On the tomographic reconstruction resolution from compressive holography

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Abstract: We report a closed analytical expression formulating the reconstruction guarantees for performing object tomography of a 3D object from its 2D recorded hologram.

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1. Introduction

Compressive sensing (CS) [1,2] has made a large impact in various research fields during the recent years. Compressive sensing theory provides an alternative to conventional signal sampling paradigm, according to which a relatively large amount of information is acquired and then most of it is discarded during the digital post processing stage. Instead, CS attempts to take the smallest number of measurements, while still guaranteeing exact signal reconstruction. Compressive sensing theory relies on the assumption that the signal we wish to acquire can be represented sparsely in some domain. Since natural images and scenes tend to have extremely sparse representations, 2D and 3D imaging are among the most beneficiary fields from the introduction of CS, specifically one of the fields that was greatly benefited from CS is digital holography (DH) [3].

In previous works, we have dealt with the reconstruction and performance guarantees for the case where a 2D object’s field was reconstructed from its subsampled 2D hologram [4,5]. Here we extend the analysis and provide, to the best of our knowledge, first reconstruction guarantees for the recovery of a 3D object tomography from its 2D recorded hologram, a framework which is also known as “compressive holography” [6]. The term compressive in this sense means that even conventional holography can be viewed as a compressive process since it projects 3D information into 2D recorded holographic measurements.

2. 3D to 2D forward model

An x-z axis schematic description of the framework is illustrated in Fig. 1. A monochromatic plane wave, with wavelength $\lambda$, shines a 3D object volume. We want to formulate the forward operator relating the 3D object volume to the 2D hologram. Therefore, we discretize the object to $N_{\text{object}} = N_x \times N_y \times N_z$ voxels, with voxel dimensions of $\Delta x \times \Delta y \times \Delta z$. The object length is given by $L = N_z \times \Delta z$. The 3D object wavefield can be recorded by standard holography methods on the CCD. In the numerical near field approximation, $\Delta x_{\text{CCD}} = \Delta x$, $\Delta y_{\text{CCD}} = \Delta y$ and the number of pixels is $N_{\text{holo}} = N_x \times N_y$.

The discrete forward model relating the 3D input object, $f$, to its 2D wavefield in the detector plane, $g$, can be expressed under Born approximation as follows:

$$g(k\Delta x, q\Delta y) = \sum_{j=1}^{N_y} \int_{-\Delta z}^{\Delta z} \mathcal{F}_{2D}^{-1} \left\{ e^{-j\Delta z\Delta k}\left[ e^{j(k\Delta x + q\Delta y)} + e^{j(k\Delta x - q\Delta y)} \right] \right\} e^{-j\frac{2\pi}{\lambda}} \mathcal{F}_{2D}\left[ f(p\Delta x, q\Delta y; r\Delta z) \right],$$

(1)
where $\mathcal{F}_{2D}$ is the discrete 2D Fourier transform. For convenience, we further assume that $\Delta x=\Delta y=\Delta$ and $N_x=N_y=N$, and therefore the discrete indices are $0 \leq p,q,m,n,k,l \leq N-1$, and $\Delta \nu_x = \Delta \nu_y = 1/(N \Delta)$. Standard numerical object reconstruction from the 2D hologram is often obtained by using Fresnel back-propagation. This reconstruction, which is obtained by focusing digitally on different object depth planes, may be distorted due to out of focus object points located in other object planes. This is well illustrated in Fig. 2.

These disturbances are the result of using an incomplete model of the system; following a 2D-2D model, linking the hologram to a single depth plane and ignoring other object planes. Clearly, applying reconstruction by any such 2D-2D model is subject to distortions if an object point disobeys the model, i.e., the object point is located in another depth plane. In order to avoid such distortions, a 3D-2D forward model relating all the voxels to the $N_{holog} = N \times N$ hologram points should be considered. As a possible solution, CS was suggested as a strategy to reconstruct 3-D objects from their 2-D recorded hologram. This principle was demonstrated in works such as [6-13], to name a few.

In order to comply with standard CS notation, we rewrite Eq. (1) as a matrix-vector multiplication, yielding:

$$
\mathbf{g} = \left[ \mathcal{F}_{2D}^{-1} e^{-j 2\pi \Delta x} \mathbf{Q}_{x,N} \mathcal{F}_{2D} \cdots \mathcal{F}_{2D}^{-1} e^{-j 2\pi \Delta x} \mathbf{Q}_{x,N} \mathcal{F}_{2D} \right] \begin{bmatrix} f_{x_1} & \cdots & f_{x_N} \end{bmatrix}^T = \Phi \mathbf{f}^T,
$$

where the matrix $\mathbf{Q}_{x,N}$ is an $N^2 \times N^2$ diagonal matrix with the appropriate quadratic phase elements along its diagonal. Compressive sensing then seeks the sparsest possible object which fulfils the measurements constraints as given in Eq. (2). This is usually done by solving an $\ell_1$-norm minimization problem, or using greedy algorithms [2].

3. **Reconstruction guarantees for 3D to 2D forward model**

The success of the CS theory stems from the mathematical reconstruction guarantees which determine when an inverse problem, which was classically considered to be ill-posed, has a unique solution, along with a supporting algorithmic framework. In order to get a successful recovery, the object should have a sparse representation in some transform domain. A signal that can be sparsely represented by $S$ non-zero terms is referred to as $S$-sparse. Typically, $S$ is much smaller than the nominal sampling rate of the 3D object ($N_{object} = N^3 \times N_z$, in our case). The number of object features which can be accurately reconstructed, $S$, is given by [2]:

$$
S \leq 0.5 \left( 1+1/\mu \right),
$$

Fig. 2. Illustration of the reconstruction artifacts obtained due to the model mismatch when reconstructing a 3D object from its 2D hologram. The input object is shown in (a), while the reconstruction from its 2D complex field amplitude is shown in (b). The distortions due to model are evident in the out of focus terms in each plane.
where $\mu$ is the coherence parameter. The coherence parameter determines the coherence between the sensing operator and the sparsifying operator. In the case where the sensing operator encodes $N^2 \times N_s$ measurement into $N^2$ measurements, the coherence parameter is theoretically bounded by: $\sqrt{(N_s - 1)/ (N^2 N_s - 1)} \leq \mu \leq 1$. The CS theory then tells us that the object can be accurately reconstructed, by evoking a convex optimization procedure [2].

Here, we report that we have recently evaluated the coherence parameter for the above described 3D-2D forward model, given by:

$$\mu \approx \frac{\Lambda^2}{\lambda \Delta z}$$

Equation (4) presents a closed analytic expression of the coherence parameter, from which we can infer the 3D object’s reconstruction guarantees, from its 2D hologram. Combining Eq. (4) with (3) shows that the number of object features which can be accurately reconstructed is given by:

$$S \leq 0.5 \left(1 + \lambda \Delta z / \Lambda^2 \right)$$

The result in Eq. (5) predicts that by increasing the wavelength or using coarser axial resolution, the number of reconstructed features that can be reconstructed accurately is increased. On the other hand, the number of reconstructed features is also inverse proportional to the pixel size, that is small pixels will yield better results. While the coherence parameter is based on a worst case analysis and is considered rather pessimistic, it gives an evaluation and a trend for actual performance results, regardless of the reconstruction method / algorithm being used.

To conclude, we have formulated the reconstruction guarantees for performing object tomography of a 3D object from its 2D recorded hologram. The result depends on the physical attributes of the object, CCD and illuminating wavelength. We believe that the result may be useful for all researches which combine compressive sensing with digital holography.

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4. References