New constructions of diagonal patchwork copulas

Fabrizio Durante a, José Antonio Rodríguez-Lallena b, Manuel Úbeda-Flores b, *

a Department of Knowledge-Based Mathematical Systems, Johannes Kepler University, A-4040 Linz, Austria
b Departamento de Estadística y Matemática Aplicada, Universidad de Almería, 04120 La Cañada de San Urbano (Almería), Spain

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ABSTRACT

We present a method for constructing symmetric copulas which generalizes the diagonal patchwork construction of copulas procedure. We also show how it is related to a new construction of a generalized Farlie–Gumbel–Morgenstern distribution and to the copula transforms.

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1. Introduction

The description of the dependence among random variables has been a subject extensively studied by several researchers. For this purpose, the concept of copula has appeared to be very useful for capturing the dependence properties of a continuous random vector apart from its marginal behavior (see [27,37]). Nowadays, copulas are largely used in finance and insurance, especially for risk models (see, e.g., [7,24,30,34]), and in hydrology (see, e.g., [23,41]). In particular, symmetric copulas have been introduced and investigated for describing situations where the components of a random vector are exchangeable (see, e.g., [8,17,29,33]). Relevant applications of such copulas have been recently found in reliability theory (see [4,21]).

Specifically, a (bivariate) copula is a function $C : [0,1]^2 \rightarrow [0,1]$ which satisfies

(C1) the boundary conditions $C(t,0) = C(0,t) = 0$ and $C(t,1) = C(1,t) = t$ for all $t \in [0,1]$, and

(C2) the 2-increasing property, i.e.,

$$V_C([u_1, u_2] \times [v_1, v_2]) := C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0$$

for all $u_1, u_2, v_1, v_2$ in $[0,1]$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$.

$V_C([u_1, u_2] \times [v_1, v_2])$ is called the C-volume of $[u_1, u_2] \times [v_1, v_2]$. We will also use the definition of $V_C$ for other functions $C$ which are not necessarily copulas. In other words, a copula is the restriction to $[0,1]^2$ of a continuous bivariate distribution function whose marginals are uniformly distributed.

* Corresponding author. Tel.: +34 950015813; fax: +34 950015167.
E-mail addresses: fabrizio.durante@jku.at (F. Durante), jarodrig@ual.es (J.A. Rodríguez-Lallena), mubeda@ual.es (M. Úbeda-Flores).
The importance of copulas is described by Sklar's Theorem (see [43]), which states that, given the random variables X and Y with joint distribution function \(H\) and marginal distribution functions \(F\) and \(G\), respectively, then there exists a copula \(C\) (which is uniquely determined on Range \(F \times \text{Range } G\)) such that \(H(x, y) = C(F(x), G(y))\) for all \(x, y\) in \([-\infty, \infty]\).

Copulas are of interest to statisticians essentially because they are a way of studying scale-free measures of dependence, and are tools to build families of bivariate distributions with given margins. In particular, recently, there has been a growing interest in the determination of copulas with given values at some fixed sections or having preassigned values on some given regions (see, for example, [9,11–13,19,32,35,38,39,42]). These methods have proved to be very useful for building copulas having some specific tail dependencies and/or asymmetries.

Here, we will consider constructions of copulas based on diagonal patchwork techniques, as introduced in [12,15,38]. Specifically, by using some special aggregation functions, we present a natural extension of these methods in a more general framework (Section 2). Thanks to this extension, we can consider from a different (and slightly more general) point of view the generalized family of Farlie–Gumbel–Morgenstern copulas (FGM, for short) introduced in [3] (see Section 3). Moreover, we can also derive a generalization of the copula transforms considered, e.g., in [20,28,36] (see Section 4).

2. Generalized diagonal patchwork copulas

In this section we extend the construction in [12,15] and [38] to obtain the so-called diagonal patchwork copulas (see their definition below) to a more general framework. First we introduce some notation. Let \(A\) be a mapping from \([0,1]^2\) to \(\mathbb{R}\). The transposed of \(A\) is the mapping \(A^\prime : [0,1]^2 \to \mathbb{R}\) defined by \(A^\prime(u,v) := A(v,u)\). The mapping \(A\) is said to be symmetric if \(A^\prime = A\), i.e., if \(A(v,u) = A(u,v)\) for all \((u,v)\) in \([0,1]^2\). Associated with the function \(A\) we can consider two functions \(A^\wedge, A^\vee : [0,1]^2 \to \mathbb{R}\), respectively defined by

\[
A^\wedge(u,v) := A(u \wedge v, u \lor v)
\]

and

\[
A^\vee(u,v) := A(u \lor v, u \wedge v),
\]

where \(u \wedge v = \min(u, v)\) and \(u \lor v = \max(u, v)\) for every \(u, v \in [0,1]\). Observe that both functions \(A^\wedge\) and \(A^\vee\) are symmetric. Moreover, \(A^\wedge = A\) if, and only if, \(A^\vee = A\) if, and only if, \(A\) is symmetric. Then \((A^\wedge)^\wedge = (A^\vee)^\vee = A^\wedge\) and \((A^\wedge)^\vee = (A^\vee)^\wedge = A^\vee\). Since \(A^\wedge = (A^\vee)^\wedge\) and \(A^\vee = (A^\wedge)^\vee\), every result about functions of the form \(A^\wedge\) can be translated in terms of functions of the type \(A^\vee\).

Now we recall a concept that generalizes the notion of copula. An aggregation function is a mapping \(A : [0,1]^2 \to [0,1]\) satisfying the following properties (see [5,6,25]):

\[
\begin{align*}
(A1) & \ A \text{ is nondecreasing in each variable.} \\
(A2) & \ A(0,0) = 0 \text{ and } A(1,1) = 1.
\end{align*}
\]

In particular, an aggregation function is 2-increasing if it also satisfies property (C2). As obvious, all copulas are 2-increasing aggregation functions, the converse implication not being true (see [14]).

The symbols \(\mathcal{A}, \mathcal{A}_2\) and \(\mathcal{C}\) will denote the sets of aggregation functions, 2-increasing aggregation functions and copulas, respectively. It is clear that, for any mapping \(A : [0,1] \to [0,1]\), \(A^\wedge\) is in \(\mathcal{A}\) (respectively, \(\mathcal{A}_2\) or \(\mathcal{C}\)) if, and only if, \(A\) is in \(\mathcal{A}\) (respectively, \(\mathcal{A}_2\) or \(\mathcal{C}\)).

The diagonal section of a mapping \(A : [0,1]^2 \to [0,1]\), \(A^\wedge\) is in \(\mathcal{A}\) (respectively, \(\mathcal{A}_2\) or \(\mathcal{C}\)) if, and only if, \(A\) is in \(\mathcal{A}\) (respectively, \(\mathcal{A}_2\) or \(\mathcal{C}\)).

The diagonal section of some copula, then it is known that the function defined on \([0,1]^2\) by

\[
K_A(u,v) := \min\left(\frac{\delta(u) + \delta(v)}{2}\right)
\]

copula whose diagonal section is \(\delta\) (see [22,26]).

Given two mappings \(A, B : [0,1]^2 \to [0,1]\) with the same diagonal section, the function \(A \oplus B\) of \(A\) and \(B\) (also called the diagonal splice of \(A\) and \(B\)) is the function \(A \oplus B\) defined by

\[
(A \oplus B)(u,v) := \begin{cases} A(u,v), & u \leq v, \\ B(u,v), & u > v. \end{cases}
\]

If \(A\) and \(B\) are aggregation functions with the same diagonal section, then it is easy to check that \(A \oplus B\) is an aggregation function as well. If \(A\) and \(B\) are copulas with the same diagonal section \(\delta\), conditions under which \(A \oplus B\) is a copula have been investigated in [12] and [38]. Specifically, Theorem 7 in [38] establishes that \(A \oplus B\) is a copula if, and only if, \(A(u,v) + B(u,v) \leq \delta(u) + \delta(v)\) for every \((u,v)\) in \([0,1]^2\) such that \(u < v\). Here we present an extension of this result to diagonal patchworks of aggregation functions (see Theorem 2.2 below). Other possible generalizations to aggregation functions have been discussed by [18]. Observe also that Eq. (2.1) can be rewritten as

\[
A^\wedge = A \oplus A^\vee
\]

and, analogously, \(A^\vee = A^\prime \oplus A^\wedge\).

Let \(T_L\) and \(T_U\) be the triangles defined by \(T_L := \{(u,v) \in [0,1]^2 \mid u \geq v\}\) and \(T_U := \{(u,v) \in [0,1]^2 \mid u \leq v\}\). Let \(\mathcal{A}_L\) denote the class of all aggregation functions \(A\) satisfying the following properties:
(L1) A is 2-increasing on $T_u$, in the sense that $V_A(R) \geq 0$ for every rectangle $R = [u_1, u_2] \times [v_1, v_2]$ entirely contained into $T_u$.

(L2) $A(1, v) = v$ for all $v \in [0, 1]$.

Analogously, we denote by $\mathcal{A}_U$ the class of all aggregation functions $A$ satisfying the following properties:

(U1) A is 2-increasing on $T_u$, in the sense that $V_A(R) \geq 0$ for every rectangle $R = [u_1, u_2] \times [v_1, v_2] \subset T_U$.

(U2) $A(u, 1) = u$ for all $u \in [0, 1]$.

Note that $A \in \mathcal{A}_U$ if, and only if, $A' \in \mathcal{A}_L$. We wonder whether there is some relationship between the set $\mathcal{A}_L$ and the sets $\mathcal{A}_L$ and $\mathcal{A}_U$. The following copula:

Example 2.1. Let us consider $A : [0, 1]^2 \rightarrow [0, 1]$ defined by $A(u, v) = u^2 + v^2$. It is easy to check that $A \in \mathcal{A}_U$ but $A \notin \mathcal{A}_L \cup \mathcal{A}_U$. So $\mathcal{A}_L \subset \mathcal{A}_L \cup \mathcal{A}_U$. On the other hand, the function $C_{12}$ of [38, Example 5] is trivially an element of both $\mathcal{A}_L$ and $\mathcal{A}_U$ but does not belong to $\mathcal{A}_U$. So $\mathcal{A}_L \cap \mathcal{A}_U \subset \mathcal{A}_U$, whence we also have that the set $\mathcal{A}_L \cap \mathcal{A}_U$ contains strictly the set of copulas: $C \in \mathcal{A}_L \cap \mathcal{A}_U$.

We show that Theorem 7 in [38] holds under more general hypotheses.

Theorem 2.2. Let $A$ and $B$ be two aggregation functions belonging to $\mathcal{A}_U$ and $\mathcal{A}_L$, respectively. Suppose that $A$ and $B$ have the same diagonal section $\delta$. Then the function $\text{A} \oplus \text{B}$ is a copula if, and only if, $A(u, v) + B(v, u) \leq \delta(u) + \delta(v)$ for every $(u, v) \in [0, 1]^2$ such that $u < v$.

Proof. Since $A$ and $B$ satisfy conditions (U2) and (L2), respectively, then condition (C1) holds for $A \oplus B$. Since $A$ and $B$ satisfy conditions (U1) and (L1), respectively, then it is easy to conclude that condition (C2) holds for $A \oplus B$ if, and only if, $V_{A \oplus B}(u, v^2) \geq 0$ for every $(u, v) \in [0, 1]^2$ with $u < v$. Finally, observe that $V_{A \oplus B}(u, v^2) \geq 0$ if, and only if, $A(u, v) + B(v, u) \leq \delta(u) + \delta(v)$, which completes the proof. □

The following example shows a non-trivial application of Theorem 2.2.

Example 2.3. Let $A$ and $B$ the functions defined on $[0, 1]^2$ by

$$
A(u, v) = \frac{u^2 + 2uv + uv^2 - u^2v}{3}, \quad B(u, v) = \frac{u^2 + uv + uv^2 - uv^2}{2}.
$$

It is immediate that the function $A$ satisfies (A2) and (U2), and does not satisfy (L2). Observe that $(\partial A/\partial u)(u, v) = (2u(1 - v) + v(2 + v))/3 \geq 0$ and $(\partial^2 A/\partial u^2)(u, v) = u(2 + 2v - u)/3 \geq 0$ for all $(u, v) \in [0, 1]^2$, whence $A$ satisfies (A1). Moreover, $(\partial^2 A/\partial u \partial v)(u, v) = 2(1 + v - u)/3 \geq 0$ for all $(u, v) \in [0, 1]^2$ such that $u \leq v$, whence $A$ also satisfies (U1). Therefore, $A$ is an aggregation function in $\mathcal{A}_U$.

Analogously, it is immediate that the function $B$ satisfies (A2) and (L2), and does not satisfy (U2). Since $(\partial B/\partial u)(u, v) = uv + v(1 - v)/2 \geq 0$ and $(\partial B/\partial v)(u, v) = v(1 - u) + u(1 + u)/2 \geq 0$ for all $(u, v) \in [0, 1]^2$, we have that $B$ satisfies (A1). Moreover, $(\partial^2 B/\partial u \partial v)(u, v) = 1/2 + u - v \geq 0$ for all $(u, v) \in [0, 1]^2$ such that $u \geq v$, whence $B$ also satisfies (L1). Therefore, $B$ is an aggregation function in $\mathcal{A}_L$ (but not in $\mathcal{A}_U$). The diagonal sections of $A$ and $B$ are given by

$$
\delta_0(t) = \delta_2(t) = t^2
$$

for all $t \in [0, 1]$.

From Theorem 2.2, $A \oplus B$ is a copula if, and only if,

$$
(u^2 + 2uv + uv^2 - u^2v)/3 + (u^2 + uv + uv^2 - uv^2)/2 \leq u^2 + v^2
$$

for every $(u, v) \in [0, 1]^2$ such that $u < v$. It is easy to check that the previous inequality is equivalent to the following

$$(v - u^2) + 5v(1 - u)(v - u) \geq 0,$$

which holds trivially for every $(u, v) \in [0, 1]^2$ such that $u < v$. So we have obtained the following copula:

$$(A \oplus B)(u, v) = \begin{cases} 
\frac{u^2 + 2uv + uv^2 - u^2v}{3}, & u \leq v, \\
\frac{u^2 + uv + uv^2 - uv^2}{2}, & u > v.
\end{cases}$$

As a particular case of Theorem 2.2, we have the following

Corollary 2.4. Let $A$ be in $\mathcal{A}_U$ with diagonal section $\delta$. Then, the function $A' \delta$ defined by Eq. (2.1) is a copula if, and only if, $A(u, v) \leq K_\delta(u, v)$ for every $(u, v) \in [0, 1]^2$ such that $u < v$. Moreover, in this case we also have $A' \delta \leq K_\delta$ on $[0, 1]^2$.

Proof. First recall Eq. (2.2). From Theorem 2.2, $A' \delta = A' \delta$ is a copula if, and only if, $A(u, v) + A'(v, u) = 2A(u, v) \leq \delta(u) + \delta(v)$ for every $(u, v) \in [0, 1]^2$ such that $u < v$. Moreover, $A(u, v) \leq A(u, 1) = u$ for all $u \in [0, 1]$. Therefore, if $A'$ is a copula, then $A(u, v) \leq K_\delta(u, v)$ whenever $0 \leq u < v \leq 1$. In the opposite direction the proof is immediate. Thus, if $A'$ is a copula, then, from the definition of $A'$, $A' \leq K_\delta$ on $T_L$. Since $A'$ and $K_\delta$ are symmetric, then we can conclude that $A' \leq K_\delta$ on the whole set $[0, 1]^2$, which completes the proof. □
Remark 2.5. As a consequence of Corollary 2.4, if $A$ is in $\mathcal{A}_L$, then we have that $A^\wedge = A^\wedge_0A$ is a copula if, and only if, $A(u, v) < K_s(u, v)$ for every $(u, v)$ in $[0, 1]^2$ such that $u > v$ (and then $A^\wedge \leq K_s$ on $[0, 1]^2$ as well).

An example of application of Corollary 2.4 is the following:

Example 2.6. Let $A$ be the aggregation function in $\mathcal{A}_L$ introduced in Example 2.3. From Corollary 2.4, the function $A^\wedge$ defined by Eq. (2.1) is a copula if, and only if, $(u^2 + 2uv + u^2v - u^2v)/3 \leq \min(u, (u^2 + v^2)/2)$ for every $(u, v)$ in $[0, 1]^2$ such that $u < v$. It is easy to check that $(u^2 + 2uv + u^2v - u^2v)/3 \leq u$ if, and only if, $u(1 - v)(3 + v - u) \geq 0$, which holds trivially for every $(u, v)$ in $[0, 1]^2$. It is also immediate that $(u^2 + 2uv + u^2v - u^2v)/3 \leq (u^2 + v^2)/2$ if, and only if, $(v - u)(v - u + 2v(1 - u)) \geq 0$, which holds trivially for every $(u, v)$ in $[0, 1]^2$ such that $u < v$. So we have obtained the following copula:

$$A^\wedge(u, v) = \begin{cases} \frac{u^2 + 2uv + u^2v - u^2v}{3}, & u \leq v, \\ \frac{v^2 + 2uv + u^2v - u^2v}{3}, & u > v. \end{cases}$$

3. Relationships with new generalized Farlie–Gumbel–Morgenstern copulas

Recently, a new extension of the FGM family of copulas has been investigated in [3]. This extension is given by the symmetric copulas of the form

$$C_{\theta f}(u, v) = uv + \theta(u \vee v)f(u)g(v)$$

(3.1)

for all $(u, v) \in [0, 1]^2$, where $\theta, f : [0, 1] \to \mathbb{R}$ are continuously differentiable functions. Necessary and sufficient conditions on $\theta$ and $f$, which ensure that the function $C_{\theta f}$ given by Eq. (3.1) is a copula, have been provided by those authors. Here we will show that copulas of this form can be obtained and extended by means of the diagonal patchwork construction from Section 2.

Observe that, by setting $g(t) := \theta(t)f(t)$ for every $t \in [0, 1]$, the function $g : [0, 1] \to \mathbb{R}$ is also continuously differentiable and the function $C_{\theta f}$ given by Eq. (3.1) is equal to the function $A^\wedge_{\theta g}$ defined by

$$A^\wedge_{\theta g}(u, v) = uv + f(u \wedge v)g(u \vee v)$$

(3.2)

for all $(u, v) \in [0, 1]^2$, where $A_{\theta g}(u, v) = uv + f(u)g(v)$

(3.3)

for all $(u, v) \in [0, 1]^2$ – recall Definition 2.1.

Next, for every non-zero continuously differentiable functions $f$ and $g$ from $[0, 1]$ to $\mathbb{R}$, we study whether the functions $A^\wedge_{\theta g}$ and $A_{\theta g}$ defined by Eqs. (3.2) and (3.3), respectively, are copulas.

As a trivial consequence of [40, Theorem 2.3], we can characterize which functions $A_{\theta g}$ are copulas, as shown in the following result.

Theorem 3.1. Let $f, g : [0, 1] \to \mathbb{R}$ be two non-zero continuously differentiable functions. Then, the function $A_{\theta g}$ defined by (3.3) is a copula if, and only if,

(i) $f(0) = g(0) = f(1) = g(1) = 0$ and

(ii) $f(u)g(v) \geq -1$ for all $(u, v) \in [0, 1]^2$.

The following result characterizes which functions $A^\wedge_{\theta g}$, given by Eq. (3.2), are copulas. As we show below, this result extends Theorem 1 in [3]. The proof of both results are similar, but we think that it is worthwhile to provide a short proof of our result.

For any $f : [0, 1] \to \mathbb{R}$ that is non-zero continuously differentiable, let $\mathcal{J}_f$ be the (nonempty) set formed by the open intervals $(a, b) \subset [0, 1]$ such that $f(u) = 0$ for all $u \in (a, b)$ and $f(a) = f(b) = 0$ (with the possible exception of the case $b = 1$: in this case $f(b)$ could be different from zero).

Theorem 3.2. Let $f, g : [0, 1] \to \mathbb{R}$ be two non-zero continuously differentiable functions. Then, the function $A^\wedge_{\theta g}$ defined by Eq. (3.2) is a copula if, and only if, the following conditions hold:

(i) $f(0) = g(1) = 0$;

(ii) $f(u)g(v) \geq -1$ for all $(u, v) \in T_0$;

(iii) $g/f$ is nonincreasing on every interval $(a, b) \in \mathcal{J}_f$.

Proof. It is easy to check that the function $A^\wedge_{\theta g}$ satisfies condition (C1) of copulas if, and only if, condition (i) is satisfied. If $A^\wedge_{\theta g}$ satisfies condition (C2) of copulas and $0 \leq u_1 < u_2 \leq v_1 < v_2 \leq 1$, then the $A^\wedge_{\theta g}$-volume of $[u_1, u_2] \times [v_1, v_2]$ is nonnegative, namely,
Thus,
\[
1 + \frac{f(u_2) - f(u_1)}{u_2 - u_1} \frac{g(v_2) - g(v_1)}{v_2 - v_1} \geq 0.
\]

If \( \nu_2 \rightarrow v_1 \) and \( u_1 \rightarrow u_2 \) in the last inequality, then condition (ii) follows. Analogously, if \( A_{f \circ g} \) satisfies condition (C2) and \( 0 \leq u < v \leq 1 \), by considering the \( A_{f \circ g} \)-volume of \([u, v]^2\) we obtain
\[
0 \leq v - u + \frac{f(v) - f(u)}{v - u} g(v) - f(u) \frac{g(v) - g(u)}{v - u}.
\]

If \( v \rightarrow u \) \((u \in [0, 1])\) in this inequality, then we have that \( f'(u)g(u) - f(u)g'(u) \geq 0 \) for every which implies that \( f/g \leq 0 \) on \((a, b)\), for every \((a, b) \in \mathcal{J}\), i.e., condition (iii) holds.

Finally, we need to prove that conditions (ii) and (iii) imply the 2-increasingness of \( A_{f \circ g} \). To verify this property it suffices to prove the nonnegativity of the \( A_{f \circ g} \)-volumes of the following types of rectangles \( R = [u_1, u_2] \times [v_1, v_2] \): (a) the rectangles included in \( T_U \); (b) the rectangles included in \( T_L \); (c) the rectangles of the form \([u, v]^2\), with \( 0 \leq u < v \leq 1 \). If \( R \subset T_U \), then it is easy to obtain that its \( A_{f \circ g} \)-volume is equal to the first member of the inequality (3.4). From the mean value theorem we have that there exist \( \lambda \in (u_1, u_2) \) and \( \beta \in (v_1, v_2) \) such that
\[
\frac{f(u_2) - f(u_1)}{u_2 - u_1} = f'(\lambda) \quad \text{and} \quad \frac{g(v_2) - g(v_1)}{v_2 - v_1} = g'(\beta),
\]
whence – by using condition (ii) – inequality (3.4) holds. The nonnegativity of the \( A_{f \circ g} \)-volumes of the rectangles included in \( T_L \) can be proved analogously. For the rectangles of the third type, by applying conditions (ii) and (iii) we have
\[
\begin{align*}
V_{A_{f \circ g}}([u, v]^2) &= (v - u)^2 + (f(v) - f(u))g(v) - f(u)(g(v) - g(u)) + \int_u^v f'(t) g(v) dt - \int_u^v f(u) g'(t) dt \\
&= (v - u)^2 + \int_u^v [f'(t) g(v)] dt + \int_u^v [f(t) g'(v)] dt + \int_u^v [f(t) g'(v)] dt - \int_u^v f(u) g'(t) dt \\
&\geq (v - u)^2 - \frac{(v - u)^2}{2} + \frac{(u - v)^2}{2} + \int_u^v [f'(t) g(t) - f(t) g'(t)] dt \geq 0,
\end{align*}
\]
which completes the proof.

Theorem 3.2 is an extension of Theorem 1 in [3] since, if two functions \( \theta \) and \( f(\phi \text{ in the mentioned paper}) \) satisfy the hypotheses of this last theorem, then the functions \( f \) and \( g = \theta f \) satisfy the hypotheses of Theorem 3.2. As a consequence, contrary to FGM family, our class of copulas of type (3.2) allows for non-zero tail dependence coefficients in the upper corner (see [3, Example 1]) as well as in the lower corner (see [3, Example 2]). Moreover, it can describe a wider range of dependence (see [3, Proposition 6]).

However, note that there are many selections of the functions \( f \) and \( g \) satisfying the hypotheses of Theorem 3.2 in such a way that the function \( \theta = g \circ f \) does not satisfy the hypotheses of Theorem 1 in [3]; for instance, when the function \( f \) vanishes at some point \( x \in [0, 1] \) such that \( g(x) = 0 \). This is the case of the following example.

Example 3.3. Let \( f \) and \( g \) be the functions defined on \([0, 1]\) by
\[
f(u) = u \left( \frac{1}{2} - u \right), \quad g(v) = \frac{v(v - 1)}{2}.
\]
It is easy to check that such functions satisfy the hypotheses of Theorem 3.2. As a consequence, the function \( A_{f \circ g} \) defined on \([0, 1]^2\) by
\[
A_{f \circ g}(u, v) = uv \left[ 1 + \frac{\min((2u - 1)(1 - v), (1 - u)(2 - v))}{4} \right]
\]
is a copula. Observe also that the function \( A_{f \circ g} \) is given in this case by
\[
A_{f \circ g}(u, v) = uv \left[ 1 + \frac{(1 - 2u)(1 - v)}{4} \right],
\]
for all \((u, v) \in [0, 1]^2\), and, from Theorem 3.1, is not a copula.

If a function \( A_{f \circ g} \), given by Eq. (3.2), is a copula, then it is immediate that the function \( A_{f \circ g} \) given by Eq. (3.3) satisfies (A2), (U1) and (U2). But it is not necessary that \( A_{f \circ g} \) is in \( \mathcal{A}_U \), i.e., \( A_{f \circ g} \) may not be nondecreasing in each variable. The following result characterizes which functions \( A_{f \circ g} \) of the type (3.3) are aggregation functions in \( \mathcal{A}_U \).
Theorem 3.4. Let \( f, g : [0, 1] \to \mathbb{R} \) be two non-zero continuously differentiable functions. Then, the function \( A_{f,g} \) defined by (3.3) is in \( \mathscr{U} \) if, and only if, conditions (i) and (ii) in Theorem 3.2 and also the following conditions are satisfied:

(iv) \( f(u)g(v) \leq f(v)g(v) \) for all \( (u, v) \in [0, 1]^2 \);
(v) \( f(u)g'(v) \leq u \) for all \( (u, v) \in [0, 1]^2 \).

Proof. Suppose that \( A_{f,g} \) is in \( \mathscr{U} \). Then we have \( u = A_{f,g}(u,1) = u + f(u)g(1) \) for all \( u \in [0, 1] \), whence \( g(1) = 0 \); and \( 0 \leq A_{f,g}(0, v) \leq A_{f,g}(0,1) = 0 \) implies that \( 0 = A_{f,g}(0, v) = f(0)g(v) \) for all \( v \in [0, 1] \), whence condition (i) follows. Let \( u_1, u_2, v \) be in \( [0, 1] \) such that \( u_2 \neq u_1 \). If \( u_2 > u_1 \), then \( 0 \leq A_{f,g}(u_2, v) - A_{f,g}(u_1, v) = (u_2 - u_1)v + f(u_2 - f(u_1))g(v) \), i.e.,

\[
g'(u)(u_2 - f(u_1))/(u_2 - u_1) \geq 0.
\]

If \( u_2 < u_1 \), this inequality can be obtained similarly. If we let \( u_2 \to u_1 \) in that inequality, then condition (iv) follows. Condition (v) can be proved similarly. Finally condition (ii) follows from condition (U1) of \( A_{f,g} \), as in the proof of Theorem 3.2.

Conversely, it is clear that condition (i) implies that properties (A2) and (U2) holds for \( A_{f,g} \). Furthermore, condition (ii) implies (U1), as in the proof of Theorem 3.2. If \( 0 \leq v \leq 1 \) and \( 0 \leq u < u_2 \leq 1 \), by integrating inequality (iv) with respect to the variable \( u \) over the interval \([u_1, u_2]\), we immediately obtain that \( A_{f,g} \) is nondecreasing in the variable \( u \). Similarly, from (v) we can derive that \( A_{f,g} \) is nondecreasing in the variable \( v \), which completes the proof. \( \square \)

It can be checked that the functions \( f \) and \( g \) in Example 3.3 satisfy conditions (iv) and (v) (in addition to conditions (i), (ii) and (iii)). Thus, the function \( A_{f,g} \) given by Eq. (3.5) is an aggregation function in \( \mathscr{U} \). On the other hand, the following example presents a copula \( A_{f,g} \) such that the function \( A_{f,g} \) is not an aggregation function.

Example 3.5. Let \( f \) and \( g \) be the functions defined on \([0, 1]\) by

\[
f(u) = u(u - 1)(u + 2)(1 - u/3), \quad g(v) = -v(1 - v)^2.
\]

It is immediate that condition (i) of Theorem 3.2 is satisfied by these functions. Observe that

\[
f'(u) = (-2/5)(2u^2 - 6u^2 - 7u + 5) \quad \text{for all } u \in [0, 1] \quad \text{and} \quad g'(v) = -3v^2 + 4v - 1.
\]

It can be checked that \( f' \) is strictly increasing in \([0, 1]\), with \( f'(0) = -2, f'(1/2) = 12/5 \) and \( f'(0) = 0 \) for some \( u_0 \in (1/3, 2/3) \); and \( g' \) is strictly increasing in \([0, 2/3]\) and strictly decreasing in \([2/3, 1]\), with \( g'(0) = -1, g'(1/3) = 0, g'(2/3) = 1/3 \) and \( g'(1) = 0 \). So, for \((u, v) \in T_U\), \( f(u,g(v)) < 0 \) if, and only if, \( v \in (1/3, 1) \) and \( u \in \min(u_0, v) \). Hence we have that \( f(u,g(v)) \geq -2/3 \) in \( T_U \), and condition (ii) of Theorem 3.2 holds. Since, for all \( t \in [0, 1] \),

\[
\left( \frac{g}{f} \right)'(t) = \frac{-65 - 10t + 5t^2}{(5 - t)^2(2 + t)^2} < 0,
\]

condition (iii) of Theorem 3.2 is also satisfied. As a consequence, the function \( A_{f,g} \) defined on \([0, 1]^2 \) by

\[
A_{f,g}(u, v) = uv + \frac{1}{5}uv(1 - u)(1 - v) - 3uv + (u \wedge v)(13 + uv - (u \wedge v))
\]

is a copula. However, \( A_{f,g} \) is a function from \([0, 1]^2 \) to \([0, 1]\) satisfying properties (A2), (U1), (U2) and (L2) which is not an aggregation function ((A1) is not satisfied; for instance, \( A_{f,g}(0.8, 0.1) > A_{f,g}(1, 0.1) \)).

A copula \( A_{f,g} \) given by Eq. (3.2) presents in several cases a singular component whose support is in the main diagonal of the unit square. The following result characterizes the absolutely continuous copulas of that type.

Theorem 3.6. Let \( f, g : [0, 1] \to \mathbb{R} \) be two non-zero continuously differentiable functions, such that the function \( A_{f,g} \) given by Eq. (3.2) is a copula. Then, \( A_{f,g} \) is absolutely continuous if, and only if, \( g/f \) is a step function on its domain, and \( f(s) = f'(s) = 0 \) for every step point \( s \) of the function \( g/f \).

Proof. After some computation, we have

\[
\int_0^u \int_0^v \frac{\partial^2 A_{f,g}}{\partial x \partial y}(x, y) \, dy \, dx = A_{f,g}(u, v) + \int_0^{u-v} (f(x)g'(x) - f'(x)g(x)) \, dx
\]

for every \((u, v) \in [0, 1]^2\). Hence, \( A_{f,g} \) is absolutely continuous if, and only if,

\[
\int_0^t (f(x)g'(x) - f'(x)g(x)) \, dx = 0
\]

for every \( t \in [0, 1] \), i.e., \( f(x)g'(x) - f'(x)g(x) = 0 \) for every \( x \in [0, 1] \) (since \( f(x)g'(x) - f'(x)g(x) \) is continuous on \([0, 1]\)). Therefore, \( A_{f,g} \) is absolutely continuous if, and only if, \( g/f \) is constant on every interval \((a, b) \in J_f\), i.e., \( g/f \) is constant on every interval \((a, b) \in J_f\). Suppose that two intervals of the form \((a, b) \) and \((b, c) \) are in \( J_f \). Then \( g(x) = \alpha f(x) \) for every \( x \in (a, b) \) and \( g(x) = \beta f(x) \) for every \( x \in (b, c) \). From the continuity of the derivatives, we also have that \( g'(b) = \alpha f'(b) = \beta f'(b) \). Then either \( \alpha = \beta \) (in this case, from the continuity of \( f \) and \( g \), we would have that \( g(x) - \alpha f(x) \) for every \( x \in [a, c] \)) or \( f'(b) = 0 \). Otherwise,
if we have an interval of \( f \) joined to a closed non-degenerated interval where the function \( f \) vanishes, then \( f' \) also vanishes on the point which is in the boundary of both intervals. Hence the proof follows. \( \square \)

A particular case of absolutely continuous copulas of the form \( A_{f,g}^{s} \) are those where \( g(t) = \alpha f(t) \) for every \( t \in [0,1] \) with \( \alpha \in [-1,1] \), which were extensively studied in [12]. Another example is the given here.

**Example 3.7.** Let \( f \) be the function defined on \([0,1]\) by \( f(u) = |u(1-u)(u-1/2)^2| \). Let \( g \) be the function defined on \([0,1]\) by

\[
g(v) = \begin{cases} f(v), & \text{if } 0 \leq v \leq 1/2, \\ 2f(v), & \text{if } 1/2 < v \leq 1. \end{cases}
\]

It is easy to check that such functions are continuously differentiable and satisfy conditions (i) and (ii) of Theorem 3.2. And, after some computation, we can prove that \( |f'(t)| \leq 1/4 \) and \( |g'(t)| \leq 1/2 \) for every \( t \in [0,1] \), whence condition (ii) of Theorem 3.2 holds for the functions \( f \) and \( g \). Therefore, the function \( A_{f,g}^{s} \) defined on \([0,1]^2\) by

\[ A_{f,g}^{s}(u,v) = \begin{cases} uv + uv(1-u)(v-1/2)^2(v-1/2)^2, & u \vee v \leq 1/2, \\ uv + 2uv(1-u)(v-1/2)^2(v-1/2)^2, & u \vee v > 1/2. \end{cases} \]

is an absolutely continuous copula. Observe also that, from Theorem 3.1, the function \( A_{f,g}^{s} \) given by

\[ A_{f,g}(u,v) = \begin{cases} uv + uv(1-u)(v-1/2)^2(v-1/2)^2, & v \leq 1/2, \\ uv + 2uv(1-u)(v-1/2)^2(v-1/2)^2, & v > 1/2 \end{cases} \]

is also an absolutely continuous copula.

4. Generalized transformations of copulas

Here, we generalize some transformation of copulas procedures considered in recent literature (see [10,20,28,36]) by using the results established in Section 2. We need some notation. Given a mapping \( A \) from \([0,1]^2\) to \([0,1]\), and three functions \( f, h, k : [0,1] \to [0,1] \), \( A_{f,h,k} \) will denote the function defined on \([0,1]^2\) by

\[ A_{f,h,k}(u,v) := f(A(h(u), k(v))). \]

(4.1)

If we apply Definition 2.1 to this function, then we obtain the function \( (A_{f,h,k})^{-} \), which we simply denote by \( A_{f,h,k}^{s} \), given, for all \((u,v) \in [0,1]^2 \), by

\[ A_{f,h,k}^{s}(u,v) = f(A(h(u) \vee v), k(u \vee v))). \]

(4.2)

Observe that \((A^{-})_{f,h,k} \) is different from \( A_{f,h,k}^{s} \). Invoking Eq. (4.2) we also have \( A_{f,h,k}^{s} = A_{f,h,k}^{s} \bigcirc A_{f,h,k}^{s} = A_{f,h,k}^{s} \bigcirc A_{f,h,k}^{s} \) (here \( A_{f,h,k}^{s} \) means \((A^{-})_{f,h,k} \) and observe the change of the order of the functions \( h \) and \( k \) at the indices).

Let \( \Theta \) be the set of the continuous non-decreasing functions \( h : [0,1] \to [0,1] \) such that \( h(1) = 1 \), and let \( \Theta_{l} \) be the set of the strictly increasing and concave functions in \( \Theta \). For every function \( h \) in \( \Theta_{l} \), the pseudo-inverse of \( h \) is the function \( h^{-1} : [0,1] \to [0,1] \) defined by

\[ h^{-1}(t) := \begin{cases} h^{-1}(t), & 0 \leq t \leq h(0), \\ 0, & 0 \leq t < h(0). \end{cases} \]

Note that \( h^{-1}(s) = s \) for all \( s \in [0,1] \), and \( h(h^{-1}(t)) = \max(t, h(0)) \).

Now we have the following result.

**Lemma 4.1.** Let \( h, k \) be functions in \( \Theta \) such that \( h \in \Theta_{l} \) and \( h \leq k \). Let \( A \) be an aggregation function in \( \mathcal{A} \). Then, the function \( A_{h^{-1},h,k}^{s} \) defined by (4.1) is also an aggregation function in \( \mathcal{A} \).

**Proof.** Since \( h \) and \( h^{-1} \) are non-decreasing, and \( A \) is non-decreasing in each argument, then it is immediate that \( A_{h^{-1},h,k}^{s} \) is non-decreasing in each variable, i.e., it satisfies condition (A1). Since \( A \) satisfies condition (U2), then we have \( A_{h^{-1},h,k}^{s}(u,1) = h^{-1}(A(h(u), k(1))) = h^{-1}(A(u)) = u \) for all \( u \in [0,1] \); thus, condition (U2) holds for \( A_{h^{-1},h,k}^{s} \). Hence, \( 0 \leq A_{h^{-1},h,k}^{s}(0,0) \leq A_{h^{-1},h,k}^{s}(0,1) = 0 \) and \( A_{h^{-1},h,k}^{s}(1,1) = 1 \), i.e., \( A_{h^{-1},h,k}^{s} \) satisfies condition (A2). Finally, in order to prove that property (U1) holds for \( A_{h^{-1},h,k}^{s} \), let \( u_1, u_2, v_1, v_2 \) be in \([0,1]\) such that \( u_1 < u_2 \leq v_1 < v_2 \) i.e., \([u_1, u_2] \times [v_1, v_2] \subset T_{U} \). We should prove that

\[ V_{A_{h^{-1},h,k}^{s}}([u_1, u_2] \times [v_1, v_2]) \geq 0. \]

If we denote \( s_1 = A(h(u_2), k(v_2)) \), \( s_2 = A(h(u_2), k(v_1)) \), \( s_3 = A(h(u_1), k(v_2)) \) and \( s_4 = A(h(u_1), k(v_1)) \), then the previous inequality is equivalent to:

\[ h^{-1}(s_1) - h^{-1}(s_2) - h^{-1}(s_3) + h^{-1}(s_4) \geq 0. \]
On the other hand, observe that \( h(u_1) \leq h(u_2) \leq h(v_1) \leq k(v_1) \leq k(v_2) \), whence the rectangle \([h(u_1), h(u_2)] \times [k(v_1), k(v_2)]\) is in \( T_U \). Since \( A \) satisfies (U1), then
\[
V_A([h(u_1), h(u_2)] \times [k(v_1), k(v_2)]) \geq 0,
\]
i.e., \( s_2 + s_3 \leq s_1 + s_4 \). Moreover, it is clear that \( \max(s_2, s_3) \leq \max(s_1, s_4) \). Since \( h^{-1} \) is convex, continuous and nondecreasing, then Proposition B2 in [31, Chapter 4] yields that \( h^{-1}(s_3) + h^{-1}(s_2) \leq h^{-1}(s_4) + h^{-1}(s_1) \), which completes the proof. \( \square \)

**Remark 4.2.** It can be checked that an analogous result can be proved for aggregation functions in \( \mathcal{A}_1 \) under the same hypotheses for \( h \) and \( k \), if \( A \in \mathcal{A}_1 \), then the function \( A_{h^{-1}, k} \) defined by (4.1) is in \( \mathcal{A}_1 \).

**Remark 4.3.** As a consequence of Lemma 4.1 and Remark 4.2, if \( k = h \) and \( A \in \mathcal{A}_1 \cap \mathcal{A}_U \), then \( A_{h^{-1}, k} \in \mathcal{A}_1 \cap \mathcal{A}_U \). Moreover, if \( A \in \mathcal{A} \), then \( A_{h^{-1}, k} \in \mathcal{A} \) (see, e.g., [20]).

In the following result we seek copulas which are functions of the form given by Eq. (4.2) (so they are symmetric).

**Theorem 4.4.** Let \( h, k \) be functions in \( \Theta \) such that \( h \in \Theta_1 \) and \( h \leq k \). Let \( C \) be a copula. Then, the function \( C_{h^{-1}, k}^{\vee} \) given by Eq. (4.1) is a copula if, and only if,
\[
C_{h^{-1}, k}^{\vee}(u, v) \leq \frac{C_{h^{-1}, k}^{\vee}(u, u) + C_{h^{-1}, k}^{\vee}(v, v)}{2}
\] (4.3)
for all \((u, v) \in [0,1]^2\) such that \( u < v \), where \( C_{h^{-1}, k}^{\vee} \) is the function defined by (4.1).

**Proof.** Every copula \( C \) is, in particular, a function in \( \mathcal{A}_U \). Thus, from Lemma 4.1, we have that \( C_{h^{-1}, k}^{\vee} \) is also a function in \( \mathcal{A}_U \). From Corollary 2.4 we have that \( C_{h^{-1}, k}^{\vee} \) is a copula if, and only if, \( C_{h^{-1}, k}^{\vee}(u, v) \leq K_{\mathcal{A}_U}(u, v) \) for all \((u, v) \in [0,1]^2\) such that \( u < v \), where \( \delta \) is the diagonal section of \( C_{h^{-1}, k}^{\vee} \), i.e., \( \delta(t) = C_{h^{-1}, k}^{\vee}(t, t) \) for all \( t \in [0,1] \). Since \( C_{h^{-1}, k}^{\vee} \) satisfies conditions (A1) and (U2), then we have that \( C_{h^{-1}, k}^{\vee}(u, v) \leq C_{h^{-1}, k}^{\vee}(u, 1) = u \), whence the proof follows. \( \square \)

We would like to stress some connections between the previous result and other constructions of copulas introduced earlier. First, when both \( h \) and \( k \) coincide with the identity mapping \( Id \) of \([0,1]\), this procedure collapses to the case described by Corollary 2.4 since \( C_{Id^{-1}, Id}^{\vee} = C \). On the other hand, when the copula \( C \) is symmetric, our method was already considered by [20] for the case \( k = h \) (see also [28,36]). We can also consider the case \( C(u, v) = H(u, u) - uv \). In this case, Theorem 4.4 yields copulas belonging to the generalized Archimedean family as introduced by [16]; in particular, some sufficient conditions that ensure that Eq. (4.3) is satisfied are given there.

As an application of Theorem 4.4, we provide the following example.

**Example 4.5.** Let \( h(t) = t \) and \( k(t) = (2 - t) \) for all \( t \in [0,1] \), and let \( C(u, v) = uv|1 + (1 - u)(1 - v)| \) for all \((u, v) \in [0,1]^2 \). (\( C \) is a member of the FGM family of copulas). Then, it is clear that \( h \in \Theta_1 \), \( k \in \Theta \) and \( h \leq k \). Since
\[
C_{h^{-1}, k}^{\vee}(u, v) = u(v(2 - v)[1 + (1 - u)(1 - v)^2])
\]
for all \((u, v) \in [0,1]^2\), then Eq. (4.3) reads
\[
2uv(2-v)[1+(1-u)(1-v)^2] \leq \frac{u^2(2-u)[1+(1-u)^3]+v^2(2-v)[1+(1-v)^3]}{2}.
\] (4.4)
Next we prove that Eq. (4.4) holds whenever \( 0 \leq u \leq v \leq 1 \). From Theorem 4.4, we have that the function \( C_{h^{-1}, k}^{\vee} \), i.e., the function defined on \([0,1]^2\) by
\[
C_{h^{-1}, k}^{\vee}(u, v) = uv(-2u v + 2u - u v + (v - u) \cdot v^2 + (-8 + 8u + 5u^2 - u^3) \cdot v + (-3u - u^2) \cdot v^2 + (9 - 3u - u^2) \cdot v^2 + (5 + u + v^2 + u v)^2) \geq 0.
\]
is a copula. After some computation, it is easy to check that Eq. (4.4) is equivalent to the following:
\[
(4u - 8u^2 + 2u - u^3) \cdot v + (8u - 8u^2 + 5u^2 - u^3) \cdot v^2 + (9 - 3u - u^2) \cdot v^2 + (5 + u + v^2 + u v)^2 \geq 0.
\]
For every \( u \in [0,1] \) we consider the function \( f_u \) defined on \([u,1]\) by \( f_u(v) = 4u - 8u^2 + 2u - u^3 + (4 + 4u - 9u^2 + 5u^2 - u^3) \cdot v + (9 - 3u - u^2) \cdot v^2 + (5 + u + v^2 + u v)^2 \). Thus, we have that Eq. (4.4) holds for every \((u, v) \in [0,1]^2\) such that \( u < v \) if, and only if, \( f_u(v) \geq 0 \) for every \( u \in [0,1] \) and \( v \in (u,1] \). Now we provide an outline of the proof of this last condition. Since \( f_u'(v) > 0 \), then \( f_u''(v) \) is increasing in \([u,1]\). If \( u \geq 5/6 \), then \( f_u''(v) > 0 \). If \( u < 5/6 \), then \( f_u''(v) < 0 \) when \( u < v < 1 - u/5 \); and \( f_u''(v) \geq 0 \) when \( 1 - u/5 \leq v \leq 1 \). Hence, after some computation, we obtain that \( f_u''(v) \leq 0 \) whenever \( u \geq (23 - \sqrt{61})/26 \); otherwise, if \( u < (23 - \sqrt{61})/26 \), then \( f_u''(v) \geq 0 \) if and only if \( u < v < 1 - u/5 - \sqrt{10 - 10u + 14u^2}/10 \). In both cases we can prove that \( f_u''(v) > 0 \) in \([u,1]\), and hence \( f_u''(v) > 0 \) in \([u,1]\). Therefore, it is easy to conclude that \( f_u(v) \geq 0 \) for every \( u \in [0,1] \) and \( v \in [u,1] \), as desired.
5. Conclusions

We have extended the diagonal patchwork procedure for copulas (see \cite{12,15,38}) by using some special binary aggregation functions. This extension has been hence applied to construct another generalization of the Farlie–Gumbel–Morgenstern family of copulas (following \cite{3}), and a modified version of the copula transforms.

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