Distributed Detection over Random Networks: Large Deviations Analysis

Dragana Bajović, Dušan Jakovetić, João Xavier, Bruno Sinopoli and José M. F. Moura

Abstract—We show by large deviations theory that the performance of running consensus is asymptotically equivalent to the performance of the (asymptotically) optimal centralized detector. Running consensus is a stochastic approximation type algorithm for distributed detection in sensor networks, recently proposed. At each time step, the state at each sensor is updated by a local averaging of its own state and the states of its neighbors (consensus) and by accounting for the new observations (innovation). We assume Gaussian, spatially correlated observations, and we allow for the underlying network to be randomly varying. This paper shows through large deviations that the Bayes probability of detection error, for the distributed detector, decays at the best achievable rate, namely, the Chernoff information rate. Numerical examples illustrate the behavior of the distributed detector for finite number of observations.

I. INTRODUCTION

We apply large deviations to study the asymptotic performance of distributed detection in sensor networks. Each node in the network senses the environment and cooperates locally with its neighbors to decide between the two hypothesis, $H_1$ and $H_0$. The nodes are connected by a generic, randomly varying network, and there is no fusion center. Specifically, we consider distributed detection via running consensus\(^1\) that has been recently proposed in [2]. With running consensus, at each time $k$, $N$ nodes update their decision variables by: 1) incorporating new observation (innovation step); and 2) mixing their decision variables locally with the neighbors (consensus step).

In this paper, we show, under Gaussian assumptions for spatially correlated noise, that the running consensus distributed detector is asymptotically optimal, as the number of observations $k$ goes to infinity. That is, the running consensus distributed detector asymptotically approaches the performance of the optimal centralized detector. We apply large deviations to study the asymptotic performance of both the (asymptotically) optimal centralized detector, which collects observations from all nodes $i$ at each time $k$, and the running consensus detector. For both detectors, the Bayes probability of error decays as $e^{-kc}$, where $C$ is the Chernoff distance between the distributions of the $N \times 1$ observation vectors under the two hypothesis. Our result for the running consensus is a distributed detection counterpart of the (Bayesian) Chernoff lemma.

We complement our theoretical findings for large $k$ (asymptotic regime) with numerical examples for the non asymptotic finite time regime. Interestingly, the numerical examples reveal that the best sensor–detector (among all $N$ sensors) performs very close to the optimal centralized detector (in the Bayes probability of error sense,) even when $k$ is very small.

We now briefly review the existing work on distributed detection. Distributed detection has been extensively studied. Prior work studies parallel fusion architectures (see, e.g., [3], [4], [5], [6], [7], [8]) where all nodes communicate with a fusion node. Also, consensus-based detection schemes have been studied (with no fusion node) in, for example, [9], [10], [11], where nodes in the network: 1) collect measurements; and 2) subsequently run the consensus algorithm to fuse their detection rules. The running consensus distributed detection has been proposed in [12]. Running consensus is different from classical consensus detection, as it incorporates new observations at each time step $k$, in real time; thus, unlike classical consensus, no delay is introduced from collecting observations to reaching consensus.

We now comment on the differences between this paper and reference [12], which also studies asymptotic optimality of distributed detection via running consensus. Reference [12] considers the Neyman-Pearson framework, while we adopt the Bayesian framework.
Reference [12] considers that, as the number of observations $k$ grows, the distribution means under the two hypothesis become closer and closer, at the rate of $1/\sqrt{k}$; consequently, as $k \to \infty$, there is an asymptotic, non zero, probability of miss, and asymptotic, non zero, probability of false alarm. In contrast, we assume that the distributions do not change with $k$ (do not approach each other.) and the Bayes probability of error decays to zero; we then examine the rate of decay of the Bayes error probability. Finally, reference [12] assumes that the observations at different sensors are independent identically distributed, with generic distribution, while we assume Gaussian; however, we allow for spatial correlation among observations—a well-suited assumption, e.g., for densely deployed wireless sensor networks (WSNs).

**Paper organization.** Section II reviews the large deviations results and the Chernoff lemma in hypothesis testing. Section III explains data and network models that we assume. Section IV introduces the (asymptotically) optimal centralized detection, as if there was a fusion node and its detection performance. Section V shows that the distributed running consensus detector asymptotically approaches in performance the optimal centralized detector. Section VI corroborates, with numerical examples, our theoretical findings from section V; it also studies the non asymptotic (finite time) regime. Finally, section VII summarizes the paper.

## II. BACKGROUND

In this section, we briefly review standard large deviations analysis for binary hypothesis testing and standard asymptotic results (in particular, Chernoff lemma) in binary hypothesis testing. We will later use these results throughout the paper.

### A. Binary hypothesis testing problem: Log-likelihood ratio test

Consider the sequence of independent identically distributed (i.i.d.) $d$-dimensional random vectors (observations) $y(k)$, $k = 1, 2, ..., $, and the binary hypothesis testing problem of deciding whether the probability measure (law) generating $y(k)$ is $\nu_0$ (under hypothesis $H_0$) or $\nu_1$ (under $H_1$). Assume that $\nu_1$ and $\nu_0$ are mutually absolutely continuous, distinguishable measures. Based on the observations $y(1), ..., y(k)$, formally, a decision test $T$ is a sequence of maps $T_k : \mathbb{R}^{kd} \to \{0, 1\}$, $k = 1, 2, ..., $, with the interpretation that $T_k(y(1), ..., y(k)) = l$ means that $H_l$ is decided, $l = 0, 1$. Specifically, consider the log-likelihood ratio (LLR) test to decide between $H_0$ and $H_1$, where the $T_k$ is given as follows:

$$D(k) := \frac{1}{k} \sum_{j=1}^{k} \log \frac{d\nu_1}{d\nu_0}(y(j))$$

where $L(k) := \log \frac{d\nu_1}{d\nu_0}(y(k))$ is the LLR (given by the Radon-Nikodym derivative of $\nu_1$ with respect to $\nu_0$ evaluated at $y(k)$), $\gamma_k$ is a chosen threshold, and $I_A$ is the indicator of event $A$. The LLR test with threshold $\gamma_k = 0$, $\forall k$, is asymptotically optimal in the sense of Bayes probability of error decay rate, as will be explained in subsection V-B.

### B. Log-likelihood ratio test: Large deviations

This subsection studies large deviations for the LLR decision test with decision variables $D(k)$ given in eqn. (1). The large deviations analysis will be very useful in estimating the exponential rate at which the Bayes probability of error decays and in showing the asymptotic optimality of the distributed running consensus detector. We first give the definition of the large deviations principle [13].

**Definition 1 (Large deviations principle (LDP))**

Consider a sequence of real valued random variables $\{\Theta(k)\}_{k=1}^{\infty} := \{\Theta(k)\}$ and denote with $\theta_k$ the probability measure of $\Theta(k)$. We say that the sequence of measures $\{\theta_k\}$ satisfies the LDP with a rate function $J : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ if the following holds:

1) For any closed, measurable set $F \subset \mathbb{R}$:

$$\limsup_{k \to \infty} \frac{1}{k} \log \theta_k(F) \leq - \inf_{t \in F} J(t)$$

2) For any open, measurable set $G \subset \mathbb{R}$:

$$\liminf_{k \to \infty} \frac{1}{k} \log \theta_k(G) \geq - \inf_{t \in G} J(t).$$

It can be shown that the sequence of LLR’s $\{L(k)\}$, conditioned on $H_l$, $l = 0, 1$, is i.i.d. Denote by $\mu_k^{l}$ the probability measure of $D(k)$ under hypothesis $H_l$. Using Cramér’s theorem ([13]), it can be shown that the sequence of measures $\{\mu_k^{l}\}$, $l = 0, 1$, satisfies the LDP with good rate function:

$$\Lambda^*_l(t) = \sup_{\lambda \in \mathbb{R}} (\lambda t - \Lambda_l(\lambda)),$$

where $\Lambda_l(\cdot)$ is the log-moment generating function of $L(k)$ under hypothesis $H_l$:

$$\Lambda_l(\lambda) = \log \mathbb{E} \left(e^{\lambda L(k)} \big| H_l \right).$$

That is, the rate function $\Lambda^*_l(t)$ is the Fenchel-Legendre (F-L) ([13]) transform of the log-moment generating

\[ \text{2Goodness of rate function is compactness of its sublevel sets.} \]
function of $L(k)$ under $H_l$. It can be shown that $\Lambda^*_1(t) = \Lambda^*_0(t) - t$. We summarize this result in the following theorem, e.g., [13]:

**Theorem 2** The sequence of measures $\{\mu_k^{(i)}\}$ of $D(k)$ under $H_l$ satisfies the LDP with good rate function given by eqn. (3).

**C. Asymptotic Bayes detection performance: Chernoff lemma**

We adopt the Bayes minimum probability of error detection. Denote by $P^e(k)$ the Bayes probability of error after $k$ samples are processed:

$$P^e(k) = P(H_0) \alpha(k) + P(H_1) \beta(k), \quad (5)$$

where $P(H_l)$ are the prior probabilities, $\alpha(k) := P(D(k) > \gamma_k H_0)$ and $\beta(k) := P(D(k) \leq \gamma_k H_1)$ are, respectively, the probability of false alarm and the probability of miss, and $\gamma_k$ is the test threshold.

We will be interested in the rate at which the Bayes probability of error decays to zero as the number of observations $k$ goes to infinity. Also, as auxiliary results, we will need the rates at which $\alpha(k)$ and $\beta(k)$ go to zero as $k \to \infty$. That is, we will be interested in the following quantities:

$$\lim_{k \to \infty} \frac{1}{k} \log P^e(k) \quad (6)$$

$$\lim_{k \to \infty} \frac{1}{k} \log \alpha(k) \quad (7)$$

$$\lim_{k \to \infty} \frac{1}{k} \log \beta(k). \quad (8)$$

Theorem 4 ([13]) states that, among all possible decision tests, the LLR test with zero threshold minimizes (6). This result is a corollary of the Theorem 3 ([13]), that asserts that, for a LLR test with fixed threshold $\gamma_k = \gamma$, $\alpha(k)$ and $\beta(k)$ indeed (simultaneously) decay to zero exponentially; also, Theorem 3 expresses the exponential rate of decay in terms of the rate functions defined in eqns. (7) and (8). Before stating the Theorem, define $\mathcal{T}_{(l)} := \mathbb{E} (L(k)|H_l), l = 0, 1$.

**Theorem 3** The LLR test with constant threshold $\gamma_k = \gamma, \gamma \in (\mathcal{T}_{(0)}, \mathcal{T}_{(1)})$ satisfies:

$$\lim_{k \to \infty} \frac{1}{k} \log \alpha(k) = -\Lambda^*_0(\gamma) < 0 \quad (9)$$

$$\lim_{k \to \infty} \frac{1}{k} \log \beta(k) = \gamma - \Lambda^*_0(\gamma) < 0. \quad (10)$$

**Theorem 4 (Chernoff lemma)** If $P(H_0) \in (0, 1)$, then:

$$\inf_T \inf_{k \to \infty} \left\{ \frac{1}{k} \log P^e(k) \right\} = -\Lambda^*_0(0), \quad (11)$$

where the infimum over all possible tests $T$ is attained for the LLR test with $\gamma_k = 0, \forall k$.

The quantity $\Lambda^*_0(0) = \Lambda^*_1(0)$ is called the Chernoff distance between the distributions of $y(k)$ under $H_0$ and $H_1$, or Chernoff information, [13].

**Asymptotically optimal test.** We introduce the following definition of the asymptotically optimal test.

**Definition 5** The decision test $T$ is asymptotically optimal if it attains the infimum in eqn. (11).

We will show that, for the distributed Gaussian hypothesis testing over random networks, the running consensus is asymptotically optimal in the sense of Definition 5.

### III. Distributed detection model: Data and Network models

This section describes: 1) the data model (subsection III-A), i.e., the observation model at each sensor in the network; and 2) the model of the network through which the sensors cooperate with the running consensus distributed detection algorithm (subsection III-B). The distributed detection algorithm is detailed in Section V.

#### A. Data model

We consider Gaussian binary hypothesis testing in spatially correlated noise. The sensors operate (in terms of sensing and communication) synchronously, at discrete time steps $k$. At time $k$, sensor $i$ measures (scalar) $y_i(k)$. Collect the sensor measurements in a vector $y(k) = (y_1(k), y_2(k), ..., y_N(k))^T$, where $N$ is the total number of sensors. Nature can be in one of two possible states: $H_1$—event occurring (e.g., target present); and $H_0$—event not occurring (e.g., target absent.) We assume the following distribution model for the vector $y(k)$:

$$y(k) = m_l + \zeta(k), \quad l = 0, 1, \quad (12)$$

where $m_l$ is the (constant) signal under hypothesis $H_l$, and $\zeta(k)$ is zero mean Gaussian additive noise. We assume that $\{\zeta(k)\}$ is an independent identically distributed (i.i.d.) sequence of $N \times 1$ random vectors with distribution $\zeta(k) \sim \mathcal{N}(0, S)$, where $S$ is a (positive definite) covariance matrix. Thus, with our model, the noise is temporally independent, but can be spatially correlated. Spatial correlation should be taken into account due to, for example, dense deployment of wireless sensor networks, while it is still reasonable to assume that the observations are independent along time. (Conditioned to $H_l$, $\{y(k)\}$ are i.i.d. with the distribution $\mathcal{N}(m_l, S)$.)

#### B. Random network model and data mixing model

We consider distributed detection via running consensus where each node at a time $k$: 1) measures $y_i(k)$; 2) exchanges its current decision variable (denote it by $x_i(k)$) with its neighbors; and 3) performs a weighted average of its own decision variable and the neighbors’ decision variables. The network connectivity is assumed
random. The weighted averaging, at each time \( k \), as with the standard consensus algorithm, is described by the \( N \times N \) weight matrix \( W(k) \). The weight matrix \( W(k) \) respects the sparsity pattern of the network, i.e., \( W_{ij}(k) = 0 \), if the link \( \{i,j\} \) is down at time \( k \). We allow the matrices \( \{W(k)\} \) to be random to account for: 1) random link failures in the network; 2) randomized averaging protocols (e.g., standard gossip [14]); and 3) random (non-constant) weights when the link is present. Random link failures occur frequently in networks supported by wireless communication (e.g., random packet dropouts in WSNs.) We assume that \( \{W(k)\} \) are i.i.d.

To simplify notation, we will drop the index \( k \) in \( W(k) \) when it is not essential. Define also \( J := (1/N)1^T \), where \( 1 \) is \( N \times 1 \) vector with unit entries. We now summarize the assumptions on the matrices \( \{W(k)\} \):

**Assumption 6** For the sequence of matrices \( \{W(k)\}_{k=1}^{\infty} \), we assume the following:

1) The sequence \( \{W(k)\}_{k=1}^{\infty} \) is i.i.d.
2) \( W \) is symmetric with probability one.
3) \( W \) is stochastic (row-sums are equal to 1 and the entries are nonnegative) with strictly positive diagonal, with probability one.
4) \( q = \rho(E[W] - J) < 1 \). Here \( \rho(\cdot) \) is the spectral norm.
5) The random matrix \( W(l) \) and the random vector \( y(s) \) are independent, \( \forall l, \forall s \).

We make the following remarks on the network connectivity model implied by Assumption 6:

1) The links \( \{i,j\} \) in the network are undirected, i.e., if node \( i \) communicates with \( j \), then also node \( j \) communicates with \( i \).
2) The link failures are temporally independent, but can be spatially correlated. Spatial correlation should be taken into account due to, for instance, interference among close links; at the same time, it is still reasonable to assume that the link failures are temporally independent.
3) Assumption 6 subsumes typical connectivity patterns for distributed averaging, e.g., 1) the erasure model studied in [15]; and 2) the standard pairwise gossip studied in [14]. Reference [16] also considers spatially correlated and temporally independent link failures, as studied in this paper.

**Network supergraph.** Define also the network supergraph as a pair \( G := (\mathcal{N}, \mathcal{E}) \), where \( \mathcal{N} \) is the set of nodes with cardinality \( |\mathcal{N}| = N \), and \( \mathcal{E} \) is the set of edges with cardinality \( |\mathcal{E}| = M \), defined as: \( E = \{(i,j) : P(W_{ij} \neq 0) > 0, i \neq j \} \). Clearly, when, for some \( \{i,j\} \), \( P(W_{ij} \neq 0) = 0 \), then the link \( \{i,j\} \notin E \) and nodes \( i \) and \( j \) never communicate. It can be shown that Assumption 6-4) is equivalent to the requirement that the supergraph \( G \) is connected. Connectedness of the supergraph means that the underlying network is connected on average; network instantiations during algorithm runs, however, do not have to be connected, like with the gossip algorithm [14], where, at a time \( k \), only one link is active.

### IV. CENTRALIZED DETECTION: BAYES OPTIMAL TEST

We first consider the centralized detection scenario, as if there was a fusion node that collects and processes all sensor observations. The decision variable \( D(k) \) and the LLR decision test are given by eqns. (1) and (2), where now, under the data assumptions in subsection III-A:

\[
L(k) = (m_1 - m_0)^T S^{-1} \left( y(k) - \frac{m_1 + m_0}{2} \right)
\]

Conditioned on either hypothesis \( H_1 \) and \( H_0 \), \( L(k) \) is distributed as

\[
\mathcal{N} \left( \mu_L^{(1)}, \sigma_L^2 \right),
\]

where

\[
\mu_L^{(1)} = -\mu_L^{(0)} = \frac{1}{2}(m_1 - m_0)^T S^{-1}(m_1 - m_0)
\]

\[
\sigma_L^2 = (m_1 - m_0)^T S^{-1}(m_1 - m_0).
\]

Define the vector \( v \in \mathbb{R}^N \) as

\[
v := S^{-1}(m_1 - m_0).
\]

Then, the LLR \( L(k) \) can be written as follows:

\[
L(k) = \sum_{i=1}^{N} v_i \left( y_i(k) - \frac{[m_1]_i + [m_0]_i}{2} \right) = \sum_{i=1}^{N} \eta_i(k)
\]

(15)

Thus, the LLR at time \( k \) is separable, i.e., the LLR is the sum of the terms \( \eta_i(k) \) that depend affinely on the individual observations \( y_i(k) \). We will exploit this fact in subsection V-A to derive the distributed, running consensus, detection algorithm.

Applying Theorem 2 to the sequence \( \{D(k)\} \) (under hypothesis \( H_1 \), \( l = 0,1 \)), we have that the sequence of measures of \( D(k) \) satisfies the LDP with good rate function

\[
I(t) : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}
\]

which, by evaluating the log-moment generating function of \( L(k) \) in (13) and its F-L transform, can be shown to be:

\[
I(t) = \frac{(t - m_L^2)^2}{2\sigma_L^4}, l = 0,1.
\]

(16)

We state this result as a Corollary 7.

**Corollary 7** The sequence \( \{D(k)\} \), under \( H_1 \), \( l = 0,1 \), satisfies the LDP with good rate function \( I(t) \), given by eqn. (16).

We remark that Theorem 4 also applies to the detection problem explained in subsection III-A. Denote
the Bayes probability of error for the centralized detector in section IV after k samples are processed by $P^c_{cen}(k)$. Due to continuity of the rate functions in (16), it can be shown that: $\lim \inf_{k \to \infty} \frac{1}{k} \log P^c_{cen}(k) = \lim \sup_{k \to \infty} \frac{1}{k} \log P^c_{cen}(k) = \lim_{k \to \infty} \frac{1}{k} \log P^c_{cen}(k)$. Thus, Theorem 4 in this case simplifies to the following corollary:

**Corollary 8.** (Chernoff lemma for optimal centralized detector) The LLC test with $\gamma_k = 0$, $\forall k$, is asymptotically optimal in the sense of definition 5. Moreover, for the LLC test with $\gamma_k = 0$, $\forall k$, we have:

$$\lim_{k \to \infty} \frac{1}{k} \log P^c_{cen}(k) = -I_{(0)}(0)$$

$$= -\frac{1}{8}(m_1 - m_0)S^{-1}(m_1 - m_0).$$

**Remark.** In the Gaussian case studied here, the LLC test with zero threshold is optimal also in the finite time regime, for all $k$, in the sense that it minimizes the Bayes probability of error, when the prior probabilities $P(H_0) = P(H_1) = 0.5$. When the prior probabilities are not equal, the LLC test is also optimal, but the threshold $\gamma_k$ will be different than zero.

V. DISTRIBUTED DETECTION ALGORITHM

A. Distributed detection via running consensus

We now present a distributed detection algorithm via running consensus. With this detection algorithm, no fusion node is required, and the underlying network topology is generic, with random link failures. The running consensus is proposed in [2], and it is a stochastic approximation type of algorithm (see [1]). Reference [2] studies the case when the observations of different sensors at a time are i.i.d. We extend the running consensus detection algorithm to the case of spatially correlated Gaussian observations.

With the running consensus distributed detector, each node $i$ makes local decisions based on its local decision variable $x_i(k)$: if $x_i(k) > 0$, then $H_1$ is accepted; if $x_i(k) \leq 0$, then $H_0$ is accepted. At each time step $k$, the local decision variable at node $i$ is improved two-fold: 1) by exchanging information with its immediate neighbors in the network; 2) by incorporating into the decision process the new observation $y_i(k)$. Recall the definition of $\eta_i(k)$ in eqn. (15). Specifically, the update of the local decision variable at node $i$ is given by the following equation:

$$x_i(k + 1) = \frac{k}{k + 1} W_{ii}(k)x_i(k) + \frac{1}{k + 1} N\eta_i(k + 1),$$

$$k = 1, 2, ...$$

$x_i(1) = N\eta_i(1)$

Here $\Omega_i(k)$ is the (random) neighborhood of node $i$ at time $k$, and $W_{ij}(k)$ are the (random) averaging weights, defined together with the $N \times N$ (random) matrices $W(k) = [W_{ij}(k)]$ in subsection III-B. Let $x(k) = (x_1(k), x_2(k), ..., x_N(k))^T$ and $\eta(k) = (\eta_1(k), ..., \eta_N(k))^T$. The algorithm in matrix form is given by:

$$x(k + 1) = \frac{k}{k + 1} W(k)x(k) + \frac{1}{k + 1} N\eta(k + 1),$$

$$k = 1, 2, ...$$

$$x(1) = N\eta(1)$$

Recall the definition of the $N \times 1$ vector $v$ in (14). The sequence of $N \times 1$ random vectors $\{\eta_l(k)\}$, conditioned to $H_l$, is i.i.d. Vector $\eta_l(k)$ (under hypothesis $H_l$, $l = 0, 1$) is Gaussian with mean $m^{(l)}_{\eta}$ and covariance $S^{(l)}$:

$$m^{(l)}_{\eta} = (-1)^{(l+1)} \text{Diag}(v) \frac{1}{2}(m_1 - m_0)$$

$$S^{(l)} = \text{Diag}(v) S \text{Diag}(v).$$

Here $\text{Diag}(v)$ is a diagonal matrix with the diagonal entries equal to the entries of $v$.

B. Asymptotic optimality of the distributed detection algorithm

In this subsection, we present our main result, which states that the distributed detection via running consensus asymptotically achieves the performance of the optimal centralized detector, in the sense that it approaches the exponential error decay rate of the (asymptotically) optimal centralized detector.

Denote the probability measure of $x_i(k)$ under hypothesis $H_l$ with $\chi_{x_i(k)}^{(l)}$. First, we show that the sequence of measures $\{\chi_{x_i(k)}^{(l)}\}$, for all nodes $i$, satisfies a LDP in subsection IV after $\chi_{x_i(k)}^{(l)}$, for all nodes $i$, satisfies a LDP with good rate function; the rate function for all nodes $i$ is the same, and it is the same as the rate function of the optimal centralized detector in eqn. (16).

We now comment on the challenges in deriving the LDP for $\chi_{x_i(k)}^{(l)}$ (that is, for $\{x_i(k)\}$ under $H_l$). First, the sequence $\{x_i(k)\}$ (under $H_l$) is not i.i.d., it is a dependent sequence. Thus, showing the LDP is more challenging than if the sequence $\{x_i(k)\}$ was independent. Second, the weight matrices $W(k)$ (or, accordingly, the network topology) are random. If the network topology was static, then $x_i(k)$ (see eqn. (18) for the distributed detection algorithm) would be Gaussian random variable, for all $k$. However, the distribution of $x_i(k)$ in the presence of random topology is no longer Gaussian, and, thus, the LDP analysis becomes more involved.

We prove that the sequence of measures for $\{x_i(k)\}$ (under $H_l$, $l = 0, 1$) satisfies the LDP using the Gartner-Ellis Theorem from large deviations theory, see [13].
proof of Theorem 9 is left for the companion journal paper. We now state Theorem 9.

**Theorem 9** Let assumption 6 hold. The sequence of measures \( \{ \mathcal{I}_{i,k} \} \), for all nodes \( i \), satisfies the large deviations principle with good rate function. The rate function is the same as for the optimal centralized detector and is given by \( I(0) \) in eqn. (16).

**Corollary 10** (Chernoff lemma for the distributed detector: Asymptotic optimality) The local decision test \( T_{k,i} := \mathcal{I}_{\{x_{i,k} \geq 0\}} \), \( k = 1,2,... \), at each node \( i \), asymptotically optimal in the sense of Definition 5. The corresponding exponential decay rate of the Bayes probability of error, at each node \( i \), is given by:

\[
\lim_{k \to \infty} \frac{1}{k} \log P_{i,\text{dis}}(k) = -I(0)(0) \tag{22}
\]

Thus, the failure probability of a link \( \{ i,j \} \in E \) increases quadratically with the distance \( \delta_{ij} \) between sensors \( i \) and \( j \), and the maximal link failure probability is 0.5. As averaging weights, we use the standard time-varying Metropolis weights \( W_{ij}(k) \), defined for \( \{ i,j \} \in E \), \( i \neq j \), as \( W_{ij}(k) = 1/(1 + \max(d_i(k), d_j(k))) \), if the link \( \{ i,j \} \) is online at time \( k \), and 0 otherwise. The quantity \( d_i(k) \) represents the number of neighbors (i.e., the degree) of node \( i \) at time \( k \). Clearly, \( W_{ij}(k) \equiv 0 \) if \( \{ i,j \} \notin E \). The diagonal weights \( W_{ii}(k) \) are given by:

\[
W_{ii}(k) = 1 - \sum_{j \in \Omega_i(k)} W_{ij}(k).
\]

The link failures are spatially and temporally independent. This network and weight model satisfy Assumption 6.

**Remarks on Corollary 10.** Corollary 10 says that, for large \( k \) (i.e., in the asymptotic regime), the Bayes probability of error at node \( i \) behaves as: \( P_{i,\text{dis}}(k) \sim \exp(-I(0)(0)) \). That is, \( P_{i,\text{dis}}(k) \), for large \( k \), decays exponentially at the best possible rate, equal to the rate \( I(0)(0) \) of the (asymptotically) optimal centralized detector. This rate does not depend on the network connectivity, provided that the supergraph \( G \) is connected on average. Intuitively, an arbitrary network, connected on average, provides sufficient information flow to achieve asymptotic optimality. We now comment on the non asymptotic finite time regime. To this end, remark that \( P_{i,\text{dis}}(k) \), in the finite time regime, can be expressed as:

\[
P_{i,\text{dis}}(k) = F_i(k)e^{-I(0)(0)}, \quad \text{where} \quad \lim_{k \to \infty} \frac{1}{k} \log F_i(k) = 0 \tag{and thus, \( F_i(k) \) has no effect when \( k \) grows large.)
\]

The sequence \( \{ F_i(k) \} \) plays a role in a finite time regime; it clearly depends on the network topology and can be, in general, different for different nodes. Section VI addresses the non asymptotic regime. It shows by numerical example that, in the finite time regime, the sequence \( F_i(k) \) does not have a very large effect, and that best distributed detector over all \( N \) sensors in the network is very close to the optimal centralized detector in the finite time regime also. The proof of Corollary 10 and a more detailed finite time analysis is left for a companion journal paper.

**VI. SIMULATIONS**

In this section, we provide a simulation example, with two goals. The first is to corroborate numerically the validity of Corollary 10, thus showing that the running consensus detection algorithm is indeed asymptotically optimal in the sense of the best achievable Bayes probability of error decay rate. The second is to study the non asymptotic regime, i.e., the finite time regime. This will show that, although suboptimal in the finite time regime, the running consensus detection algorithm shows good performance.

**A. Simulation setup**

**Network model.** We consider a supergraph \( G \) with \( N = 40 \) nodes and \( M = 176 \) edges. Nodes are uniformly distributed on a unit square and nodes within distance less than a radius \( r \) are connected by an edge. We model the failure probability of a link \( \{ i,j \} \in E \) as follows:

\[
P_{ij} = c \cdot \frac{\delta_{ij}^2}{r^2}, \quad c = 0.5.
\]

Thus, the failure probability of a link \( \{ i,j \} \in E \) increases quadratically with the distance \( \delta_{ij} \) between sensors \( i \) and \( j \), and the maximal link failure probability is 0.5. As averaging weights, we use the standard time-varying Metropolis weights \( W_{ij}(k) \), defined for \( \{ i,j \} \in E \), \( i \neq j \), as \( W_{ij}(k) = 1/(1 + \max(d_i(k), d_j(k))) \), if the link \( \{ i,j \} \) is online at time \( k \), and 0 otherwise. The quantity \( d_i(k) \) represents the number of neighbors (i.e., the degree) of node \( i \) at time \( k \). Clearly, \( W_{ij}(k) \equiv 0 \) if \( \{ i,j \} \notin E \). The diagonal weights \( W_{ii}(k) \) are given by:

\[
W_{ii}(k) = 1 - \sum_{j \in \Omega_i(k)} W_{ij}(k).
\]

The link failures are spatially and temporally independent. This network and weight model satisfy Assumption 6.

**Data model: Monte Carlo simulations.** We randomly generate the distribution parameters of the observations \( y(k) \); that is, \( m_1, m_0, \) and \( S \) are generated randomly. Prior probabilities are \( P(H_0) = P(H_1) = 0.5 \); thus, both in finite time \( k \) and in the asymptotic regime, the (centralized) LLR test with zero threshold is Bayes optimal. We perform 10,000 Monte Carlo simulations to numerically estimate the Bayes error probabilities \( P_{\text{cen}}(k) \) and \( P_{i,\text{dis}}(k) \).

**B. Asymptotic analysis**

This subsection verifies Corollary 10. To this end, we estimate the following quantity:

\[
\epsilon(k) = \frac{1}{k} \log P_{i,\text{cen}}^e(k) - \max_{i=1,\ldots,N} \left\{ \frac{1}{k} \log P_{i,\text{dis}}^e(k) \right\} \tag{23}
\]

The quantity \( \epsilon(k) \) says how far is the worst sensor at time \( k \) from the optimal centralized detector, in terms of the error probability decay rate. According to Corollary 10, \( \epsilon(k) \) converges to zero as \( k \to \infty \). Figure 1 (top) plots \( \epsilon(k) \) along iterations. We can see that it indeed converges to zero.

**C. Non asymptotic–finite time analysis**

This subsection examines the non asymptotic regime, i.e., the case of finite time \( k \). We consider the following quantities: \( \max_{i=1,\ldots,N} P_{i,\text{dis}}(k), \min_{i=1,\ldots,N} P_{i,\text{dis}}(k) \)
networks. In running consensus, the state at each node spatially correlated Gaussian noise and randomly varying performance of the running consensus detection algo-

\[ P_{\text{cen}}(k) = \max_{i=1,\ldots,N} P_{i,\text{cen}}(k) \] (maximal distributed), \( P_{\text{dis}}(k) \) (minimal distributed) and \[ \frac{1}{N} \sum_{i=1}^{N} F_{i,\text{dis}}(k) \] (average distributed).

and \[ \frac{1}{N} \sum_{i=1}^{N} F_{i,\text{dis}}(k) \]. That is, we consider the maximal, the minimal, and the average error probability across all nodes in the network. Figure 1 (bottom) plots these quantities together with the Bayes probability of error for the centralized detector, \( P_{\text{cen}}(k) \). Naturally, in the finite time regime, the decentralized detector is suboptimal. We can see that the hidden factors \( F_i(k) \) (discussed in subsection V-B, last paragraph) show an effect in the finite time regime. However, we can see that the minimal probability of error across nodes, \( \min_{i=1,\ldots,N} P_{i,\text{dis}}(k) \), for the distributed detector, is very close to \( P_{\text{cen}}(k) \), even in the finite time regime, and even when \( k \) is very small.

VII. SUMMARY

We applied large deviations theory to analyze the performance of the running consensus detection algorithm, a distributed detection algorithm. We consider spatially correlated Gaussian noise and randomly varying networks. In running consensus, the state at each node is updated at each time step by: 1) exchanging information with the immediate neighbors in the network; and 2) incorporating into the decision process new local observations. The underlying network topology is random. We showed that running consensus asymptotically approaches the optimal centralized detector. That is, the Bayes probability of detection error at each sensor decays exponentially at the best achievable rate, the Chernoff information. We complemented our theoretical findings in the asymptotic regime with numerical examples for the non asymptotic (finite time \( k \)) regime. Numerical examples reveal that the best distributed detector–sensor, among all sensors in the network, is very close to the optimal centralized detector (in terms of Bayes probability of error) even when \( k \) is small.

REFERENCES


