A Numerical Algorithm for Nonlinear $L_2$-gain Optimal Control With Application to Vehicle Yaw Stability Control*

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Abstract—This paper is concerned with $L_2$-gain optimal control approach for coordinating the active front steering and differential braking to improve vehicle yaw stability and cornering control. The vehicle dynamics with respect to the tire slip angles is formulated and disturbances are added on the front and rear cornering forces characteristics modelling, for instance, variability on road friction. The mathematical model results in input-affine nonlinear system. A numerical algorithm based on conjugate gradient method to solve $L_2$-gain optimal control problem is presented. The proposed algorithm, which has backward-in-time structure, directly finds the feedback control and the “worst case” disturbance variables. Simulations of the controller in closed-loop with the nonlinear vehicle model are shown and discussed.

I. INTRODUCTION

Over the past two decades, there has been tremendous progress in the development of various control strategies for vehicle dynamics stabilization. In [1] and [2] linear $H_\infty$ controllers that stabilize the vehicle against uncertainty have been proposed. The $\mu$-synthesis has been applied to the linearized vehicle model in [3], obtaining a controller which provides robust stability against perturbations generated in various driving conditions. In [4] a fuzzy-logic controller is proposed to improve vehicle handling and stability when non-linearities are present in the model. A dynamic control allocation approach for vehicle yaw stabilization scheme has been presented in [5]. Also, different model predictive control (MPC) strategies have been explored in the vehicle dynamics context: time-varying MPC [6], [7], hybrid MPC [8], and switched MPC [9].

However, to the knowledge of the authors, the problem of a nonlinear $L_2$-gain vehicle stability control has not been investigated yet.

In this paper, the nonlinear $L_2$-gain optimal control problem of stabilizing the vehicle dynamics using differential braking and active front steering is considered. We formulate the vehicle dynamics with respect to the tire slip angles. A simplified “magic formula” for the tire model is used. The disturbance (uncertainty) is added on the front and rear cornering forces characteristics. This is a reasonable disturbance, which models, instance, a road friction different of what expected, due to the presence of ice/snow/gravel on the road.

It is well known [10] that the $L_2$-gain optimal control problem requires solving a Hamilton-Jacobi-Issaacs equation (HJIE). The analytic solution of HJIE is difficult or impossible to find in most cases. In [11] the HJIE for systems with input constraints is derived. Authors have introduced a two-player policy iteration scheme that results in a framework that allows the use of neural networks to approximate optimal policies and value functions. In [12] an application of neural networks to find a closed-form representation of the feedback strategies and the value function that solves the associated HJIE is presented.

In our approach, the nonlinear $L_2$-gain optimal control problem is transformed into a nonlinear finite-horizon optimal state feedback control problem with min-max cost. In contrast to the approaches based on neural networks for an approximate solution of HJIE [11], [12], in this paper the tuning of basis functions weights is based on the direct minimization of the performance criterion with respect to the control input, with simultaneous maximization of the same performance criterion with respect to the disturbance. A conjugate gradient approach is used for minimization/maximization of the performance criterion, while the performance criterion gradients are calculated exactly using chain rule for ordered derivatives. Since the control, disturbance and state variables are treated as dependent variables (coupled via plant equations), the final algorithm has a backward-in-time structure similar to the back-propagation-through-time (BPTT) [13] algorithm.

The algorithm presented in this paper is an extension of the recent work in [14], [15] toward finite-horizon $L_2$-gain optimal state feedback control. In [14], and [15] a conjugate gradient-based BPTT-like algorithm for optimal open-loop control of nonlinear multivariable systems with control and state constraints is presented. The algorithm performance is illustrated on a realistic high-dimensional vehicle dynamics model. The optimization results have demonstrated favourable features of the algorithm in terms of accuracy, robust numerical stability, and relatively fast execution.

As a further algorithmic improvement, in this work Jacobian matrices are calculated using automatic differentiation (AD). Application of AD comparing with numerical differentiation [15] provide significant reduction of the algorithm computational time.

The rest of the paper is organized as follows. In Section

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II by formulating the vehicle dynamics using tire slip angles
and steering angle as the states and by considering a "magic
formula" approximation of the tire forces with respect to the
tire slip angles, the mathematical model results in an input-
affine nonlinear system. In Section III, the state feedback
nonlinear $L_2$-gain control problem is transformed into a
feedback min-max optimal control problem. We formulate
the feedback as a linear combination of basis function.
The backward-in-time min-max control algorithm is derived.
The effectiveness of the proposed algorithm is illustrated
on a nonlinear benchmark example with analytic solution.
In Section IV the proposed algorithm is implemented to
coordinate active front steering and differential braking in
a driver-assist steering system that aims at stabilizing the
vehicle. Analysis of the simulation results are given. Finally,
Section V concludes the paper.

**Notation:** The notation used is fairly standard. Matrices are
represented in bold upper case. All vectors are intended
as column vectors and represented in bold lower case. Scalars
are represented in italic lower case. The symbol $^T$ denotes
transposition, $I$ is identity matrix and $0$ is null matrix of
appropriate dimensions. col$(\cdot)$ denotes the operator which
puts its arguments into a single column vector and row$(\cdot)$
denotes the operator which puts its arguments into a single
row vector. We use diag$(\cdot)$ to denote a diagonal matrix with
specified entries on the main diagonal and zeros elsewhere.
The derivative of a vector size $m$ with respect to a vector of
size $n$ is a matrix of size $n \times m$. This also means that
the derivative of a scalar with respect to a vector is a column
vector. Definitions of various matrix/vector derivatives can be
found in [16]. The operator $\| \cdot \|$ denotes the Euclidean
norm. We avoid to explicitly show the dependence of the
variables from the time when not needed.

II. VEHICLE YAW STABILITY CONTROL

A. Vehicle steering model

This subsection describes the vehicle model used for
control design and simulations. We consider normal "on-
road" driving maneuvers, where the vehicle dynamics can be
conveniently approximated by the bicycle model [17] shown
in Fig. 1. The approximated model has the advantage of
reduced complexity over a four-wheel vehicle model, while
still being able to capture the relevant dynamics.

For small steering angles, $\cos \delta \simeq 1$, the vehicle behaviour
can be described by the differential equations [9]

$$
\begin{align*}
\dot{\alpha}_f &= \frac{F_f + F_r}{m v_x} - \frac{v_x (\alpha_f - \alpha_r + \delta)}{a + b} + \frac{a (a F_f - F_r + Y)}{v_x I_x} - \varphi, \\
\dot{\alpha}_r &= \frac{F_f + F_r}{m v_x} - \frac{v_x (\alpha_f - \alpha_r + \delta)}{a + b} - \frac{b (a F_f - F_r + Y)}{v_x I_x}, \\
\delta &= \varphi,
\end{align*}
$$

(1)

where $\alpha_f$ [rad] and $\alpha_r$ [rad] are the tire side-slip angles
at the front and at the rear tires, respectively, $Y$ [Nm] is the
yaw moment obtained by differential braking, $\varphi = \delta$ [rad/s]
is the steering angle rate, $I_x$ [kgm$^2$] is the vehicle inertia
along the $z$-axis, $m$ [kg] is the vehicle mass, $a$ [m] and
$b$ [m] are the distances of the front and rear wheel axles
from the vehicle center of mass, respectively, $v_x$ [m/s] is the
longitudinal velocity at the wheels equal to the one at the
center of mass.

The front and rear tire forces $F_f$ [N], $F_r$ [N], respectively,
are nonlinear functions of tire slip angles $\alpha_f$ and $\alpha_r$. In this
paper the tire model is based on a simplified form [18] of the
"magic formula" [19], i.e.,

$$
F_j = \mu D_j \sin \left[ C_j \arctan \left( -B_j \alpha_j \right) \right],
$$

(2)

where $j = f$ for the front tires, and $j = r$ for the rear
tires, $\mu$ is the tire friction coefficient, $B_j$, $C_j$ and $D_j$ are tire
model parameters. The numerical values of these parameters
are given in Section IV. Fig. 2 depicts front and rear tire
forces versus side-slip angles for fixed values of the friction
coefficients.

Usually, the major element of uncertainty in the vehicle
dynamics is the road friction coefficient $\mu$. We assume its
value is within a certain known interval. The variations in
the coefficient $\mu$ can be represented by $\mu = \mu(1+p_\mu \Delta_\mu)$, where
$\mu$ is the so-called nominal value of $\mu$, $p_\mu$ is the maximum
relative uncertainty with $-1 \leq \Delta_\mu \leq 1$ being the relative
variation. If we choose the disturbance vector components
as $d_j = \Delta_j \bar{\mu} D_j \sin \left[ C_j \arctan \left( -B_j \alpha_j \right) \right]$, it follows that

$$
F_j = \bar{\mu} D_j \sin \left[ C_j \arctan \left( -B_j \alpha_j \right) \right] + p_\mu d_j.
$$

(3)
B. Control problem formulation

Let \( x = [x_1 \ x_2 \ x_3]^T = [\alpha_f \ \alpha_r \ \delta]^T, \ u = [u_1 \ u_2]^T = [Y \ \phi]^T \) and \( d = [d_1 \ d_2]^T = [d_f \ d_r]^T \) be the state, the control input, and the disturbance input of the vehicle dynamics, respectively. With the previously defined vectors, the vehicle model (1), (3) can be easily transformed into a control-oriented affine nonlinear dynamical system.

The control objective is to find control inputs \( Y \) (vehicle’s yaw moment) and \( \varphi \) (steering angle rate) and to determine the “worst case” disturbance (with respect to the friction coefficient) to avoid the vehicle dynamics (1) to be unstable.

The limits on the braking torques induce constraints on the achievable yaw moment, that, for the maneuvers of interest, are enforced by

\[
-Y_{max} \leq Y \leq Y_{max}. \tag{4}
\]

In addition, we consider constraints on the steering angle rate

\[
-\varphi_{max} \leq \varphi \leq \varphi_{max}. \tag{5}
\]

In order to achieve desired behavior, the following cost function is defined

\[
J = \int_0^{t_f} \left( ||z||^2 - \gamma^2 ||d||^2 + \sum_{k=1}^{4} K_k g_k^2 H^{-}(g_k) \right) dt, \tag{6}
\]

where \( ||z||^2 = x^T Q_x x + u^T Q_u u \), and \( g_k \) are the components of the vector function consisting of control inequality constraints. \( Q_x \) and \( Q_u \) are positive definite weight matrices.

\( H^{-}(\cdot) \) is the Heaviside step function defined as \( H^{-}(\cdot) = 0 \) if \( \cdot \geq 0 \), and \( H^{-}(\cdot) = 1 \) if \( \cdot < 0 \). Note that although the Heaviside step function \( H^{-}(\cdot) \) is not continuous, the penalty terms of the form \( (\cdot)^2 H^{-}(\cdot) \) in equation (6) are continuously differentiable functions. The penalty function coefficients \( K_k \) should be chosen sufficiently large to provide accurate constraints satisfaction.

III. NONLINEAR \( L_2 \)-GAIN OPTIMAL CONTROL DESIGN

Consider the affine nonlinear dynamical system of the form,

\[
x(t) = f(x) + G_1(x)u(t) + G_2(x)d(t), \quad x(0) = x_0,
\]

\[
z(t) = \begin{bmatrix} g_3(x) \\ u(t) \end{bmatrix}, \quad f(0) = 0, \quad g_3(0) = 0, \tag{7}
\]

where \( x \in \mathbb{R}^{n_o} \) is the state vector, \( u \in \mathbb{R}^{n_u} \) is the control input, \( d \in \mathbb{R}^{n_d} \) is the vector representing internal/external disturbance, \( z \in \mathbb{R}^{n_z} \) is the to-be-controlled output or penalty variable. The functions \( f(\cdot), G_1(\cdot), G_2(\cdot), g_3(\cdot) \) are smooth functions of \( x \). It is assumed that \( d \in \mathcal{L}_2[0, t_f], \) \( t_f \geq 0 \), where \( \mathcal{L}_2[0, t_f] \) denotes the standard Lebesgue space of vector valued square integrable functions over \( [0, t_f] \). Note that, the vehicle model (1), (3), is easily formulated as (7).

The objective is to determine a state-feedback controller \( u(x) \), for the case when all the states of the system are available, and determine the “worst case” disturbance internal/external variables \( d(x) \), such that the finite \( L_2 \)-gain from \( d \) to \( z \) is less than or equal to some positive number \( \gamma > 0 \). In other words, for every initial condition \( x(0) = x_0 \),

\[
\int_0^{t_f} ||z||^2 dt \leq \gamma^2 \int_0^{t_f} ||d||^2 dt + J(x_0). \tag{8}
\]

The original idea behind this approach is to formulate the \( L_2 \)-gain optimal control problem as a differential game in which \( u \) and \( d \) are two opposing players [20]. It is well known [21] that this problem is equivalent to the solving the min-max optimization problem

\[
J^* = \min_{u} \max_{d} \left\{ \int_0^{t_f} (||z||^2 - \gamma^2 ||d||^2) dt \right\} \tag{9}
\]

subject to (7).

Problem (9) is solved by the feedback [10]

\[
u^*(x) = -G_1^T(x) \frac{\partial V}{\partial x}, \quad d^*(x) = \frac{1}{\gamma^2} G_2^T(x) \frac{\partial V}{\partial x}, \tag{10}
\]

where \( V \) is a smooth function of the corresponding HJIIE with \( V(0) = 0 \).

Lemma 1: If (i) the nonlinear system (7) is asymptotically stable with \( d = 0 \) and \( u = u^* \), and (ii) has \( \mathcal{L}_2 \)-gain less than \( \gamma \) when \( d \neq 0 \), and (iii) the cost function (9) is smooth, then the closed-loop dynamics are asymptotically stable.

Proof: See [10].

A. Algorithm derivation

In order to numerically solve problem (9) subject to (7), we consider a special form of \( u \) and \( d \) from (10) as follows

\[
u(x) = \Theta(x) \pi(t), \quad d(x) = \Psi(x) \rho(t), \tag{11}
\]

where

\[
\Theta(x) \equiv \text{diag} \left( \theta_1(x), \ldots, \theta_{n_o}(x) \right),
\]

\[
\Psi(x) \equiv \text{diag} \left( \psi_1(x), \ldots, \psi_{n_a}(x) \right),
\]

with

\[
\theta_i(x) \equiv \text{row} \left( \theta_i^1(x), \ldots, \theta_i^{n_d}(x) \right),
\]

\[
\psi_i(x) \equiv \text{row} \left( \psi_i^1(x), \ldots, \psi_i^{n_d}(x) \right),
\]

are vectors of basis functions on a compact set \( \Omega \subset \mathbb{R}^{n_o} \) with \( \theta_i^j(x) \in C^1(\Omega), \psi_i^j(x) \in C^1(\Omega), \) and \( \theta_i^j(0) = 0, \psi_i^j(0) = 0 \). Furthermore,

\[
\pi(t) \equiv \col \left( p^1(t), \ldots, p^{n_o}(t) \right),
\]

\[
\rho(t) \equiv \col \left( r^1(t), \ldots, r^{n_a}(t) \right),
\]

are the vectors of basis functions time-varying weights.

Weierstrass’s theorem [22] guarantees that any continuous function on a bounded domain in \( \mathbb{R}^{n_o} \) can be approximated by a complete independent basis set. Standard usage of Weierstrass’s approximation theorem exploits polynomial basis functions. Non-polynomial basis sets have been considered in [23], where it is shown that linear combination
of basis functions with time-varying weights can be used to uniformly approximate continuous time-varying functions.

Hence, with control and disturbance variables from (11), the final min-max optimization problem is

\[
J^* = \min_{\pi} \max_{\rho} \left\{ \int_0^{t_f} \left( \|g_3\|^2 + w^T Q(x) w \right) dt \right\},
\]  

subject to

\[
\dot{x}(t) = f(x) + \Gamma(x) w(t), \quad x(0) = x_0,
\]

where

\[
w(t) = \left[ \pi^T(t) \quad \rho^T(t) \right]^T,
\]

\[
\Gamma(x) = \begin{bmatrix} G_1(x) \Theta(x) & G_2(x) \Psi(x) \end{bmatrix},
\]

\[
Q(x) = \begin{bmatrix} \Theta^T(x) \Theta(x) & 0 \\ 0 & -\gamma^2 \Psi^T(x) \Psi(x) \end{bmatrix}.
\]

1) Time discretization: To compute the numerical approximation of the nonlinear \( L_2 \)-gain optimal control problem, we discretize the system dynamics (13) and the performance criterion (12) basing on the explicit Adams method.

Assume that the time interval \([0, t_f]\) is divided into \(N - 1\) sub-intervals of equal length. Then, the time grid consists of points \(t_i = i\tau\) for \(i = 0, 1, 2, \ldots, N-1\), where \(\tau = t_f/N\) is the time step length.

The discrete-time form of (13) is

\[
x(i+1) = \phi_d(x(i), w(i)), \quad x(0) = x_0,
\]

where \(w(i)\) denotes the weights sequence over the intervals 0, 1, 2, \ldots, \(N-1\), respectively, while

\[
\phi_d(x(i), w(i)) = x(i) + \tau \sum_{j=1}^{k} a_j \phi(i-j+1),
\]

is the \(k\)-th order Adams approximation of the continuous-time state equation for \(i = k-1, k, k+1, \ldots\), and initial conditions \(x(0) = x_0, x(1) = x_1, \ldots, x(k-1) = x_{k-1}\), where \(\phi(i) = f(x(i)) + \Gamma(x(i)) w(i)\). \(a_j\) are constant coefficients (for their numerical values see [24]).

The explicit Adams method (15) is a \(k\)-th order vector difference equation, which can be conveniently transformed into the following discrete-time state-space form

\[
\hat{x}(i+1) = \hat{\phi}(\hat{x}(i), w(i)), \quad \hat{x}(0) = \hat{x}_0,
\]

where \(\hat{x}(i)\) is the extended \((n_a = n_0 \cdot k)\)-dimensional state vector

\[
\hat{x}(i) = [x_1(i) \quad x_2(i) \ldots x_{na-1}(i) \quad x_{na}(i)]^T,
\]

and

\[
\hat{\phi} = [x_1(i) + \tau a_1 \phi_1(i) + \tau x_{na+1}(i) \ldots a_k \phi_{na}(i)]^T.
\]

Lemma 2: Adams method (16) is convergent if and only if \(\hat{x}\) is stable and consistent.

Proof: See [24, p. 392].

The Adams method of the \(k\)-th order, as a multistep method, requires the knowledge of \(k\) initial conditions.

In this work, to determine these initial conditions the fourth-order Runge-Kutta method [24] is used.

The discrete-time form of the performance criterion is given by

\[
J(x_0) = \tau \sum_{i=0}^{N-1} F(\hat{x}(i), w(i)),
\]

where \(F(\cdot)\) is the sub-integral function of (12) in the \(i\)-th sampling interval.

2) Conjugate gradient algorithm: The numerical algorithm for tuning the \(\pi\) weights is based on the direct minimization of the performance criterion with simultaneous tuning of the \(\rho\) weights through the maximization of the same performance criterion.

In this work the optimization is performed by a conjugate gradient descent/ascent algorithm in the following form

\[
s(l+1)(i) = -g(l+1)(i) + \beta(l)s(l)(i),
\]

\[
w(l+1)(i) = w(l)(i) + \eta(l)s(l)(i),
\]

where

\[
g(i) = \left( \frac{\partial J}{\partial w(i)} \right)^T = \left[ \left( \frac{\partial J}{\partial \pi(i)} \right)^T \left( -\frac{\partial J}{\partial \rho(i)} \right)^T \right]^T,
\]

for \(i = 0, 1, 2, \ldots, N-1\) and \(l = 1, 2, \ldots, M\). \(N\) is the number of time instants, \(M\) is the number of gradient algorithm iterations, and \(s\) is the search direction vector.

The maximization of the performance criterion with respect to \(\rho\) is obtained simply by changing the sign in front of the gradient of the cost function.

The standard method for computing \(\eta(l)\) is the line search algorithm which requires one-dimensional minimization of the performance criterion. This is a computationally expensive method which may require many evaluations of the performance criterion during one iteration of the gradient algorithm. Also, if the performance criterion is not appropriately scaled, the line search algorithm may exhibit poor convergence properties [15]. In order to avoid these issues, in this work we use the SuperSAB approach [25] which requires only the information of the gradient directions in two consecutive iterations of the gradient algorithm.

The algorithm is modified in terms of using a scalar convergence rate \(\eta(l)\) (as oppose to a matrix formulation), in order to avoid discontinuities in vector \(w\). The modified SuperSAB algorithm is given by

\[
\eta(l) = \begin{cases} 
  d^+ \eta(l-1) & \text{if} \quad g^T(l) g(l-1) > 0, \\
  d^- \eta(l-1) & \text{if} \quad g^T(l) g(l-1) < 0, \\
  \eta(l-1) & \text{if} \quad g^T(l) g(l-1) = 0,
\end{cases}
\]

where \(0 < d^- < 1 < d^+\) are dilatation coefficients (decreasing/increasing factors).

The scalar \(\beta(l)\) is determined by

\[
\beta(l) = \frac{\mu g^T(l+1) g(l+1) + [1-\mu] g^T(l+1) [g(l+1) - g(l)]}{\nu g^T(l) g(l) + [1-\nu] s^T(l) [g^T(l) - g(l)]},
\]

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where \( \mu \in [0, 1] \) and \( \nu \in [0, 1] \). If the scalars \( \mu \) and \( \nu \) take only their extreme values, 0 or 1, then four possible combinations are obtained: the Fletcher-Reeves method [26] for \( \mu = 1 \) and \( \nu = 1 \), the Polak-Ribiere method [27] for \( \mu = 0 \) and \( \nu = 1 \), the Hestenes-Stiefel method [28] for \( \mu = 0 \) and \( \nu = 0 \), the Dai-Yuan method [29] for \( \mu = 1 \) and \( \nu = 0 \).

It is important to say that, in order to ensure numerical stability of the algorithm, the parameter \( \beta^{(l)} \) is limited to \( \beta_{\text{max}} \). If the parameter \( \beta^{(l)} \) has a constant value, \( 0 < \beta^{(l)} < 1 \), then the conjugate gradient algorithm becomes equivalent to a standard gradient algorithm with momentum.

3) Gradient calculation: The gradient of the performance criterion (19) with respect to \( w \) in the \( l \)-th iteration of the gradient algorithm is given by

\[
\frac{\partial J}{\partial w(j)} = \tau \sum_{i=0}^{N-1} \frac{\partial F(i)}{\partial w(j)} = \tau \left( \frac{\partial F(j)}{\partial w(j)} + \sum_{i=j+1}^{N-1} \frac{\partial F(i)}{\partial w(j)} \right),
\]

for \( j = 0, 1, 2, \ldots, N-1 \), where (because of the causality principle) the terms with \( i < j \) are equal to zero.

The terms in the sum on the right-hand side of (24) depend on \( w(j) \) implicitly through \( \hat{x}(i) \) for \( i > j \), which gives

\[
\frac{\partial F(i)}{\partial w(j)} = \frac{\partial \hat{x}(i)}{\partial w(j)} \frac{\partial F(i)}{\partial \hat{x}(i)}.
\]  

(25)

From (16) it follows that

\[
\frac{\partial \hat{x}(i)}{\partial w(j)} = \frac{\partial \hat{x}(i)}{\partial w(j)} \frac{\partial \hat{F}(i)}{\partial \hat{x}(i)}.
\]  

for \( i = j + 2, \ldots, N-1 \). The above iterative expression is the chain rule for ordered derivations.

Next, let us denote the second term in the bracket of the right side of equation (24) by

\[
\sigma(j) = \sum_{i=j+1}^{N-1} \frac{\partial F(i)}{\partial w(j)}.
\]  

(27)

Then substituting (25) in (28)

\[
\sigma(N-2) = \frac{\partial \hat{x}(N-1)}{\partial w(N-2)} \frac{\partial F(N-1)}{\partial \hat{x}(N-1)},
\]

and substituting (26) in (29) we get

\[
\sigma(N-2) = \frac{\partial \hat{F}(N-2)}{\partial w(N-2)} \frac{\partial F(N-1)}{\partial \hat{x}(N-1)}.
\]  

(30)

Further, for \( j = N-3 \), \( i = N-1 \), \( N-2 \) it follows

\[
\sigma(N-3) = \frac{\partial F(N-2)}{\partial w(N-3)} + \frac{\partial F(N-1)}{\partial w(N-3)},
\]

(31)

then substituting (25) in (31)

\[
\sigma(N-3) = \frac{\partial \hat{x}(N-2)}{\partial w(N-3)} \frac{\partial F(N-2)}{\partial \hat{x}(N-2)} + \frac{\partial \hat{x}(N-1)}{\partial w(N-1)} \frac{\partial F(N-1)}{\partial \hat{x}(N-1)}
\]

(32)

and substituting (26) in (32) we obtain

\[
\sigma(N-3) = \frac{\partial \hat{F}(N-3)}{\partial w(N-3)} \left[ \frac{\partial F(N-2)}{\partial x(N-2)} + \frac{\partial F(N-1)}{\partial x(N-1)} \right] + \frac{\partial \hat{F}(N-2)}{\partial w(N-2)} \frac{\partial F(N-1)}{\partial \hat{x}(N-1)}.
\]

(33)

This procedure can be further continued as follows

\[
\sigma(N-4) = \frac{\partial \hat{F}(N-4)}{\partial w(N-4)} \left[ \frac{\partial F(N-3)}{\partial x(N-3)} + \frac{\partial F(N-2)}{\partial x(N-2)} \right] + \frac{\partial \hat{F}(N-3)}{\partial w(N-3)} \frac{\partial F(N-2)}{\partial \hat{x}(N-2)} \frac{\partial F(N-1)}{\partial \hat{x}(N-1)}.
\]

\[
\vdots
\]

\[
\sigma(j) = \frac{\partial \hat{F}(j)}{\partial w(j)} \omega(j),
\]

\[
\omega(j) = \frac{\partial \hat{F}(j+1)}{\partial x(j+1)} + \frac{\partial \hat{F}(j+1)}{\partial x(j+1)} \omega(j+1), \quad \omega(N-1) = 0.
\]

(34)

The final backward-in-time recursive algorithm for calculation of the gradient in (20) has the form

\[
\omega(N-1) = 0,
\]

\[
\omega(j) = \frac{\partial \hat{F}(j+1)}{\partial x(j+1)} + \frac{\partial \hat{F}(j+1)}{\partial x(j+1)} \omega(j+1),
\]

(35)

\[
\sigma(j) = \frac{\partial \hat{F}(j)}{\partial w(j)} \omega(j), \quad \frac{\partial F(j)}{\partial w(j)} = \tau \left( \frac{\partial F(j)}{\partial w(j)} + \sigma(j) \right),
\]

for \( j = N-2, N-3, \ldots, 0 \). Obviously, from equations (13) and (16), it follows that

\[
\frac{\partial \hat{F}(j)}{\partial w(j)} = \Gamma^{-1}(\hat{x}(j)),
\]

and from (12) and (19), it follows that

\[
\frac{\partial F(j)}{\partial w(j)} = 2 Q(x(j)) w(j).
\]

4) Jacobians calculation: The extended Jacobian matrix \( \partial \hat{F}(\cdot)/\partial \hat{x}(\cdot) \) or the Adams method can be expressed basing on (16) as function of the basic Jacobian matrix \( \partial \hat{F}(\cdot)/\partial \hat{x}(\cdot) \). Similarly, the gradient of the function in the summation in (19) with respect to the extended state vector is related to the basic gradient.

The basic Jacobian matrix \( \partial \hat{F}(\cdot)/\partial \hat{x}(\cdot) \) and gradient \( \partial \hat{F}(\cdot)/\partial \hat{x}(\cdot) \) can be calculated using AD. AD is now a widely used tool within scientific computing. A variety of tools exist for AD. In this work, TOMLAB/MAD [30] mathematical software is used.
B. Benchmark example with analytic solution

We show now by an example for which the analytical solution of the HJIE can be computed that the solution found by our numerical algorithm is close to the analytical solution.

Consider the second order nonlinear system [31]

\[
\begin{align*}
\dot{x}_1 &= -\frac{29x_1 + 87x_1x_2^2 - 2x_2 + 3x_2^2}{4} + u_1 + \frac{d}{2}, \\
\dot{x}_2 &= -\frac{x_1 + 3x_1x_2^2}{4} + 3u_2 + d,
\end{align*}
\]

(36)

with

\[
z = \begin{bmatrix} \sqrt{2} (2x_1 + 6x_1x_2^2) \\ \sqrt{2} (4x_2 + 6x_1^2 x_2) \end{bmatrix}.
\]

If \( \gamma = 1 \) then by analytically solving the corresponding HJIE, the feedback laws are

\[
\begin{align*}
u^*_1(x) &= -x_1 - 3x_1x_2^2, \\
u^*_2(x) &= -6x_2 - 9x_1^2 x_2, \\
d^*(x) &= \frac{1}{2} x_1 - 2x_2 + 3x_1^2 x_2 + \frac{3}{2} x_1 x_2^2.
\end{align*}
\]

(37)

In our numerical algorithm we choose the basis functions

\[
\begin{align*}
\theta^1(x) &= \theta^2(x) = \psi^1(x) = \\
&= \begin{bmatrix} x_2 \\ x_2^2 \\ x_1 \\ x_1 x_2 \\ x_1 x_2^2 \\ x_1^2 x_2 \\ x_1^2 x_2^2 \\ x_2^3 \\ x_1 x_2^3 \\ x_2^4 \end{bmatrix}.
\end{align*}
\]

(38)

The terminal time is \( t_f = 3 \) sec and the number of optimization time intervals is \( N = 3000 \) so that the sampling interval is \( \tau = 0.001 \) sec. The fourth-order Adams method is used, and the conjugate gradient Polak-Ribiere method is applied. The numerical values of the algorithm parameters in (22) and (23) are chosen as \( d^+ = 1.05 \), \( d^- = 0.95 \), \( \eta^{(0)} = 1.0 \), \( \beta_{max} = 1.0 \).

The simulation results for initial condition \( x_0 = [1 \ 0.5]^T \), and \( \pi^{(0)} = \rho^{(0)} = 0 \) as initial basis functions weights are shown on Fig. 3. As it can be seen from Fig. 3 the numerical solution has a small error compared to the analytical solution.

![Fig. 3. Simulation results for system (36).](image)

IV. VEHICLE YAW CONTROL APPLICATION

In the previous section we have developed an approach for solving the \( L_2 \)-gain optimal control problem for general affine nonlinear systems using a conjugate gradient based numerical algorithm with backward-in-time structure. In this section the application of proposed algorithm to the vehicle yaw stability control problem is presented.

The state feedback \( L_2 \)-gain optimal control strategy synthesized by proposed numerical algorithm for vehicle stability control was tested in simulation in closed loop with vehicle dynamics model based on (1), which has been validated in experimental vehicle tests in [9]. For the considered passenger vehicle, we have \( m = 2050 \) kg, \( I_z = 3344 \) kNm², \( a = 1.47 \) m, \( b = 1.43 \) m, \( v_s = 15 \) m/s.

The coefficients of the “magic formula” (2) are chosen as \( D_j = 0.5 \cdot m \cdot g \), where \( g = 9.81 \) m/s², \( C_f = 1.2 \), \( B_f = 8.5 \), \( C_r = 1.5 \) and \( B_r = 10.2 \). We consider the nominal value \( \bar{p} = 0.7 \). It is assumed that the road friction coefficient \( \mu \) can vary up to \( \pm 40\% \) its nominal value, i.e., \( p_\mu = 0.4 \).

The bounds of the actuators are set as \( Y_{max} = 1000 \) [Nm] and \( \varphi_{max} = 0.5 \) [rad/s].

Moreover, the control inputs basis functions are chosen as

\[
\begin{bmatrix}
Y \\
\varphi
\end{bmatrix} = \begin{bmatrix}
\theta^1(\alpha_f, \alpha_r) \\
0
\end{bmatrix} \begin{bmatrix}
\phi^1(t) \\
\phi^2(t)
\end{bmatrix},
\]

(39)

where

\[
\theta^1(\alpha_f, \alpha_r) = \begin{bmatrix} \alpha_f \ 
\alpha^2_f 
\alpha_f \alpha_r 
\alpha^2_f \alpha_r 
\alpha^2_f^2 \alpha_r 
\alpha^2_f \alpha^2_r 
\end{bmatrix},
\]

and the disturbance inputs basis functions are chosen as

\[
\begin{bmatrix}
d_f \\
d_r
\end{bmatrix} = \begin{bmatrix}
\psi^1(\alpha_f) \\
0
\end{bmatrix} \begin{bmatrix}
\phi^1(t) \\
\phi^2(t)
\end{bmatrix},
\]

(40)

where

\[
\psi^1(\alpha_f) = \bar{\beta}_\mu D_f \sin(C_f \arctan(B_f \alpha_f)), \quad \psi^2(\alpha_r) = \bar{\beta}_\mu D_r \sin(C_r \arctan(B_r \alpha_r)).
\]

The terminal time is \( t_f = 3 \) sec and the number of time intervals is \( N = 3000 \) so that the sampling interval is \( \tau = 0.001 \) sec. The conjugate gradient Polak-Ribiere method is used, and the number of iterations of the gradient algorithm is \( M = 1000 \). The numerical values of the algorithm parameters are chosen as \( d^+ = 1.15 \), \( d^- = 0.85 \), \( \eta^{(0)} = 10.0 \), \( \beta_{max} = 1.0 \). The initial basis functions weights are set to zero.

The weighting factors of the cost function (6) are chosen as follows. The state weighting matrix is \( Q_x = \text{diag} (1, 1, 10) \). The weights on inputs are \( Q_u = \text{diag} (10^{-6} , 1) \). The penalty function factors are \( K_1 = K_2 = 0.5 \) and \( K_3 = K_4 = 10 \), large enough to satisfy constraints (4) and (5). The scalar value \( \gamma^2 \) is set to 1.

An important point we want to mention here is that the components of control vector \( Y \) and \( \varphi \) are different in magnitude. This means that a unique convergence rate for all the control variables in the conjugate gradient algorithm could not be effective. Because of this we use a scaling factor for each component of control vector in conjugate gradient algorithm.
Simulation results are shown in Figures 4-8. It can be seen in Figure 4 that fast stabilization from the initial conditions \( x_0 = [0.15 \ 0.25 \ 0]^T \) is achieved. It can be observed from Figure 5 that the proposed controller yields control inputs \( |u| \leq u_{\text{max}} \). The basis function weights \( r^1_1 \) and \( r^2_1 \) of the disturbance inputs \( d_1 \) are considered as normalized uncertainty term \( \Delta_j \) of the vehicle dynamics so that \(-1 \leq r^1_1, r^2_1 \leq 1 \) as shown in Figure 6. In Figure 7 time histories of front and rear tire forces are shown.

Furthermore, in a second series of simulation tests we analyzed the robustness to disturbance variations for the case \( \mu = \{ \text{worst case}, 0.95, 0.45 \} \). In Figure 8 it can been seen that the proposed robust optimal controller provides a good degree of robustness, with very limited variability of the trajectories despite large variations in the parameters.

![Fig. 4. Time histories of front and rear tire slip angles (left) and steering angle (right).](image)

![Fig. 5. Time histories of yaw moment, steering angle (upper plots) and basis functions weights (lower plots).](image)

![Fig. 6. Time histories of disturbance inputs (upper plots) and basis functions weights (lower plots).](image)

![Fig. 7. Time histories of front and rear tire forces.](image)

![Fig. 8. Time histories of front and rear tire slip angles in the robustness simulations.](image)

V. CONCLUSION

In this paper an application of a numerical algorithm for \( \mathcal{L}_2 \)-gain vehicle yaw stability control problem has been presented. The proposed algorithm directly finds the control inputs (vehicle’s yaw moment and steering angle rate) in presence of uncertainty in vehicle dynamics (e.g., in the road friction coefficient) to avoid the vehicle dynamics to become unstable. The closed-loop dynamics response is evaluated in computer simulations, on a model validated in experimental tests in [9].

While the individual methods such as backward-in-time techniques, conjugate gradient optimization algorithms, Adams method for solving ODEs, and AD are known from the literature, in our approach they are integrated together to provide an effective, novel algorithm for numerical solution of the \( \mathcal{L}_2 \)-gain optimal control problems.

Comparison of the algorithm with other existing methods is a subject of ongoing work and future publications. Also, in future work the proposed algorithm for \( \mathcal{L}_2 \)-gain optimal control will be extended with a dynamic observer providing optimal output feedback control. Further improvements of the algorithm application in terms of using a more precise vehicle dynamics model will be considered as well. The
algorithm can be applied to higher-order systems with some increase in complexity.

REFERENCES


