have calculated \( c \), this is a matter of some further algebraic manipulation. The derivation shows that the fixed point of \( M \) not only exists, but that it is also unique. We note that \( c \) is the complex combination of \( p_1 \) and \( p_2 \). □

5 Remarks

We hereby wish to make a brief but important note. One need not be satisfied working with any given bounding circle \( C = B(c, r) \) of \( F \), since it can be easily improved. Having such a circle and taking some finite \( L \)-level iterate of it by the Hutchinson operator, ie. \( H^L(C) \), we can take the smallest enclosing circle \((c', r')\) of the \( n \)-centers \( H^L(\{c\}) \), and then \((c', r' + \lambda^L r)\) will be a tighter bounding circle of \( F \). For large enough \( L \), we can get within any \( \varepsilon > 0 \) accuracy of the fractal. So we do not need any complex algorithms to arrive at the tightest bounding circle - the minimum bounding circle and sphere algorithms resolve the problem.

Upon considering the various circles, one may contemplate whether there is any good reason to introduce these circumcircles for the case of two or three maps. There is however, in that they may be potentially tighter than the general bounding circle. According to our numerical experiments, in most cases the circumcircles are somewhat tighter, yet the general bounding circle still remains competent. In a few cases, it is even tighter than a circumcircle, so for applications, we advise the programming of all three circles.

![Figure 7: The circumcircle (red) vs. the general bounding circle (blue).](image)

Throughout our paper, we have emphasized the potential of bounding circles to resolve virtually any problem in the plane numerically, thus being a “panacea”. We conclude with an
illustration of the numerical solution to the fractal-line intersection problem, which suggests analogous solutions to other geometrical questions. In this and other specific cases, the algorithms may be designed ideally for the lowest possible algorithmic complexity.

Figure 8: Intersecting a fractal with a line numerically.

6 Conclusion

We have introduced three kinds of explicitly-defined bounding circles for IFS fractals with similitude contractions. The general bounding circle encloses polyfractals, and it even generalizes to higher dimensions. The circumcircle of trifractals generalizes the analogous circle for triangles, hinting at a novel polygonal view of IFS fractals. The circumcircle of bifractals generalizes the concept of the interval for Cantor sets on the real line. Indeed tri- and bifractals may hold a special place in the theory of Fractal Geometry, and may require as much attention as triangles in Euclidean Geometry. Cyclic polyfractals, where the fixed points lie on a circle, may be analogously rich in research potential.

Our investigations indeed intend to spawn future mathematical research, and our paper has several inspirational half-thoughts that may be elaborated upon. To point out some, the algorithm for the fractal-line intersection problem may have an optimal design, whether in 2D or 3D. There is a wide variety of other geometrical problems, having an optimal algorithmic solution using bounding circles. On a more theoretical level, cyclic polyfractals may also possess an explicit bounding circle, expressible in terms of the circle along which the fixed points lie. The ultimate question of the explicit tightest bounding circle of any IFS fractal lingers. Extending results to higher dimensions remains a continual challenge.

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References


