On the Diameter of Geometric Path Graphs of Points in Convex Position

Jou–Ming Chang\(^1,\)† Ro–Yu Wu\(^2\)

\(^1\) Department of Information Management, National Taipei College of Business, Taipei, Taiwan. (spade@mail.ntcb.edu.tw)

\(^2\) Department of Industrial Engineering Management, Lunghwa University of Science and Technology, Taoyuan, Taiwan. (eric@mail.lhu.edu.tw)

Abstract

For a set \(S\) of \(n\) points in convex position in the plane, let \(\mathcal{P}(S)\) denote the set of all plane spanning paths on \(S\). The geometric path graph \(G_n\) of \(S\) is the graph with \(\mathcal{P}(S)\) as its vertex set and two vertices \(P,Q \in \mathcal{P}(S)\) are adjacent if one of the corresponding paths can be obtained from the other by means of a single edge replacement. Recently, Akl et al. [On planar path transformation, Inform. Process. Lett. 104 (2007) 59–64] showed that the diameter of \(G_n\) is at most \(2n - 5\). In this note, we derive the exact diameter of \(G_n\) for \(n \geq 4\).

Keywords: Flips; Plane spanning paths; Geometric path graphs; Diameter;

\(\mathcal{P}(S)\) denote the set of all planar paths on \(S\). Flipping an edge on a path \(P \in \mathcal{P}(S)\) means that a new path \(P' \in \mathcal{P}(S)\) is obtained from \(P\) by a single edge replacement (i.e., one edge \(e \in P\) is removed and another edge \(f \notin P\) is inserted). In this way, we say that \(P'\) is obtained from \(P\) by means of a flip. A geometric path graph on \(S\), denoted by \(G(S)\), is the graph consisting of all paths in \(\mathcal{P}(S)\) as its vertices and two vertices are joined by an edge if their corresponding paths can be transformed to each other by using a flip. Note that, for any set \(S\) on \(n\) points in convex position in the plane, \(G(S)\) is isomorphic to \(G(H_n)\) where \(H_n\) is a regular \(n\)-gon. Hereafter, we denote by \(G_n\) the geometric path graph \(G(H_n)\). For instance, Figure 1 shows the geometric path graph on a set of 4 points.

1. Introduction

Given a set \(S\) of \(n\) (\(\geq 3\)) points in the plane, a plane spanning path (or planar path for short) of \(S\) is a path \(P\) whose edges connect all points of \(S\) with straight line segments and such that no two edges of \(P\) cross. Let

\*This research was partially supported by National Science Council under the Grants NSC95-2115-M-141-001-MY2.

†Corresponding author.

Figure 1: A geometric path graph \(G_4\).
More recently, Akl et al. [3] showed that there are totally \( n2^n - 3 \) planar paths in \( \mathcal{P}(S) \) if \( S \) contains \( n \) points in convex position, and every planar path in the set can be transformed into another planar path by means of at most \( 2n - 5 \) flips. This implies that \( G_n \) is connected and has diameter no more than \( 2n - 5 \). In addition, they also proposed an algorithm employing flips and 2-flips (i.e., an operation for path transformation using two edges replacement) to generate all directed planar paths on a set of \( n \) points in convex position and such that the time required between two successive paths is \( O(n) \). Experimental results are also given in [3] for discussing the connectedness of geometric path graphs of points in general position. For more results related to planar paths and their generalization, called plane spanning trees, on a set of points in convex position, and such that the time required between two canonical paths is \( O(n^2) \). Experimental results are also given in [3] for discussing the connectedness of geometric path graphs of points in general position. For more results related to planar paths and their generalization, called plane spanning trees, on a set of points in convex position and in general position, the reader can refer to [4, 6, 7].

In this note, we provide extreme cases of planar path transformation to show the following result:

**Theorem 1** For \( n \geq 4 \), the geometric path graph of a set of \( n \) points in convex position has diameter exactly \( 2n - 5 \) if \( n = 4 \), and \( 2n - 6 \) otherwise.

### 2 Preliminaries

For a set \( S \) of \( n \) points in convex position, we denote \( \text{ch}(S) \) as the boundary of the convex hull of \( S \). For any point \( p \in S \), the neighbors of \( p \) on \( \text{ch}(S) \) in clockwise direction and counter-clockwise direction are denoted by \( p^+ \) and \( p^- \), respectively. Let \( P = p_0 p_1 \cdots p_{n-1} \) be a planar path in \( \mathcal{P}(S) \). An edge \( p_ip_{i+1} \in P \) is said to be an arc if it lies on \( \text{ch}(S) \), and a chord otherwise. The quality of \( P \), denoted as \( \rho(P) \), is the number of arcs that exist in \( P \). An edge of \( \text{ch}(S) \) which does not appear in \( P \) is called a lacking arc of \( P \). An observation has been pointed out in [3, 7] that \( p_0p_1 \) and \( p_{n-2}p_{n-1} \) must always be arcs. For this reason, we call \( p_0p_1 \) and \( p_{n-2}p_{n-1} \) the terminal arcs of \( P \). Define \( \mathcal{P}_i(S) = \{ P \in \mathcal{P}(S) : \rho(P) = i \} \) for \( i = 2, 3, \ldots, n - 1 \). In particular, a planar path \( P \in \mathcal{P}_{n-1}(S) \) is called a canonical path and a planar path \( P \in \mathcal{P}_2(S) \) is called a zig-zag path. Obviously, all edges in a zig-zag path are chords except the two terminal arcs. For any two planar paths \( P, Q \in \mathcal{P}(S) \), the distance \( d(P, Q) \) is the least number of flips necessary to transform one path into the other. Clearly, if \( P \) and \( Q \) are two canonical paths, then \( d(P, Q) = 1 \).

Suppose that \( P = p_0p_1 \cdots p_{n-1} \) is a non-canonical path in \( \mathcal{P}(S) \). Let \( i > 1 \) be the smallest integer such that \( p_{i-1}p_i \) is a chord in \( P \). Clearly, \( p_0p_i \in \text{ch}(S) \); otherwise, the set \( \{ p_{i+1}, p_{i+2}, \ldots, p_{n-1} \} \) is separated into different sides of \( p_{i-1}p_i \) in the plane and thus \( P \) cannot form a planar path. Then, we can construct a path by removing edge \( p_{i-1}p_i \) from and adding edge \( p_0p_i \) to \( P \) such that the resulting path \( p_{i-1}p_{i-2} \cdots p_1p_0p_ip_{i+1} \cdots p_{n-1} \) forms another planar path in \( \mathcal{P}(S) \). This shows that there is at least a planar path \( Q \in \mathcal{P}(S) \) with \( \rho(Q) = \rho(P) + 1 \) such that \( Q \) can be obtained from \( P \) by using a flip. Since \( \rho(P) \geq 2 \), it further implies that any non-canonical path can be transformed into a canonical path in at most \( n - 3 \) flips. From this reasoning together with the fact that any two canonical paths have distance 1, an upper bound of the diameter of \( G_n \) is achieved in [3] as follows.

**Lemma 2.1** [3] Let \( S \) be a set of \( n \geq 4 \) points in convex position. For any two planar paths \( P, Q \in \mathcal{P}(S) \), \( d(P, Q) \leq 2n - 5 \).

### 3 Proof of the Main Result

Because we can inspect the graph in Figure 1 to see the correctness of Theorem 1 for \( n = 4 \), throughout the rest we assume that \( S \)
is a set of n points in convex position in the plane for $n \geq 5$.

**Lemma 3.1** For each $i = 2, 3, \ldots, n - 1$, $|\mathcal{P}_i(S)| = n\binom{n-3}{i-2}$.

**Proof.** Consider a path $P = p_0p_1 \cdots p_{n-1} \in \mathcal{P}_i(S)$. Since there are $n - 3$ edges of $P$ that can be either arcs or chords except the two terminal arcs and $\rho(P) = i$, the number of planar paths starting from $p_0$ with quality $i$ is $\binom{n-3}{i-2}$. Since we have $n$ different choices of $p_0$ in $S$, the result directly follows. \qed

From Lemma 3.1, we obtain another proof to show that the number of planar paths in $\mathcal{P}(S)$ is totally

$$\sum_{i=2}^{n-1} |\mathcal{P}_i(S)| = \sum_{i=2}^{n-1} n\binom{n-3}{i-2} = n2^{n-3}.$$ 

**Lemma 3.2** For any planar path $P \in \mathcal{P}_i(S)$ where $2 \leq i \leq n - 3$, there exist exactly two planar paths $Q_1, Q_2 \in \mathcal{P}_{i+2}(S)$ such that $d(P, Q_1) = d(P, Q_2) = 1$. Moreover, there exists a unique path $R \in \mathcal{P}_{i+2}(S)$ such that $d(Q_1, R) = d(Q_2, R) = 1$.

**Proof.** Consider a path $P = p_0p_1 \cdots p_{n-1} \in \mathcal{P}_i(S)$. Since $i \leq n - 3$ and $n \geq 5$, $P$ contains at least two chords. Let $s > 1$ be the smallest integer and $t < n - 2$ be the largest integer such that $p_{s-1}p_s$ and $p_{t-1}p_t$ are chords in $P$. Note that $s \leq t$ and clearly $p_0p_s, p_tp_{n-1} \in \text{ch}(S)$; otherwise, $\{p_{s+1}, p_{s+2}, \ldots, p_{n-1}\}$ (respectively, $\{p_0, p_1, \ldots, p_{t-1}\}$) is separated into different sides of $p_{s-1}p_s$ (respectively, $p_{t-1}p_t$) in the plane and thus $P$ is not a planar path. Applying a flip on $P$, we can find two planar paths in $\mathcal{P}_{i+1}(S)$ as follows:

$$Q_1 = p_{s-1}p_s \cdots p_{t-1}p_t p_0p_s p_{s+1} \cdots p_{n-1}$$

and

$$Q_2 = p_0p_1 \cdots p_{s-2}p_s p_{s+1} \cdots p_{t+1}.$$ 

Moreover, it is easy to verify that only $Q_1$ and $Q_2$ in the set $\mathcal{P}_{i+1}(S)$ can be obtained from $P$ by using a flip.

By the same reasoning, for $j = 1, 2$, there exist exactly two planar paths in $\mathcal{P}_{i+2}(S)$ for which $Q_j$ can transform into each of them via a flip. Since $p_{t}p_{t+1}$ is a chord in $Q_1$ and $p_{s-1}p_{s}$ is a chord in $Q_2$, we can further check that $R = p_s \cdots p_{s-2}p_{s-1}p_0p_s \cdots p_{t}p_{n-1}p_{n-2} \cdots p_{t+1}$ is the unique path in $\mathcal{P}_{i+2}(S)$ and such that $d(Q_1, R) = d(Q_2, R) = 1$. \qed

Let $P \in \mathcal{P}_i(S)$ with $2 \leq i \leq n - 2$ (i.e., $P$ is a non-canonical path). For each $j$ with $i \leq j \leq n - 1$, we denote $\mathcal{P}_{i,j}(P)$ as the set containing all planar paths in $\mathcal{P}_j(S)$ that can be obtained from $P$ by means of $j - i$ flips, i.e., $\mathcal{P}_{i,j}(P) = \{ Q \in \mathcal{P}_j(S) : d(P, Q) = j - i \}$. In particular, $\mathcal{P}_{i,i}(P) = \{ P \}$. Now, we compute the size of $\mathcal{P}_{i,j}(P)$ as follows.

**Lemma 3.3** For $2 \leq i \leq j \leq n - 1$, if $P \in \mathcal{P}_i(S)$ is a planar path, then $|\mathcal{P}_{i,j}(P)| = j - i + 1$.

**Proof.** We prove the lemma by induction on $j$. Clearly, $|\mathcal{P}_{i,i}(P)| = 1$, and by Lemma 3.2 we have $|\mathcal{P}_{i,i+1}(P)| = 2$. Thus the lemma holds for $j = i$ and $j = i + 1$. Assume that the lemma holds for $i \leq j < k$ with $i + 2 \leq k \leq n - 1$. We now show that the lemma holds for $j = k$. From the induction hypothesis, we have $|\mathcal{P}_{i,k-1}(P)| = k - i$ and $|\mathcal{P}_{i,k}(P)| = k - i + 1$. Also, by Lemma 3.2 we know that for any two planar paths $Q_1, Q_2 \in \mathcal{P}_{i,k-1}(P)$, if there exists a planar path $Q_0 \in \mathcal{P}_{i,k-2}(P)$ with $d(Q_0, Q_1) = d(Q_0, Q_2) = 1$, then we can find a unique path $R \in \mathcal{P}_{i,k}(P)$ such that both $Q_1$ and $Q_2$ can transform into $R$ using a flip. Since there are two ways to transform a planar path $Q \in \mathcal{P}_{i,k-1}(P)$ into two different paths of $\mathcal{P}_{i,k}(P)$ using a flip and we have known that $\mathcal{P}_{i,k-2}(P)$ contains $k - i - 1$ planar paths, the number of planar paths in $\mathcal{P}_{i,k}(P)$ is $2(k - i) - (k - i - 1) = k - i + 1$. \qed
For example, Figure 2 shows a partial structure of $G_7$ to illustrate Lemma 3.3. There are five canonical paths that can be transformed from the zig-zag path $P = 1273645$ via four flips. In general, for a zig-zag path $P = p_0p_1 \cdots p_{n-1} \in \mathcal{P}_2(S)$, the set $\mathcal{P}_{2,n-1}(P)$ contains $n - 2$ canonical paths which can be obtained from $P$ by using $n - 3$ flips. More precisely, $\mathcal{P}_{2,n-1}(P) = \{ ch(S) \setminus p_ip_{i+2} : 0 \leq i \leq n - 3 \}$, where $p_ip_{i+2}$ is a lacking arc. In addition, for every planar path $Q \in \mathcal{P}_{2,1}(P)$ where $2 \leq i \leq n - 2$, all chords present in $Q$ come from the set $\{ p_ip_{i+1} : 1 \leq i \leq n - 3 \}$ and are consecutive.

**Lemma 3.4** For any two zig-zag paths $P_1, P_2 \in \mathcal{P}_2(S)$, there exists at least a canonical path $Q \in \mathcal{P}_{n-1}(S)$ such that $d(P_1, Q) = d(P_2, Q) = n - 3$.

**Proof.** By Lemma 3.3 $|\mathcal{P}_{2,n-1}(P_1)| = |\mathcal{P}_{2,n-1}(P_2)| = n - 2$. Since $n \geq 5$, $\mathcal{P}_{2,n-1}(P_1) \cap \mathcal{P}_{2,n-1}(P_2) \neq \emptyset$. □

According to Lemma 3.4, we reduce the upper bound of the diameter of a geometric path graph $G_n$ for $n \geq 5$ as follows.

**Corollary 3.5** Let $S$ be a set of $n \geq 5$ points in convex position. For any two planar paths $P, Q \in \mathcal{P}(S)$, $d(P, Q) \leq 2n - 6$.

In what follows, we will show that there exist two zig-zag paths in $\mathcal{P}_2(S)$ such that their distance attains the bound. It is easy to see that, for describing a zig-zag path, if a terminal arc of the path is designated, then all succeeding edges are determined immediately. This suggests that we can depict a zig-zag path $P$ using a specific configuration such as $P = (p_0, +)$ or $P = (p_0, -)$ as representation, where $p_0$ denotes the starting point of the path which follows a sign to indicate the direction on $ch(S)$ (e.g., plus for clockwise and minus for counter-clockwise). For example, Figure 3 shows two particular zig-zag paths $\hat{P} = (p_0, +)$ and $\hat{Q} = (p_0^-, +)$. Clearly, $\hat{P}$ and
\[ \hat{Q} \text{ have no edges in common. This observation further implies the following.} \]

![Two zig-zag paths \( \hat{P} = (p_0, +) \) and \( \hat{Q} = (p_0^-, +) \).](image)

**Lemma 3.6** Let \( \hat{P} = (p_0, +) \) and \( \hat{Q} = (p_0^-, +) \) be two zig-zag paths. If \( P \in \mathcal{P}_{2,n-2}(\hat{P}) \) and \( Q \in \mathcal{P}_{2,n-2}(\hat{Q}) \), then \( P \cap ch(S) \neq Q \cap ch(S) \).

**Proof.** Let \( \hat{P} = p_0p_1 \cdots p_{n-1} \) be a zig-zag path and \( P \in \mathcal{P}_{2,n-2}(\hat{P}) \). Suppose that \( e = p_ip_{i+1}, 1 \leq i \leq n-3 \), is the sole chord present in \( P \). Clearly, the set of lacking arcs of \( P \) is \( \{p_ip_i^+, p_{i+1}p_{i+1}^+\} \) if \( i \) is an odd integer, and \( \{p_ip_i^-, p_{i+1}p_{i+1}^-\} \) otherwise. That is, each lacking arc of \( P \) contains an end point of \( e \) as its end point. Since \( \hat{P} \) and \( \hat{Q} \) have no chords in common, it implies that the chord present in \( P \) and the chord present in \( Q \) are different. Moreover, \( P \cap ch(S) \neq Q \cap ch(S) \) follows from the fact that \( P \) and \( Q \) have distinct lacking arc sets.

**Lemma 3.7** Let \( P, Q \in \mathcal{P}_{n-2}(S) \) be two planar paths. Then, \( d(P, Q) = 1 \) if and only if \( P \cap ch(S) = Q \cap ch(S) \).

**Proof.** Let \( P = p_0p_1 \cdots p_{n-1} \in \mathcal{P}_{n-2}(S) \) with the chord \( p_ip_{i+1} \) where \( 1 \leq i \leq n-3 \). Clearly, there is only one way to flip an edge of \( P \) and such that the resulting planar path is still in \( \mathcal{P}_{n-2}(S) \). Thus, \( d(P, Q) = 1 \) and \( Q \in \mathcal{P}_{n-2}(S) \) if and only if \( Q = p_ip_{i-1} \cdots p_0p_{n-1}p_{n-2} \cdots p_{i+1} \) with the chord \( p_0p_{n-1} \). The latter means that \( P \) and \( Q \) have the same arcs.

**Lemma 3.8** Let \( \hat{P} = (p_0, +) \) and \( \hat{Q} = (p_0^-, +) \) be two zig-zag paths. If \( P \in \mathcal{P}_{2,i}(\hat{P}) \) and \( Q \in \mathcal{P}_{2,i}(\hat{Q}) \) are any two planar paths for \( 2 \leq i \leq n-2 \), then \( d(P, Q) \geq 2(n-i-1) \).

**Proof.** The proof is by induction on \( i \). Clearly, if \( i = n-2 \), then \( d(P, Q) \geq 2 \) follows from Lemma 3.6 and Lemma 3.7. Suppose that the lemma holds for \( k < i \leq n-2 \) with \( 2 \leq k \leq n-3 \). We now show that the lemma holds for \( i = k \). Let \( P' \in \mathcal{P}_{2,k+1}(\hat{P}) \) with \( d(P, P') = 1 \) and \( Q' \in \mathcal{P}_{2,k+1}(\hat{Q}) \) with \( d(Q, Q') = 1 \) be any two planar paths. By the induction hypothesis, \( d(P', Q') \geq 2(n-k-2) \). We claim that there is no transformation using less than \( 2(n-k-1) \) flips between \( P \) and \( Q \). That is, \( d(P, Q) = d(P', Q') + 2 \geq 2(n-k-1) \).

Note that \( P \) and \( Q \) contain exactly \( k \) arcs and \( n-1-k \) chords, respectively. Further, \( P \) and \( Q \) have no chords in common. Let \( N(P) \) be the set of planar paths which can be obtained from \( P \) using a flip. Since \( k \leq n-3 \), \( P \) contains at least two chords. This implies that there is no path of \( N(P) \) that has exactly \( k \) arcs. Thus, every planar path of \( N(P) \) is either in \( \mathcal{P}_{2,k+1}(P) \) or in \( \mathcal{P}_{k-1}(S) \). To complete the proof, we only need to consider a transformation \( T \) that converts \( P \) into \( Q \) and passes through an intermediary path in \( N(P) \cap \mathcal{P}_{k-1}(S) \). Let \( R \in N(P) \cap \mathcal{P}_{k-1}(S) \) be any planar path obtained from \( P \) using a flip. Clearly, \( R \) contains \( n-k \) chords and at least \( n-1-k \) of them are absent in \( Q \). By contrast, at least \( n-2-k \) chords of \( Q \) are absent in \( R \). Since no any chord of \( R \) can be directly replaced with a chord of \( Q \) by a flip, \( d(R, Q) \geq 2n-3-2k \). This further implies that the number of flips in \( T \) is at least \( (2n-3-2k)+1 = 2(n-k-1) \).
Corollary 3.9 Let \( S \) be a set of \( n \geq 5 \) points in convex position. If \( \hat{P} = (p_0, +) \) and \( \hat{Q} = (p_0^-, +) \) are two zig-zag paths in \( \mathcal{P}_2(S) \), then \( d(\hat{P}, \hat{Q}) \geq 2n - 6 \).

As a consequence, the correctness of Theorem 1 directly follows from Corollary 3.5 and Corollary 3.9.

4 Conclusion

For a given set \( S \) of points in the plane, a plane spanning tree (or crossing-free spanning tree) of \( S \) is a tree \( T \) whose edge connected all points of \( S \) with straight line segments and such that no two edges of \( T \) cross [1,2,4,6]. A geometric tree graph is the graph consisting of all plane spanning trees as its vertices and two vertices are adjacent if one of the corresponding trees can be obtained from the other by means of a flip. Geometric tree graphs were studied by Avis and Fukuda [4] for enumerating plane spanning trees. A result in that paper showed that every geometric tree graph is connected and has diameter at most \( 2n - 4 \) for any set of \( n \) points in general position. García et al. [5] showed that the number of plane spanning trees is minimum when points are in the convex case. Meanwhile, Hernando et al. [6] also concentrated their study on the convex case and acquired several results including minimum degree, maximum degree, radius, center, connectivity, and hamiltonicity of geometric tree graphs. In particular, a lower bound of diameter at about \( \lceil 3n/2 \rceil - 5 \) was derived. Currently, we attempt to reduce the gap between the upper bound and the lower bound of diameter of geometric tree graphs.

References