Exact calculations of extended logical operations on fuzzy truth values

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Abstract

In this paper we propose computationally simple, pointwise formulas for extended t-norms and t-conorms on fuzzy truth values. The complex convolutions of the extended operations are shown to be equivalent to simple pointwise expressions for several special cases. Linear fuzzy truth values are defined and it is shown that the extended Łukasiewicz operations preserve linearity. Since linear fuzzy truth values are common in representing linguistic modifiers, the results simplify fuzzy truth value-based reasoning methods. The results can also be applied immediately to type-2 fuzzy set operations.

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1. Introduction

The concept of fuzzy truth values was introduced by Bellman and Zadeh [4]. According to their interpretation, every statement can be regarded as relatively (locally) true to another statement by the following equivalence:

“x is A is f-true” ⇔ ∃B, “x is B” is true and B = F(f, A),

where A and B are classical fuzzy sets (which are identified by their membership functions) of the universe of discourse. The function f i.e. the degree of truth of “x is A” assuming that “x is B” is true can be calculated by (according to the extension principle)

\[ f_{B\mid A}(u) = \begin{cases} \bigvee_{A(x)=u} B(x) & \text{if } A^{-1}(u) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases} \]

The theory of fuzzy truth values had many applications, mainly in the field of approximate reasoning [5,1–3,12,7]. Fuzzy truth values are the next level of generalization of truth values following classical two-valued, type-1 and interval-valued
fuzzy logic. Although every level allowed a more subtle representation of truth, and induced various interpretations, they required more and more complex computations.

In recent years, the popularity of type-2 fuzzy sets has been rapidly increasing. Type-2 fuzzy sets extend the concept of classical (type-1) fuzzy sets in the following way. While the latter assign a specific membership degree to each element on its domain, a type-2 fuzzy set has fuzzy membership degrees. These are fuzzy subsets of the unit interval, i.e. fuzzy truth values. A very important special class of type-2 fuzzy sets are interval-valued fuzzy sets.

Type-2 fuzzy systems show promising results in outperforming type-1 fuzzy systems. Nowadays only interval-valued fuzzy systems are used instead of fully type-2 ones, mainly because of efficiency reasons. For example, the calculation of the conjunction of interval-valued fuzzy sets is far less complex than that of general type-2 fuzzy sets. The main bottleneck of type-2 fuzzy systems is the computational complexity of set operations, like logical operations, type-reduction or defuzzification. A consequence of the strong connection between type-2 fuzzy sets and fuzzy truth values is that the results on the latter can be interpreted in both.

In this paper we show formulas with low computational complexity for logical operations on specific classes of non-interval-valued fuzzy truth values. Our goal is not to approximate the resultant fuzzy truth value (e.g. a conjunction of two) with simple functions, but to give explicit, pointwise formulas for efficient calculations. The results can be directly applied to type-2 fuzzy systems and to reasoning systems based on fuzzy truth values.

The paper is organized as follows. Section 2 gives basic definitions and a brief overview of recent and relevant literature. In Section 3, first we discuss general theorems on transformations of extended logical operations, then we consider the special cases of monotonic, left-, right- and endmaximal fuzzy truth values. Here, we also provide sufficient conditions on whether the extended operations preserve continuity of fuzzy truth values. Section 4 deals with an important special case. We introduce linear fuzzy truth values, and investigate the extended Łukasiewicz operations on interactive linear fuzzy truth values.

2. Preliminaries

2.1. Let the unit interval be denoted by $I$

**Definition 1.** A t-norm is a binary operation $\triangle : I \times I \to I$ that is commutative, associative, increasing in each variable, and has unit element 1.

A t-conorm is a binary operation $\triangledown : I \times I \to I$ that is commutative, associative, increasing in each variable, and has unit element 0.

**Definition 2.** The residual implication $\triangleright : I \times I \to I$ associated with a t-norm $\triangle$ is defined by

$$x \triangleright y = \bigvee_{x \triangle z \leq y} z.$$ 

The residual coimplication $\triangleleft : I \times I \to I$ associated with a t-conorm $\triangledown$ is defined by

$$x \triangleleft y = \bigwedge_{x \triangledown z \geq y} z.$$ 

2.2. Operations on fuzzy truth values

**Definition 3.** Fuzzy truth values are mappings of $I$ onto itself. The set of fuzzy truth values is denoted by $F$.

Fuzzy truth values have many interpretations. They can be used as fuzzy truth-qualifications, modifier functions, fuzzy quantifiers and many more. The classical example of the use of fuzzy truth values interpreted as truth-qualifications is the following. Suppose two truth-qualified facts

$$(x \text{ is } A) \text{ is } f,$$

$$(y \text{ is } B) \text{ is } g,$$
where $A, B$ are fuzzy sets, and $f, g \in \mathcal{F}$ are truth-qualifications. The conjunction and the disjunction of these facts are the truth-qualified compound statements

$((x \text{ is } A) \text{ and } (y \text{ is } B))$ is $f \bullet g$,

$((x \text{ is } A) \text{ or } (y \text{ is } B))$ is $f \triangledown g$,

where $f \bullet g$ and $f \triangledown g$ are compound truth-qualifications, and $\bullet$ and $\triangledown$ are operations on truth-qualifications, meaning the conjunction and disjunction of them. For example, suppose the facts ‘John is tall’ is fairly true, and ‘John is strong’ is very true. The compound statement ‘John is tall and strong’ is then qualified by the conjunction of ‘fairly true’ and ‘very true’.

Mizumoto and Tanaka [8,9] and Baldwin and Guild [1,2] were the first who tackled the problem of calculating compound fuzzy truth values. They considered the following formulas—derived directly from Zadeh’s extension principle—for the conjunction and the disjunction of fuzzy truth values

$$(f \sqcap g)(z) = \bigvee_{z = x \land y} (f(x) \land g(y)), \quad (1)$$

$$(f \sqcup g)(z) = \bigvee_{z = x \lor y} (f(x) \land g(y)). \quad (2)$$

In fact, these are the extended minimum and maximum operations on the set of fuzzy truth values. In recent type-2 fuzzy logic literature they are referred as meet and join. Baldwin and Guild proposed pointwise formulas for their calculations:

$$f \sqcap g = (f \land g^R) \lor (f^R \land g), \quad (3)$$

$$f \sqcup g = (f \land g^L) \lor (f^L \land g), \quad (4)$$

where the unary operations R and L have the following definitions.

**Definition 4.** For all fuzzy truth values $f$ let

$$f^R(x) = \bigvee_{y \geq x} f(y) \quad \text{and} \quad f^L(x) = \bigvee_{y \leq x} f(y). \quad (5)$$

Note, that $f^L$ and $f^R$ are monotonic functions (see Fig. 1), and that $f^{LR} = (f^L)^R = (f^R)^L$ is a constant function which takes the supremum of $f$.

The algebraic properties of the operations $\sqcap$ and $\sqcup$ (especially on normal and convex fuzzy truth values) are thoroughly investigated recently by Walker and Walker [13].

The above operations, i.e. (1) and (2) inherently assume the non-interactivity of their arguments. This non-interactivity is represented by the t-norm $\land$ inside the supremum between $f(x)$ and $g(y)$. Non-interactivity is a similar notion to
independence in probability theory, it means that the fuzzy truth values in question have no effect on each other. Interactivity is usually modeled by a t-norm.

Godo et al. [7] were the first who considered operations on interactive fuzzy truth values, and generalized the formulas of conjunction and disjunction. Instead of extended min and max they extended an arbitrary t-norm $\triangle$ and a t-conorm $\triangledown$, which also served to realize interactivity in their setting:

$$(f \triangle g)(z) = \bigvee_{z=x \triangle y} (f(x) \triangle g(y)),\tag{6}$$

$$(f \triangledown g)(z) = \bigvee_{z=x \triangledown y} (f(x) \triangledown g(y)).\tag{7}$$

There can be two other definitions for the disjunction. Instead of the extension principle, these are based on the De Morgan identity between $\triangle$ and $\triangledown$ assuming different negation operations:

$$(f \triangledown_2 g)(z) = \bigwedge_{z=x \triangledown y} (f(x) \triangledown g(y)),\tag{8}$$

$$(f \triangledown_3 g)(z) = \bigwedge_{z=x \triangle y} (f(x) \triangle g(y)).\tag{9}$$

The three disjunctions (Eqs. (7)–(9)) are implied by the following three possible definitions of a negated fuzzy truth value:

- $f^*_1(x) = f(x')$ (e.g. fairly true–fairly false),
- $f^*_2(x) = (f(x'))'$ (e.g. fairly true–very true),
- $f^*_3(x) = (f(x))'$ (e.g. fairly true–very false),

where $'$ denotes a strong negation on $I$ (see Fig. 2).

In this paper, according to the extension principle and analogously to Walker and Walker [13] and Godo et al. [7], but generalizing their definitions, we will consider the conjunction, the disjunction, and the negation of fuzzy truth values as convolutions of the t-norm $\triangle_1$, t-conorm $\triangledown$ and negation $'$ with respect to the t-norms $\triangle$, $\triangle_2$ and $\triangledown$. The next definition is fundamental, we will refer to it often.

**Definition 5.** Let $f, g \in \mathcal{F}$ be fuzzy truth values, $\triangle, \triangle_1$ and $\triangle_2$ t-norms, $\triangledown$ a t-conorm, and $'$ a strong negation. The conjunction and disjunction of fuzzy truth values are functions $\mathcal{F} \times \mathcal{F} \rightarrow \mathcal{F}$:

$$(f \triangle g)(z) = \bigvee_{z=x \triangle_1 y} (f(x) \triangle_2 g(y)),\tag{10}$$

$$(f \triangledown g)(z) = \bigvee_{z=x \triangledown y} (f(x) \triangle g(y)).\tag{11}$$
The negation of a fuzzy truth value is a function $F \rightarrow F$:

$$
\begin{align*}
    f^*(z) &= \bigvee_{z=y'} f(y) = f(z') = f_\mu(x) = \begin{cases} 
        (x/\mu) & \text{if } \mu \neq 0, \\
        0 & \text{otherwise.}
    \end{cases}
\end{align*}
$$

(10)

For specific operations we will use indexes on $\triangledown$ and $\sqcup$. For example, the extended product operation between non-interactive fuzzy truth values will be denoted by $\triangledown^P$, i.e.

$$
(f \triangledown^P g)(z) = \bigvee_{z=x \land y} (f(x) \land g(y)).
$$

Analogously to $P$, $W$ will denote the Łukasiewicz operations. Clearly, the operations $\sqcap$ (meet) and $\sqcup$ (join) could alternatively be denoted by $\triangledown^\wedge$ and $\triangledown^\lor$.

Note, that the above definitions are generalizations of (6) and (7), and that the negation of a fuzzy truth value $f$ defined as the unary convolution of $f$ w.r.t. $\triangledown$ coincides with $f^*_1$. We will use the terms strict conjunction (disjunction) and nilpotent conjunction (disjunction) for the convolutions with strict/nilpotent t-norm $\triangledown_1$ and t-conorm $\triangledown$, independently of $\triangledown_2$, i.e. the interactivity between them.

Godo et al. have shown properties of the conjunction $\triangledown$, in the special case $\triangledown_1 = \triangledown_2$, for fuzzy truth values of the form

$$
\begin{align*}
    h_\mu(x) = \begin{cases} 
        (x/\mu) \land 1 & \text{if } \mu \neq 0, \\
        0 & \text{otherwise.}
    \end{cases}
\end{align*}
$$

Proposition 6 (Godo et al. [7]). If $\triangledown = \triangledown_1 = \triangledown_2$, the following hold for the conjunction $\triangledown$:

- it is commutative and associative,
- $h_\mu \triangledown h_\nu \geq h_\mu \lor h_\nu$,
- $h_\mu \sqcap h_\nu = h_\mu \lor h_\nu = h_{\mu \lor \nu}$.

It is easy to see that by duality, the operation $\triangledown$ also satisfies

- commutativity and associativity,
- $h_\mu \triangledown h_\nu \leq h_\mu \land h_\nu$,
- $h_\mu \sqcup h_\nu = h_\mu \land h_\nu = h_{\mu \land \nu}$.

3. Generalized extended operations on continuous fuzzy truth values

In [13] the authors show a pointwise expression similar to (3) and (4) for the extended minimum and maximum in case of product-interactive fuzzy truth values.

Theorem 7 (Walker and Walker [13]). If $\triangledown_1 = \land$, $\triangledown_2 = \lor$, and $\triangledown_2$ is the product, then the following hold for all $f, g \in F$:

$$
\begin{align*}
    (f \triangledown^P g)(z) &= \bigvee_{z=x \land y} (f(x)g(y)) = ((f^R g) \lor (fg^R))(z), \\
    (f \triangledown^P g)(z) &= \bigvee_{z=x \lor y} (f(x)g(y)) = ((f^L g) \lor (fg^L))(z).
\end{align*}
$$

Since any strict t-norm is isomorphic to the product, the theorem applies to all strict t-norms as well. It is easy to see that Theorem 7 also holds for the Łukasiewicz t-norm instead of the product. So the next theorem generalizes the results of [1,2,13].
**Theorem 8.** If $\triangle_1 = \wedge$, $\triangledown = \vee$, and $\triangle_2 = \bigtriangleup$ is an arbitrary continuous and Archimedean t-norm, then the following hold for all $f, g \in F$:

\[
(f \bigtriangleup \wedge g)(z) = \bigvee_{z = x \wedge y} (f(x) \bigtriangleup g(y)) = ((f \bigtriangleup g)(x) \bigvee (f \bigtriangleup g)(y))(z),
\]

\[
(f \bigtriangledown \vee g)(z) = \bigvee_{z = x \vee y} (f(x) \bigtriangledown g(y)) = ((f \bigtriangledown g)(x) \bigvee (f \bigtriangledown g)(y))(z).
\]

(11)

**Proof.** Let $f, g \in F$.

\[
(f \bigtriangleup \wedge g)(z) = \bigvee_{z = x \wedge y} (f(x) \bigtriangleup g(y)) = (f \bigtriangledown g)(z),
\]

\[
(f \bigtriangledown \vee g)(z) = \bigvee_{z = x \vee y} (f(x) \bigtriangledown g(y)) = (f \bigtriangleup g)(z).
\]

The disjunction can be proved analogously. □

This theorem covers the extended minimum and maximum operators in case of $\bigtriangleup$-interactivity, where $\bigtriangleup$ is an arbitrary continuous and Archimedean t-norm. From now on, we restrict our investigations to extensions of continuous and Archimedean t-norms and t-conorms. The following theorems show transformations of the convolutions of Definition 5 and serve as a basis for computational simplifications.

**Theorem 9.** If $\triangle_1$ and $\triangle_2$ are t-norms, s.t. $\triangle_1$ is continuous and Archimedean, then the following hold for all $f, g \in F$.

For $z > 0$:

\[
(f \bigtriangleup g)(z) = \bigvee_{x \geq z} (f(x) \bigtriangleup_2 g(x \triangledown_1 z)) = \bigvee_{y \geq z} (f(y \triangledown_1 z) \bigtriangleup_2 g(y)).
\]

(12)

If $\triangle_1$ is strict then for $z = 0$:

\[
(f \bigtriangleup g)(0) = (f(0) \bigtriangleup_2 g(0)) \bigvee (f \bigtriangledown_1 g(0)),
\]

(13)

and if $\triangle_1$ is nilpotent then for $z = 0$:

\[
(f \bigtriangleup g)(0) = \bigvee_{x} (f(x) \bigtriangleup_2 g(x')) = \bigvee_{y} (f \bigtriangledown_1 g(y)).
\]

(14)

where $x \triangledown_1$ denotes the residual implication of $\triangle_1$, and $x' = (x \triangledown_1 0)$ is the strong negation corresponding to $\triangledown_1$.

**Proof.** Continuous and Archimedean t-norms have a generator functional form, so let

\[
x \bigtriangleup_1 y = \varphi^{-1} \{ \varphi(0) \wedge (\varphi(x) + \varphi(y)) \},
\]
where \( \varphi : [0, 1] \rightarrow [0, \infty] \) is a strictly decreasing and continuous function with \( \varphi(1) = 0 \). So, by definition

\[
(f \triangle g)(z) = \bigvee_{\varphi(z) = (\varphi(x) + \varphi(y)) \land \varphi(0)} (f(x) \triangle_2 g(y)).
\]

By supposing \( z > 0 \), i.e. \( \varphi(z) < \varphi(0) \), the constraint is \( \varphi(z) = \varphi(x) + \varphi(y) \). This implies \( \varphi(x) \leq \varphi(z) \), i.e. \( x \geq z \), since \( \varphi \) is non-negative and strictly decreasing. Furthermore, \( y = \varphi^{-1}(\varphi(z) - \varphi(x)) = x \triangleright_1 z \), where \( \triangleright_1 \) is the residual implication of \( \triangle_1 \). Analogously, \( y > z \), and \( x = \varphi^{-1}(\varphi(z) - \varphi(y)) = y \triangleright_1 z \). So in case \( z > 0 \) we have

\[
(f \triangle g)(z) = \bigvee_{x \geq z} (f(x) \triangle_2 g(x \triangleright_1 z)) = \bigvee_{y \geq z} (f(y \triangleright_1 z) \triangle_2 g(y)).
\]

Suppose \( z = 0 \) and \( \triangle_1 \) is strict. Since \( x \triangleright_1 y = 0 \) if and only if \( x \land y = 0 \), Theorem 8 can be applied with \( z = 0 \). Now, suppose \( z = 0 \) and \( \triangle_1 \) is nilpotent. In this case \( x \triangleright_1 y = 0 \) if and only if \( y \leq (x \triangleright_1 0) = x' \), thus we have

\[
(f \triangle g)(0) = \bigvee_{x \in [0,1]} (f(x) \triangle_2 g(y)) = \bigvee_{x \leq x'} \left( f(x) \triangle_2 \bigvee_{y \leq x'} g(y) \right) = \bigvee_x (f(x) \triangle_2 g^L(x')).
\]

An equivalent condition to \( y \leq x' \) is \( y' \geq x \), and so by similar transformations

\[
(f \triangle g)(0) = \bigvee_y (f^L(y') \triangle_2 g(y)). \quad \square
\]

Note that the symmetry in (12) stems from the commutativity of \( \triangle_1 \). See Figs. 3–5 for examples of extended t-norms in a general setting. A similar theorem holds for extended Archimedean t-conorms, which can be proved in an analogous manner. Figs. 6–8 show examples of extended t-conorms.

**Theorem 10.** If \( \triangle \) is a t-norm, \( \nabla \) is a continuous and Archimedean t-conorm, then the following hold for all \( f, g \in \mathcal{F} \).

For \( z < 1 \):

\[
(f \triangledown g)(z) = \bigvee_{x \leq z} (f(x) \triangle g(x \triangle z)) = \bigvee_{y \leq z} (f(y \triangle z) \triangle g(y)),
\]

where \( \triangle \) denotes the residual coimplication of \( \nabla \). If \( \nabla \) is strict then for \( z = 1 \):

\[
(f \triangledown g)(1) = (f^L(1) \triangle L(1)) \lor (f(1) \triangle L(1)),
\]

Fig. 3. The extended minimum of fuzzy truth values with different interactivity. (a) \( f \cap g \), (b) \( f \triangle_\lambda^g \), (c) \( f \triangle_\lambda^W g \).
Fig. 4. The extended product of fuzzy truth values with different interactivity. (a) $f \bigtriangleup_P g$, (b) $f \bigstar_P g$, (c) $f \bigtriangleup_W g$.

Fig. 5. The extended Łukasiewicz t-norm of fuzzy truth values with different interactivity. (a) $f \bigtriangleup_W g$, (b) $f \bigstar_W g$, (c) $f \bigtriangleup_W g$.

Fig. 6. The extended maximum of fuzzy truth values with different interactivity. (a) $f \bigtriangledown g$, (b) $f \bigtriangledown_P g$, (c) $f \bigtriangledown_W g$.

Fig. 7. The extended algebraic sum of fuzzy truth values with different interactivity. (a) $f \bigtriangledown_P g$, (b) $f \bigtriangledown_P g$, (c) $f \bigtriangledown_W g$. 
Fig. 8. The extended Łukasiewicz t-conorm of fuzzy truth values with different interactivity. (a) $f 	riangleleft_W^g$, (b) $f 	riangleleft_W^P g$, (c) $f 	riangleleft_W^W g$.

and if $\triangledown$ is nilpotent then for $z = 1$:

$$
(f \triangleleft g)(1) = \bigvee_x (f(x) \triangle g^R(x')) = \bigvee_y (f^R(y') \triangle g(y)),
$$

(17)

where $\triangle$ denotes the residual coimplication of $\triangledown$, and $x' = (x \triangledown 1)$ is a strong negation.

3.1. Results on left- and right-maximal and monotonic fuzzy truth values

Theorems 9 and 10 are in general do not considerably decrease computational complexity of the extended operations. In this subsection, we restrict our investigations to special classes of fuzzy truth values. With these restrictions, corollaries of the above theorems are shown with practical results.

**Definition 11** (Nieminen [11], Walker and Walker [13]). A fuzzy truth value $f$ is endmaximal if $f^L = f^R$, left-maximal if $f^L = f^{LR}$, right-maximal if $f^R = f^{LR}$ and normal if $f^{LR} = 1$.

It is easy to see the following.

**Proposition 12.** For all $f \in \mathcal{F}$,

1. $f$ is right-maximal iff $f(1) = f^R(0)$.
2. $f$ is left-maximal iff $f(0) = f^L(1)$.

From now on, $\mathcal{F}^+$ and $\mathcal{F}^-$ will denote the set of non-decreasing and non-increasing continuous fuzzy truth values. Note, that if $f \in \mathcal{F}^+$ (resp., $\mathcal{F}^-$) then it is also right-maximal (left-maximal). Monotonic fuzzy truth values are widespread in modeling linguistic modifiers such as ‘true’, ‘very true’, ‘more or less true’, ‘false’, ‘very false’, ‘more or less false’, etc.

Next, we give pointwise expressions for the operations $\triangledown$ and $\vartriangle$ in case of left- and right-maximal and monotonic fuzzy truth values. Note, that if $f \in \mathcal{F}^+$ then $f^R$ is the constant function which takes the value $f(1)$ everywhere, and $f^L = f$. Analogously, if $f \in \mathcal{F}^-$ then $f^L(x) = f(1)$ for all $x \in \mathcal{I}$, and $f^R = f$. An immediate consequence of Theorem 8 is the following.

**Corollary 13.** For all right-maximal $f, g \in \mathcal{F}$

$$
f \triangledown \wedge g = (f^{LR} \triangle g) \lor (f \triangle g^{LR}),
$$

(18)

moreover, if $f, g \in \mathcal{F}^+$

$$
(f \triangledown^P g)(x) = f(x) \triangle g(x).
$$

(19)

Analogously, for all left-maximal $f, g \in \mathcal{F}$

$$
f \triangledown \vartriangle g = (f^{LR} \triangle g) \lor (f \triangle g^{LR}),
$$

(20)
moreover if $f, g \in \mathcal{F}^-$

$$(f \land g)(x) = f(x) \triangle g(x). \tag{21}$$

Note, that according to (19) and (21) the maximum and minimum of two monotone fuzzy truth values can be calculated pointwise with the t-norm representing their interactivity (Figs. 9 and 10).

**Corollary 14.** For all left-maximal $f, g \in \mathcal{F}$

$$(f \land g)(0) = f(0) \triangledown g(0). \tag{22}$$

For all right-maximal $f, g$:

$$(f \triangledown g)(1) = f(1) \triangledown g(1). \tag{23}$$

Important corollaries of Theorem 9 and 10 are the following.

**Corollary 15.** If $f$ is right-maximal and $g \in \mathcal{F}^-$, then

$$(f \land g)(x) = f^{LR}(x) \triangledown g(x), \tag{24}$$

and $f \land g \in \mathcal{F}^-$. Furthermore, if $f$ is also normal, then $f \land g = g$, i.e. $f$ acts as a unit element.

**Proof.** If $z > 0$ then

$$(f \land g)(z) = \bigvee_{x \geq z} (f(x) \triangledown g(x \triangle z)).$$
It always has a supremum at $x = 1$, so

$$(f \triangle g)(z) = f(1) \triangle_2 g(1 \triangle_1 z) = f(1) \triangle_2 g(z).$$

In case $\triangle_1$ is strict and $z = 0$, 

$$(f \triangle g)(0) = (f(0) \triangle_2 g^R(0)) \lor (f^R(0) \triangle_2 g(0))$$

$$= (f(0) \triangle_2 g(0)) \lor (f(1) \triangle_2 g(0)) = f(1) \triangle_2 g(0).$$

In case $\triangle_1$ is nilpotent and $z = 0$, 

$$(f \triangle g)(0) = \bigvee_x (f(x) \triangle_2 g^L(x \triangleright_1 0)) = \bigvee_x (f(x) \triangle_2 g(0))$$

$$= f(1) \triangle_2 g(0). \quad \Box$$

We have a similar result for extended disjunctions, which can be proved analogously.

**Corollary 16.** If $f$ is left-maximal and $g \in \mathcal{F}^+$, then

$$(f \nabla g)(x) = f^{LR}(x) \nabla g(x),$$

and $f \nabla g \in \mathcal{F}^+$. Furthermore, if $f$ is also normal, then $f \nabla g = g$.

Note, that since all non-decreasing fuzzy truth values are right-maximal (and all non-increasing fuzzy truth values are left-maximal), the above results apply to monotonic ones, too. These two corollaries express the natural intuition that the conjunction (resp. disjunction) of a positive, ‘true-like’ and a negative, ‘false-like’ fuzzy truth value is the ‘weaker’ (resp. ‘stronger’) one. This is in accordance with Boolean logic and type-1 fuzzy logic, too (Fig. 11).

### 3.2. Continuity of operations on fuzzy truth values

In this section we give sufficient conditions for the continuity of the compound fuzzy truth values $f \triangle g$ and $f \nabla g$. Let $\mathcal{F}_c$ denote the set of continuous fuzzy truth values.

**Proposition 17.** The strict conjunction $f \triangle g$ of $f, g \in \mathcal{F}_c$ is continuous if $f$ or $g$ is left- or right-maximal.

**Proof.** To prove the continuity of $f \triangle g$, it suffices to show

$$\lim_{z \to 0} (f \triangle g)(z) = (f \triangle g)(0),$$

since according to (12) for all $z > 0$, $(f \triangle g)(z)$ is continuous. Recall, that for any strict conjunction

$$(f \triangle g)(0) = (f(0) \triangle_2 g^R(0)) \lor (f^R(0) \triangle_2 g(0)).$$
Now,
\[
\lim_{z \to 0} (f \ast g)(z) = \lim_{z \to 0} \bigvee_{x \geq z} (f(x) \Delta_2 g(x \triangleright 1 z))
\]
\[
= \bigvee_{x} (f(x) \Delta_2 g(x \triangleright 1 0))
\]
\[
= (f(0) \Delta_2 g(1)) \lor \bigvee_{x > 0} (f(x) \Delta_2 g(0))
\]
\[
= (f(0) \Delta_2 g(1)) \lor (f^R(0) \Delta_2 g(0)), \tag{27}
\]
since \(\triangleright 1\) is a strict residual implication, i.e.
\[
x \triangleright 1 0 = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise.} \end{cases}
\]

If \(g\) is right-maximal i.e. \(g(1) = g^R(0)\), then clearly (27) equals to \((f \ast g)(0)\).

Furthermore, if \(g\) is left-maximal i.e. \(g(0) = g^L(0)\), then in both (26) and (27) the second term dominates the first one, i.e.
\[
f^R(0) \Delta_2 g(0) \geq f(0) \Delta_2 g^R(0)
\]
and
\[
f^R(0) \Delta_2 g(0) \geq f(0) \Delta_2 g(1)
\]
and so (26) and (27) are equal.

It can be shown analogously, that the left- or right-maximality of \(f\) is also sufficient for the continuity of \(f \ast g\). \(\square\)

**Proposition 18.** The nilpotent conjunction \(f \ast g\) of \(f, g \in \mathcal{F}_c\) is continuous if \(f \in \mathcal{F}_c^+\) or \(g \in \mathcal{F}_c^+\), i.e. if \(f\) or \(g\) is monotone increasing.

**Proof.** It suffices to show that
\[
\lim_{z \to 0} (f \ast g)(z) = (f \ast g)(0).
\]

Recall, that for any nilpotent conjunction
\[
(f \ast g)(0) = \bigvee_{x} (f(x) \Delta_2 g^L(x \triangleright 1 0)). \tag{28}
\]

Now,
\[
\lim_{z \to 0} (f \ast g)(z) = \lim_{z \to 0} \bigvee_{x \geq z} (f(x) \Delta_2 g(x \triangleright 1 z))
\]
\[
= \bigvee_{x} (f(x) \Delta_2 g(x \triangleright 1 0))
\]
\[
= \bigvee_{x} (f(x) \Delta_2 g(x')). \tag{29}
\]

It is easy to see, that (28) and (29) are equal if \(g \in \mathcal{F}_c^+\), i.e. \(g = g^L\).

It can be shown analogously, that the monotonicity of \(f\) is also sufficient for the continuity of \(f \ast g\). \(\square\)
The next two propositions can be proved analogously.

**Proposition 19.** The strict disjunction \( f \downarrow g \) of \( f, g \in \mathcal{F}_c \) is continuous if \( f \) or \( g \) is left- or right-maximal.

**Proposition 20.** The nilpotent disjunction \( f \downarrow g \) of \( f, g \in \mathcal{F}_c \) is continuous if \( f \in \mathcal{F}_c^- \) or \( g \in \mathcal{F}_c^- \).

### 4. Extended Łukasiewicz operations on linear fuzzy truth values

The Łukasiewicz t-norm \( 0 \lor (x + y - 1) \) will be denoted by \( W \). Its residual implication \( 1 \land (1 - x + y) \) and coimplication \( 0 \lor (y - x) \) will be denoted by \( \triangleright W \) and \( \lhd W \).

**Definition 21.** Let the Łukasiewicz conjunction and disjunction of fuzzy truth values be

\[
(f \uparrow_W g)(z) = \bigvee_{z=(x+y-1) \lor 0} ((f(x) + g(y) - 1) \lor 0),
\]

\[
(f \downarrow_W g)(z) = \bigvee_{z=(x+y) \land 1} ((f(x) + g(y) - 1) \lor 0).
\]

Note, that this is a special case of Definition 5. In this setting the arguments are interactive, and this interactivity is represented by the Łukasiewicz t-norm. Here we do not discuss the non-interactive case, for references, see Nguyen’s theorem in [10]. Fullér and Keresztfalvi [6] generalized the Nguyen theorem to non-interactive arguments, but did not provide explicit formulas for computations. Here, we provide pointwise, easy-to-compute formulas for the above operations on linear fuzzy truth values.

**Definition 22.** Let \( L \subset \mathcal{F}_c \) be the set of linear fuzzy truth values characterized by

\[
f_{a,b} \in L \iff f_{a,b}(x) = \begin{cases} x - a & \text{if } a < b, \\ b - a & \text{if } a > b. \end{cases}
\]  

where \( a \neq b, x \in [0, 1] \) and \( \{t\}_a^b = a \lor t \land b \).

Let \( L^+ \subset \mathcal{F}_c^+ \) denote the set of non-decreasing, and \( L^- \subset \mathcal{F}_c^- \) the set of non-increasing linear fuzzy truth values.

A linear \( f_{a,b} \) is non-decreasing iff \( a < b \) and non-increasing iff \( a > b \).

The set of normal, non-decreasing (non-increasing) linear fuzzy truth values is \( L^+_1 \) (resp. \( L^-_1 \)) and characterized by \( b \leq 1 \) (resp. \( b \geq 0 \)).

The next theorem states that the Łukasiewicz conjunction of non-decreasing linear fuzzy truth values can be calculated by a computationally simple pointwise formula instead of a convolution.

**Theorem 23.** The following hold for all \( f_i = f_{a_i,b_i} \in \mathcal{L}^+ \) (\( i = 1, 2 \)).

\[
(f_1 \uparrow_W f_2)(z) = (f_1(1) \triangle_W f_2([b_1]_z \triangleright_W z)) \lor (f_2(1) \triangle_W f_1([b_2]_z \triangleright_W z)),
\]

where \( \{x\}_z = z \lor x \land 1 \).

The following Lemma is required for the proof.

**Lemma 24.** Let \( f_1 = f_{a_1,b_1} \in \mathcal{L}^+ \) and \( f_2 = f_{a_2,b_2} \in \mathcal{L}^- \) be two linear fuzzy truth values, let \( S \) denote their sum, \( S(x) = f_{a_1,b_1}(x) + f_{a_2,b_2}(x) \). Then,

\[
S^R(z) = \bigvee_{x \geq z} S(x) = S([b_1]_z) \lor S([b_2]_z), \quad \forall z \in [0, 1],
\]

where \( \{x\}_z = z \lor x \land 1 \).
Proof. Based on the relationship between \( b_1, b_2 \) and \( z \), there are four cases:

(1) If \( b_1 \lor b_2 \leq z \), then \( S(x) = 1 + f_2(x) \) for all \( x \geq z \). Thus, \( S^R(z) = S(z) \).
(2) If \( b_1 \leq z < b_2 \), then similarly to the previous case \( S^R(z) = S(z) = S(b_2 \land 1) \).
(3) If \( b_2 \leq z < b_1 \), then \( S^R(z) = S(z) \lor S(b_1 \land 1) \).
(4) If \( z < b_1 \land b_2 \), then \( S^R(z) = S(b_1 \land 1) \lor S(b_2 \land 1) \). □

Now, we are ready to prove Theorem 23.

Proof. Since \( \mathcal{L}^+ \subset \mathcal{F}^+_c \), according to Proposition 18 \( f_1 \uplus_w f_2 \) is continuous, and so the case \( z = 0 \) of Theorem 9 need not be considered separately. Then,

\[
(f_1 \uplus_w f_2)(z) = \bigvee_{z = (x+y-1) \lor 0} (f_1(x) + f_2(y) - 1) \lor 0
\]

\[
= \bigvee_{x \geq z} (f_1(x) + f_2(1 - x + z) - 1) \lor 0
\]

\[
= 0 \lor \left( -1 + \bigvee_{x \geq z} (f_1(x) + f'_2(x)) \right),
\]

where \( f'_2(x) = f_2(1 - x + z) \). This way \( f'_2 \in \mathcal{L}^- \) is also linear with parameters

\[
a'_2 = 1 - a_2 + z \quad \text{and} \quad b'_2 = 1 - b_2 + z.
\]

By Lemma 24, the supremum is either at \( x = \{b_1\}_z \) or \( x = \{b'_2\}_z \), and

\[
(f_1 \uplus_w f_2)(z) = 0 \lor (-1 + (f_1(\{b_1\}_z) + f'_2(\{b_1\}_z)) \lor (f_1(\{b'_2\}_z) + f'_2(\{b'_2\}_z))).
\]

Consider the following equalities:

\[
f'_2(\{b_1\}_z) = f_2(\{b_1\}_z \triangledown_w z),
\]

\[
f_1(\{b'_2\}_z) = f_1(\{b_2\}_z \triangledown_w z),
\]

\[
f'_2(\{b'_2\}_z) = f_2(\{b'_2\}_z),
\]

\[
f_1(\{b_1\}_z) = f_1(1).
\]

To prove the third one, for example,

\[
f'_2(\{b'_2\}_z) = f_2(1 - (z \lor (1 - b_2 + z) \land 1) + z)
\]

\[
= f_2((1 - z) \land (b_2 - z) \lor 0 + z) = f_2(z \lor b_2 \land 1) = f_2(\{b'_2\}_z).
\]

It follows that

\[
(f_1 \uplus_w f_2)(z) = 0 \lor (-1 + (f_1(1) + f_2(\{b_1\}_z \triangledown_w z)) \lor (f_1(\{b'_2\}_z \triangledown_w z) + f_2(1)))
\]

\[
= (f_1(1) \triangle_w f_2(\{b_1\}_z \triangledown_w z)) \lor (f_1(\{b'_2\}_z \triangledown_w z) \triangle_w f_2(1)). \quad\square
\]

Example. To clarify Theorem 23 consider the following example. Let \( f_1 = f_{a_1,b_1} = f_{6,.7} \) and \( f_2 = f_{a_2,b_2} = f_{7,1} \) be linear fuzzy truth values. The pointwise calculation of \( (f_1 \uplus_w f_2)(.5) \) is as follows (see Fig. 12(a)). According to Theorem 23,

\[
(f_1 \uplus_w f_2)(.5) = (f_1(1) \triangle_w f_2(\{.7\}_5 \triangledown_w .5)) \lor (f_2(1) \triangle_w f_1(\{1\}_5 \triangledown_w .5)),
\]

where \( \{x\}_y = y \lor x \land 1 \). Furthermore, we have

\[
(f_1 \uplus_w f_2)(.5) = (1 \triangle_w f_2(.7 \triangledown_w .5) \lor (1 \triangle_w f_1(1 \triangledown_w .5))
\]

\[
= (0 \lor (1 + f_2(1 \land (1 - .7 + .5) - 1))) \lor (0 \lor (1 + f_1(1 \land (1 - 1 + .5)) - 1))
\]

\[
= (0 \lor (1 + f_2(.8)) - 1)) \lor (0 \lor (1 + f_1(.5) - 1))
\]

\[
= (0 \lor (1 + .33 - 1)) \lor (0 \lor (1 + 0 - 1)) = .33.
\]
According to Proposition 18, the Łukasiewicz conjunction of non-decreasing linear fuzzy truth values is continuous in any case. Although, it is not always linear (see for example Fig. 13), linearity is preserved for normal fuzzy truth values.

**Corollary 25.** For all \( f_i = f_{a_i, b_i} \in \mathcal{L}_1^+ \) (\( i = 1, 2 \))

\[
(f_1 \triangleleft_w f_2)(z) = f_1(b_2 \triangleright w z) \lor f_2(b_1 \triangleright w z).
\] (34)

Furthermore, \( f_1 \triangleleft_w f_2 \) is also linear with parameters

\[
a_{\triangleleft w} = (a_1 + b_2 - 1) \land (a_2 + b_1 - 1),
\]

\[
b_{\triangleleft w} = b_1 + b_2 - 1.
\]

**Proof.** If \( f_i \) are normal then \( f_i(1) = 1 \) and since \( b_i \leq 1 \), \( ([b_i]_z \triangleright w z) = (b_i \triangleright w z) \), hence (34) holds.

Furthermore, note that for example \( f_1(b_2 \triangleright w z) = f'_1(z) \), where

\[
a'_1 = a_1 + b_2 - 1 \quad \text{and} \quad b'_1 = b_1 + b_2 - 1.
\]

Indeed,

\[
f_1(b_2 \triangleright w z) = \frac{1 \land (1 - b_2 + z) - a_1}{b_1 - a_1} = \frac{b_2 \land z - (a_1 + b_2 - 1)}{b_1 + b_2 - 1 - (a_1 + b_2 - 1)}
\]

\[
= f'_1(b_2 \land z) = f'_1(z),
\]
since for all $b_2 \leq z$, $f_1'(z) = 1$. Similarly, $f_2'$ is such that
\[ a'_2 = a_2 + b_1 - 1 \quad \text{and} \quad b'_2 = b_1 + b_2 - 1. \]

Thus $f'_i$ are linear fuzzy truth values with equal upper endpoints. Their maximum is the one with the minimal lower endpoint. □

Theorem 23 can only be applied to fuzzy truth values in $\mathcal{L}^-$. For the Łukasiewicz conjunction on $\mathcal{L}^-$ two cases need to be considered, since its continuity is not guaranteed. The next theorem can be easily proved analogously to Theorem 23 considering Theorem 9 and Corollary 14.

**Theorem 26.** The following hold for all $f_i = f_{a_i,b_i} \in \mathcal{L}^-$ $(i = 1, 2)$. For $z > 0$,
\[
(f_1 \triangledown_w f_2)(z) = (f_1(z) \triangledown_w f_2([b_1]_\triangledown_w z)) \lor (f_2(z) \triangledown_w f_1([b_2]_\triangledown_w z)),
\]
and if $z = 0$, then
\[
(f_1 \triangledown_w f_2)(0) = f_1(0) \triangledown_w f_2(0).
\]

The next corollary gives necessary and sufficient conditions on the continuity of the Łukasiewicz conjunction between the elements of $\mathcal{L}^-$.

**Corollary 27.** For $f_i = f_{a_i,b_i} \in \mathcal{L}^-$, the Łukasiewicz conjunction $f_1 \triangledown_w f_2$ is continuous if and only if $b_1 + b_2 \geq 1$.

**Proof.** $(f_1 \triangledown_w f_2)(z)$ may be discontinuous only at $z = 0$. Thus, $f_1 \triangledown_w f_2$ is continuous if and only if the equality
\[
(f_1(0) \triangledown_w f_2(1 - [b_1])) \lor (f_2(0) \triangledown_w f_1(1 - [b_2])) = f_1(0) \triangledown_w f_2(0)
\]
holds. It is equivalent to
\[
f_2(1 - [b_1]) = f_2(0) = f_2([b_2]) \quad \text{and} \quad f_1(1 - [b_2]) = f_1(0) = f_1([b_1]).
\]
These equations hold iff $[b_1] + [b_2] \geq 1$, which is equivalent to the condition $b_1 + b_2 \geq 1$. □

The Łukasiewicz disjunction of linear fuzzy truth values can be handled in a totally analogous manner. We provide the following results without proofs.

**Lemma 28.** Let $f_1 = f_{a_1,b_1} \in \mathcal{L}^+$ and $f_2 = f_{a_2,b_2} \in \mathcal{L}^-$ be two linear fuzzy truth values, let $S$ denote their sum, $S(x) = f_{a_1,b_1}(x) + f_{a_2,b_2}(x)$. Then,
\[
S^\ominus(z) = \bigvee_{x \leq z} S(x) = S([b_1]^\ominus) \lor S([b_2]^\ominus) \quad \forall z \in [0,1],
\]
where $\{x\}^\ominus = 0 \lor x \land z$.

**Theorem 29.** For all $f_i = f_{a_i,b_i} \in \mathcal{L}^-$ $(i = 1, 2)$
\[
(f_1 \triangledown_w f_2)(z) = (f_1(0) \triangledown_w f_2([b_1]_\triangledown_w z)) \lor (f_2(0) \triangledown_w f_1([b_2]_\triangledown_w z)),
\]
where $\{x\}^\ominus = 0 \lor x \land z$.

**Example.** Let $f_1 = f_{a_1,b_1} = f_{8,.5}$ and $f_2 = f_{a_2,b_2} = f_{6,.2}$ be linear fuzzy truth values. The pointwise calculation of $(f_1 \triangledown_w f_2)(.7)$ is as follows (see Fig. 12(b)). According to Theorem 29,
\[
(f_1 \triangledown_w f_2)(.7) = (f_1(0) \triangledown_w f_2([.5]_\triangledown_w .7)) \lor (f_2(0) \triangledown_w f_1([.2]_\triangledown_w .7)),
\]
where \( \{x\}^y = 0 \lor x \land y \). Furthermore, we have
\[
(f_1 \downarrow W f_2)(.7) = (1 \triangle W f_2(.5 <_W .7)) \lor (1 \triangle W f_1(1 \triangleright W .5)) \\
= (0 \lor (1 + f_2(1 \land (1 - .7 + .5)) - 1)) \lor (0 \lor (1 + f_1(1 \land (1 - 1 + .5)) - 1)) \\
= (0 \lor (1 + f_2(.8)) - 1)) \lor (0 \lor (1 + f_1(.5) - 1)) \\
= (0 \lor (1 + .33 - 1)) \lor (0 \lor (1 + 0 - 1)) = .33.
\]

**Corollary 30.** For all \( f_i = f_{a_i,b_i} \in L^-_1 (i = 1, 2) \)
\[
(f_1 \downarrow W f_2)(z) = f_1(b_2 <_W z) \lor f_2(b_1 <_W z).
\]

Furthermore, \( f_1 \downarrow W f_2 \) is also linear with parameters
\[
a \downarrow W = (a_1 + b_2) \lor (a_2 + b_1),
\]
\[
b \downarrow W = b_1 + b_2.
\]

**Theorem 31.** The following hold for all \( f_i = f_{a_i,b_i} \in L^+ (i = 1, 2) \). For \( z < 1 \),
\[
(f_1 \downarrow W f_2)(z) = (f_1(z) \triangle W f_2([b_1] <_W z)) \lor (f_2(z) \triangle W f_1([b_2] <_W z)).
\]
and if \( z = 1 \), then
\[
(f_1 \downarrow W f_2)(1) = f_1(1) \triangle W f_2(1).
\]

The Łukasiewicz disjunction is always continuous on the elements of \( L^- \). On \( L^+ \) the following holds.

**Corollary 32.** For \( f_i = f_{a_i,b_i} \in L^+ \), the Łukasiewicz disjunction \( f_1 \downarrow W f_2 \) is continuous if and only if \( b_1 + b_2 \leq 1 \).

Next, we provide some further properties of the operation \( \triangleright W \) on linear fuzzy truth values. Similar ones can be easily obtained for \( \downarrow W \). The proofs are straightforward considering the above results. The constant function 1 is denoted by 1.

**Proposition 33.** The following hold for the Łukasiewicz conjunctions of non-decreasing linear fuzzy truth values:

1. idempotency
\[
f_{a,b} \triangleright W f_{a,b} = f_{a,b} \iff b = 1.
\]

2. unit elements
\[
f_{a_1,b_1} \triangleright W f_{a_2,b_2} = f_{a_1,b_1} \iff b_2 = 1 and 1 - a_2 \leq b_1 - a_1, \]
i.e. \( f_{a_2,b_2} \) is steeper than \( f_{a_1,b_1} \).

3. nilpotency
\[
f_{a_1,b_1} \triangleright W f_{a_2,b_2} = 1 \iff b_1 + b_2 \leq 1.
\]

4. zero element
\[
1 \triangleright W f = 1.
\]

5. monotonicity
\[
f_{a_1,b_1} \triangleright f_{a_2,b_2} \text{ i.e. } a_1 \leq a_2 \text{ and } b_1 \leq b_2 \implies f_{a_1,b_1} \triangleright W g \geq f_{a_2,b_2} \triangleright W g.
\]
5. Conclusions

We have discussed extended t-norms and t-conorms on continuous and interactive fuzzy truth values. Sufficient conditions were given on the continuity of the resultant functions. We have shown easy-to-implement pointwise formulas for the conjunction and disjunction of fuzzy truth values with different monotonicity.

As an important special case, we have considered the extended Łukasiewicz operations on interactive linear fuzzy truth values. It was shown that the complex convolutions of the extended Łukasiewicz operations are equivalent to simple operations on the parameters on the linear fuzzy truth values. We have given necessary and sufficient conditions when these operations preserve continuity and linearity.

The results can be directly applied to type-2 fuzzy systems and to reasoning systems based on fuzzy truth values.

References