Comment on: “Image thresholding using type II fuzzy sets”. Importance of this method

H. Bustince *, E. Barrenechea, M. Pagola, J. Fernandez, J. Sanz

Departamento de Automática y Computación, Universidad Pública de Navarra, Campus Arrosadia s/n, P.O. Box 31006, Pamplona, Spain

**A R T I C L E  I N F O**

Article history:
Received 4 March 2008
Received in revised form 30 March 2010
Accepted 3 April 2010

Keywords:
Type II fuzzy set
Interval-valued fuzzy set
Interval-valued fuzzy entropy
Fuzzy entropy
Image thresholding

**A B S T R A C T**

In this work we develop some reflections on the thresholding algorithm proposed by Tizhoosh in [16]. The purpose of these reflections is to complete the considerations published recently in [17,18] on said algorithm. We also prove that under certain constructions, Tizhoosh's algorithm makes it possible to obtain additional information from commonly used fuzzy algorithms.

© 2010 Elsevier Ltd. All rights reserved.

**1. Introduction**

From our point of view, the main contribution of Tizhoosh's paper (see [16]): “Image thresholding using type-II fuzzy sets” is the use of interval-valued fuzzy sets in the algorithm for image thresholding. These sets enable him to modelize the lack of knowledge of the expert when facing the task of constructing the membership function of the fuzzy set that best represents the image (see [2,9,10]).

We believe that this contribution [16] is important. Nevertheless we would like to note the following considerations:

- The author does not relate the concept of ultrafuzziness with the concept of IV entropy (see [3]) introduced 10 years before.
- In the title of the paper the concept of Type II fuzzy set is used (see [20]). However in the paper only interval-valued fuzzy sets, which are a particular case of Type II fuzzy sets (see [11,12]), are used.

However, these are minor details. In any case, we must thank Vlachos and Sergiadis for the observations on this paper that they use in order to relate the concepts presented by Tizhoosh with others existent in the literature.

With respect to the relation between interval-valued fuzzy sets and Atanassov’s intuitionistic fuzzy sets (see [1]), we consider that, although mathematically these concepts are equivalent, when they are applied to image processing in the sense proposed by Tizhoosh [17], their interpretation is totally different. The reason is the following: Tizhoosh uses interval-valued fuzzy sets to modelize the imprecision or lack of knowledge of the expert in the choice of the membership function associated with the image (see Fig. 1). However, when Atanassov's sets are used we speak of "membership function" and "non-membership function", but never of imprecision associated with the choice of each one of the functions (membership and non-membership). In the best of cases, Atanassov's intuitionistic index could be interpreted as a measure of imprecision in the selection of both functions at the same time (see [13]).

In addition to these considerations, in this paper we intend to prove that under certain constructions, the algorithm for image thresholding using interval-valued fuzzy sets, makes it possible to:

1. Select the best threshold from two thresholds obtained by means of the area algorithm (see [6]), which is a particular version the algorithm of Huang and Wang proposed in 1995 (see [10]);
2. Represent the pixels of the image for which two experts, who use the area algorithm, do not agree on their membership to the background or to the object.
2. Preliminaries

Definition 1. A fuzzy set \( \tilde{A} \) defined on a finite and non-empty \( U = \{u_1, \ldots, u_n\} \) is given by (see [19])

\[
\tilde{A} = \{(u_i, \mu_{\tilde{A}}(u_i)) | u_i \in U\}
\]

where \( \mu_{\tilde{A}} : U \rightarrow [0,1] \) is the membership function.

In this work we shall denote by \( FS(U) \) the set of all fuzzy sets defined on a finite referential \( U \).

In 1972 Deluca and Termini [8] formalized the properties of fuzzy entropy through the following axioms:

Definition 2. A real function \( E : FS(U) \rightarrow [0,1] \) is called an entropy on \( FS(U) \), if \( E \) has the following properties:

\[
\begin{align*}
(\text{E1}) & \quad E(\tilde{A}) = 0 \text{ if and only if } \tilde{A} \text{ is non-fuzzy;} \\
(\text{E2}) & \quad E(\tilde{A}) = 1 \text{ if and only if } \tilde{A} = \{(u_i, \mu_{\tilde{A}}(u_i) = 0.5) | u_i \in U\}; \\
(\text{E3}) & \quad E(\tilde{A}) \leq E(\tilde{B}) \text{ if } \tilde{A} \text{ refines } \tilde{B}; \text{ that is, } \mu_{\tilde{A}}(u_i) \leq \mu_{\tilde{B}}(u_i) \text{ when } \mu_{\tilde{B}}(u_i) \leq 0.5 \text{ and } \mu_{\tilde{B}}(u_i) \geq \mu_{\tilde{B}}(u_i) \text{ when } \mu_{\tilde{B}}(u_i) \geq 0.5; \\
(\text{E4}) & \quad E(\tilde{A}) = E(\tilde{A_N}), \text{ where } \tilde{A_N} = \{(u_i, 1 - \mu_{\tilde{A}}(u_i)) | u_i \in U\}.
\end{align*}
\]

Definition 3. Let \( L([0,1]) = \{[M_a, M_b]|(M_a, M_b) \in [0,1]^2 \text{ and } M_a \leq M_b\} \) be the set of all closed subintervals of the closed interval \([0,1]\). An interval-valued fuzzy set (IVFS) \( A \) on the universe \( U \) is defined as

\[
A = \{(u_i, M_a(u_i)) | u_i \in U\}
\]

where the function \( M_a : U \rightarrow L([0,1]) \) defines the degree of membership of an element \( u_i \in U \) to \( A \).

We denote by \( IVFS(U) \) the set of all interval-valued fuzzy sets on \( U \).

There are two different definitions in the literature of the concept of an interval-valued fuzzy entropy (IV entropy). The first, \( E_I \), was presented in 1996 in [3], and the second, \( E_I \), in 2001 in [15]. The difference between the two definitions lies in the fact that \( E_I \) is a measure of how far an IVFS is from a fuzzy set, whereas \( E_I \) is a measure of how far an IVFS is from a crisp set. Therefore, \( E_I \) is based on the ideas of Sambuc (see [14]) and \( E_I \) is based on the concept of fuzzy entropy. In this work we are only going to use \( E_I \).

Definition 4. A real function \( E_I : IVFS(U) \rightarrow \mathbb{R}^+ \) is called an entropy on \( IVFS(U) \) if \( E_I \) has the following properties:

\[
\begin{align*}
(\text{IF1}) & \quad E_I(A) = 0 \text{ if and only if } A \in FS(U); \\
(\text{IF2}) & \quad E_I(A) = 1 \text{ if and only if } M_a(u_i) = 0 \text{ and } M_b(u_i) = 1 \text{ for all } u_i \in U; \\
(\text{IF3}) & \quad E_I(A) = E_I(A_B) \text{ for all } A \in IVFS(U), \text{ where } A_B = \{(u_i, M_a = [1 - M_{\tilde{B}}(u_i), 1 - M_{\tilde{B}}(u_i)]) | u_i \in U\}; \\
(\text{IF4}) & \quad E_I(A) \leq E_I(B) \text{ if } A \subseteq B; \text{ that is, if } M_{\tilde{A}}(u_i) \leq M_{\tilde{B}}(u_i) \text{ and } M_{\tilde{B}}(u_i) \leq M_{\tilde{B}}(u_i) \text{ for all } u_i \in U, \text{ then } E_I(A) \geq E_I(B).
\end{align*}
\]

In [3], these entropies \( E_I \) are studied in depth and a theorem for their construction from functions \( f : [0,1] \rightarrow [0,1] \) is presented. It is also said there that the most commonly used expression is the one obtained when we take \( f(x) = x \) so that

\[
E_I(A) = \frac{1}{2} \sum_{i=1}^{n} M_a(u_i) - M_b(u_i)
\]

for all \( A \in IVFS(U) \). It is easy to see that (1) is the indetermination index introduced by Sambuc in 1975.

In [18] it is verified that the concept of measure of ultrafuzziness that Tizhoosh uses in his algorithm is the concept of interval-valued fuzzy entropy \( E_I \) presented in Definition 4. Moreover, the expression used in [16] to calculate the degree of ultrafuzziness coincides with Sambuc’s indetermination index; that is, with (1).

We must point out that for us, the best threshold is the threshold associated with the fuzzy set or with the interval-valued fuzzy set with lowest entropy (fuzzy or interval-valued depending on the case in study). Furthermore, in the algorithms and in the constructions that we present in the following section we are going to use the concept of automorphism, which is defined as follows (see [5]): A continuous, strictly increasing function \( \varphi : [0,1] \rightarrow [0,1] \) with boundary conditions \( \varphi(0) = 0, \varphi(1) = 1 \) is called an automorphism of the interval \([0,1] \subseteq \mathbb{R} \).

3. Three fuzzy algorithms for thresholding images

Within the framework of fuzzy sets theory the most popular algorithms for calculating the threshold associated with an image \( Q \), given on a scale of \( L \) levels of gray, are those that use the concept of fuzzy entropy \([2,9,10]\). In this sense one of the most commonly used is the following:

Algorithm 1.

(a) Assign \( L \) fuzzy sets \( \tilde{Q}_t \) to each image \( Q \). Each one is associated to a level of intensity \( t, (t = 0,1,\ldots,L-1) \), of the grayscale \( L \) used.

(b) Calculate the fuzzy entropy of each one of the \( L \) fuzzy sets \( \tilde{Q}_t \) associated with \( Q \).

(c) Take, the best threshold gray level \( t \), associated with the fuzzy set corresponding to the lowest entropy.

The reason why fuzzy entropy is used in Algorithm 1 is the following: Thresholding an image is a process of finding the intensity \( t \in [1,\ldots,L-1] \) such that by using one (zero) to represent the pixels with intensities greater than or equal to \( t \) and zero (one) for the rest, we obtain the binary image that best separates the background and the object. For this reason, from all the fuzzy sets constructed in Algorithm 1 (step (a)), we take the one that has the...
lowest fuzzy entropy; that is, the one whose membership function values are closest to zero or to one.

The main problem of Algorithm 1 is step (a). The question considered is: which is the best function we should take in order to assign the membership of each element/pixel of the image to the associated fuzzy set? [16]. This problem has led many authors to propose different versions of Algorithm 1. The area algorithm (see [6]), that we show below, is one of these versions. This version has been constructed using the relations between the information measures studied in [7].

Algorithm 2. Area algorithm
(a) Select two automorphisms \( \varphi_1 \) and \( \varphi_2 \).
(b) FOR \( t = 0 \) TO \( L − 1 \) DO
   \( b-1 \) Calculate: \( m_d(t) = \frac{\sum_{q=0}^{L-1} \varphi_1(q)}{\sum_{q=0}^{L-1} \varphi_2(q)} \) and \( m_u(t) = \frac{\sum_{q=L}^{L-1} \varphi_1(q)}{\sum_{q=L}^{L-1} \varphi_2(q)} \)
   \( b-2 \) Calculate the area:
   \[ \begin{align*}
   A(\tilde{Q}_d, \varphi_1, \varphi_2) &= \sum_{q=0}^{L-1} h(q)\varphi_1^{-1}(1 - 0.5(\varphi_2(q) - \varphi_2(m_u(t)))) \\
   &+ \sum_{q=L}^{L-1} h(q)\varphi_2^{-1}(1 - 0.5(\varphi_2(q) - \varphi_2(m_d(t))))
   \end{align*} \]
ENDFOR
(c) Take as the best threshold the value of \( t \) associated with the greatest value of \( A(\tilde{Q}_d, \varphi_1, \varphi_2) \).

Evidently, the area algorithm is simpler and more time efficient than Algorithm 1. Please note that \( A(\tilde{Q}_d, \varphi_1, \varphi_2) \) increases proportionally as \( q \) is closer to \( m_d(t) \) or to \( m_u(t) \). (\( h(q) \) is the number of pixels of the image with intensity \( q \).

The algorithm that we show next is the one put forward by Tizhoosh:

Algorithm 3. Tizhoosh’s algorithm
(a) Assign \( L \) fuzzy sets \( \tilde{Q}_d \) (with \( t \in \{0, \ldots, L−1\} \)) to each image \( Q \);
(b) Take \( x \in (1, \infty) \);
(c) Associate with each fuzzy set \( \tilde{Q}_d \) its corresponding IVFS \( \tilde{Q}_f \) constructed with the following method (see [4]):
\[ \tilde{Q}_d = \{(q, M_{\tilde{Q}_d}(q)) \mid q \in [0, \ldots, L−1]\} \]
given by \( M_{\tilde{Q}_d}(q) = [\mu_{\tilde{Q}_d}(q), \lambda_{\tilde{Q}_d}(q)] \), (see Fig. 1).
(d) For each IVFS \( \tilde{Q}_f \) calculate \( \varepsilon_f \):
\[ \varepsilon_f(Q_d) = \frac{1}{\sum_{q=0}^{L-1} h(q)\mu_{\tilde{Q}_d}(q)} \]
(e) Take as threshold the intensity \( t \) corresponding to IVFS \( \tilde{Q}_d \) with lowest value of \( \varepsilon_f \).

In Algorithm 3 (step (a)) we are going to use the following method to construct the fuzzy \( \tilde{Q}_d \) sets:

(i) Select the automorphism \( \varphi \);
(ii) FOR \( t = 0 \) TO \( L−1 \) DO
   Calculate: \( m_d(t) \) and \( m_u(t) \)
   FOR \( q = 0 \) TO \( L−1 \) DO
   \[ \mu_{\tilde{Q}_d}(q) = \begin{cases} 
   1 - 0.5 \varphi\left(\frac{q}{L-1}\right) - \varphi\left(\frac{m_d(t)}{L-1}\right) & \text{if } q \leq t \\
   1 - 0.5 \varphi\left(\frac{q}{L-1}\right) - \varphi\left(\frac{m_u(t)}{L-1}\right) & \text{if } q > t 
   \end{cases} \]
ENDFOR

The reason why we use this construction is the following: If \( \varepsilon_f(Q_d) \rightarrow 0 \), then \( \mu_{\tilde{Q}_d}(q) \approx \mu_{\tilde{Q}_d}(q) \) for all \( q \in [0, \ldots, L−1] \), considering that the membership functions constructed in (2) are such that: \( 0.5 \leq \mu_{\tilde{Q}_d}(q) \leq 1 \) (see [9]); we have that \( \mu_{\tilde{Q}_d}(q) \rightarrow 1 \) for all \( q \in [0, \ldots, L−1] \), therefore \( q \approx m_d(t) \) or \( q \approx m_u(t) \).

4. Relation between the algorithms

In [6,7] it is proven that the area algorithm is a particular case of Algorithm 1. Regarding the relation between the area algorithm and Tizhoosh’s algorithm we have the following results:

Proposition 1. Let \( x \geq 1 \). In the area algorithm the following inequality holds for each \( t \in \{1, \ldots, L−1\} \):
\[ A(\tilde{Q}_d, x^q, x) = \sum_{q=0}^{L−1} h(q)\left(1 - 0.5\left|\frac{q}{L-1} - \frac{m_d(t)}{L-1}\right|\right)^{1/x} \]
\[ + \sum_{q=L}^{L−1} h(q)\left(1 - 0.5\left|\frac{q}{L-1} - \frac{m_u(t)}{L-1}\right|\right)^{1/x} \geq \]
\[ A(\tilde{Q}_d, x^{1/q}, x) = \sum_{q=0}^{L−1} h(q)\left(1 - 0.5\left|\frac{q}{L-1} - \frac{m_d(t)}{L-1}\right|\right)^{1/x} \]
\[ + \sum_{q=L}^{L−1} h(q)\left(1 - 0.5\left|\frac{q}{L-1} - \frac{m_u(t)}{L-1}\right|\right)^{1/x}. \]

Proof. Direct. We only need to bear in mind that \( x^{1/q} \leq x \) for all \( x \in [0,1] \) with \( x \geq 1 \).

Theorem 1. If in step (a) of Tizhoosh’s algorithm we use the constructions presented in (2) with \( \varphi(x) = x \) for all \( x \in [0,1] \), then
\[ \varepsilon_f(Q_d) = \frac{1}{\sum_{q=0}^{L−1} h(q)} \left(A(\tilde{Q}_d, x^q, x) - A(\tilde{Q}_d, x^{1/q}, x)\right). \]

Proof. Direct.

Theorem 2. Let \( x \geq 1 \) and let \( \varphi_f(x) = x \). Let \( t_{1,t} \) and \( t_s \) be the thresholds calculated with the area algorithm taking \( \varphi_f(x) = x^2 \) and \( \varphi_f(x) = x^{1/q} \), respectively. Let \( \tilde{Q}_d \) and \( \tilde{Q}_t \) be fuzzy sets constructed with the method (2) taking \( \varphi_f(x) = x \). Under these conditions:
\[ \varepsilon_f(Q_d) \geq \varepsilon_f(Q_t) \]
where \( Q_{L_{1,q}} = \{(q, \mu_{\tilde{Q}_{L_{1,q}}}(q), \lambda_{\tilde{Q}_{L_{1,q}}}(q)) \mid q \in [0, \ldots, L−1]\} \) and \( Q_{L_{1,q}} = \{(q, \mu_{\tilde{Q}_{L_{1,q}}}(q), \lambda_{\tilde{Q}_{L_{1,q}}}(q)) \mid q \in [0, \ldots, L−1]\} \).

Proof. \( t_{1,t} \) is the threshold calculated by means of the area algorithm, therefore for all \( t \in [0,1, \ldots, L−1] \) we have
\[ A(\tilde{Q}_{L_{1,q}}, x^q, x) = \sum_{q=0}^{L−1} h(q)\mu_{\tilde{Q}_{L_{1,q}}}(q) \leq \sum_{q=0}^{L−1} h(q)\mu_{\tilde{Q}_{L_{1,q}}}(q) \]
in particular for \( t = t_s \) we have
\[ A(\tilde{Q}_{L_{1,q}}, x^{1/q}, x) = \sum_{q=0}^{L−1} h(q)\mu_{\tilde{Q}_{L_{1,q}}}(q) \leq \sum_{q=0}^{L−1} h(q)\mu_{\tilde{Q}_{L_{1,q}}}(q) \]
By the same reasoning, for \( t_s \) we have
\[ A(\tilde{Q}_{L_{1,q}}, x^{1/q}, x) = \sum_{q=0}^{L−1} h(q)\mu_{\tilde{Q}_{L_{1,q}}}(q) \leq \sum_{q=0}^{L−1} h(q)\mu_{\tilde{Q}_{L_{1,q}}}(q) \]
Therefore,
\[ \varepsilon_f(Q_d) - \varepsilon_f(Q_t) = \frac{1}{\sum_{q=0}^{L−1} h(q)} \sum_{q=0}^{L−1} h(q)(\mu_{\tilde{Q}_{L_{1,q}}}(q) - \mu_{\tilde{Q}_{L_{1,q}}}(q)) \]
\[ - \frac{1}{\sum_{q=0}^{L−1} h(q)} \sum_{q=0}^{L−1} h(q)(\mu_{\tilde{Q}_{L_{1,q}}}(q) - \mu_{\tilde{Q}_{L_{1,q}}}(q)) \]

A consequence of Theorem 2 is the following. Let us take for Tizhoosh’s algorithm the membership function given in (2), with \( \phi(x) = x^2 \). Let us consider again Tizhoosh’s algorithm with the membership function given in (2), but with \( \phi(x) = x^{1/2} \). In this way, we have obtained two best thresholds, which in principle could be different. But, by Tizhoosh’s algorithm, the best threshold of them will be the one allowing to get an IVFS with the smallest IV entropy in step (c) of the algorithm. And, by the previous result, this will be the threshold corresponding to \( \phi(x) = x^{1/2} \). But we can say even more.

**Corollary 1.** Under the constructions of Theorem 2 if we execute Tizhoosh’s algorithm, then the threshold \( t_H \) calculated is such that

\[
\mathcal{E}_F(Q_{t_H}) \leq \mathcal{E}_F(Q_{t_{1/2}}) \leq \mathcal{E}_F(Q_{t_{1/2}})
\]

**Proof.** Direct. \( \Box \)

So, in fact, the threshold we get with Tizhoosh’s algorithm is always as good as or better than the ones obtained by the algorithm of the areas applied first to the lower bounds or the upper bounds of the IVFSs we get with our methods.

Furthermore, if “we draw with one (zero) the pixels whose intensity is between \( t_{1/2} \) and \( t_s \) and with zero (one) the rest”, then it is made clear that for one of the thresholds the pixels represented are the background (object) and for the other threshold they are the object (background) that is, they are the pixels for which the expert who has provided threshold \( t_{1/2} \) and the one who has provided threshold \( t_s \) do not come to an agreement.

**Example 1.** The images in this example are a selection of the ones used by Tizhoosh in his paper. For this reason, we do not show the corresponding thresholded image.

---

**Table 1**

<table>
<thead>
<tr>
<th>IMAGE</th>
<th>( t_{1/2} )</th>
<th>( \mathcal{E}<em>F(Q</em>{t_{1/2}}) )</th>
<th>( t_s )</th>
<th>( \mathcal{E}<em>F(Q</em>{t_s}) )</th>
<th>( t_H )</th>
<th>( \mathcal{E}<em>F(Q</em>{t_H}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blocks</td>
<td>48</td>
<td>0.047714</td>
<td>43</td>
<td>0.047650</td>
<td>42</td>
<td>0.047647</td>
</tr>
<tr>
<td>Rice</td>
<td>55</td>
<td>0.042187</td>
<td>52</td>
<td>0.042102</td>
<td>51</td>
<td>0.042100</td>
</tr>
<tr>
<td>Wheel</td>
<td>86</td>
<td>0.040018</td>
<td>81</td>
<td>0.039963</td>
<td>79</td>
<td>0.039957</td>
</tr>
<tr>
<td>Newspaper</td>
<td>199</td>
<td>0.036476</td>
<td>201</td>
<td>0.036433</td>
<td>201</td>
<td>0.036433</td>
</tr>
</tbody>
</table>

---

We take \( \alpha = 2 \). In the first column of Table 1 we present the names of the images object of the study. In the other columns we show the values of \( t_{1/2} \), \( \mathcal{E}_F(Q_{t_{1/2}}) \), \( t_s \), \( \mathcal{E}_F(Q_{t_s}) \), \( t_H \) and \( \mathcal{E}_F(Q_{t_H}) \), respectively. Bearing in mind Theorem 2 and Corollary 1, Tizhoosh’s algorithm advises us to use, for example for the blocks, first \( t_{1/2} \) and between \( t_{1/2} \) and \( t_s \) the threshold \( t_s = 43 \). Following this reasoning, for the image of the rice we must take the threshold of 51 first and in any other case 52, etc.

In the first row of Fig. 2 are the original images. In the second we present, for each image, the percentage of pixels between the corresponding values of \( t_{1/2} \) and \( t_s \); that is, the percentage of pixels such that for one threshold they belong to the background and for the other they belong to the object. In the last row we draw these pixels in white.

---

5. Conclusions

In this work we have proven that when we use the same constructions (2) in order to execute Tizhoosh’s algorithm and the area algorithm (and therefore Algorithm 1), then Tizhoosh’s algorithm enables us to select the best threshold from the two thresholds calculated by means of the area algorithm when it is applied to the lower extremes and the upper extremes, respectively, of the IVFSs constructed. Moreover with these constructions, the threshold calculated by means of Tizhoosh’s algorithm is always the best (the lowest IV entropy).

We consider that these arguments justify the use of Tizhoosh’s algorithm in image processing and therefore they also justify the use of interval-valued fuzzy sets in the calculation of the threshold of an image.
In the near future we intend to analyze the relation between the algorithms presented in this paper when we use any type of construction, not just for (2).

References


