**Abstract.** In this paper we introduce a multi-stage decision making procedure where decision makers’ opinions are weighted by their contribution to the agreement after they sort alternatives into a fixed finite scale given by linguistic categories, each one having an associated numerical score. We add scores obtained for each alternative using an aggregation operator. Based on distances among vectors of individual and collective scores, we assign an index to decision makers showing their contributions to the agreement. Opinions of negative contributors are excluded and the process is reinitiated until all decision makers contribute positively to the agreement. To obtain the final collective weak order on the set of alternatives, we weigh the scores that decision makers assign to alternatives by indices corresponding to their contribution to the agreement.

**Keywords:** group decision making, consensus, aggregation operators, metrics

1. **Introduction**

Some decision problems require a group of decision makers to rank a set of alternatives to generate a collective decision. This outcome is usually obtained by applying a given decision procedure that aggregates individual opinions. If the number of alternatives is high, decision makers may find it difficult to rank feasible alternatives. According to Dummett [5], “If there are, say, twenty possible outcomes, the task of deciding the precise order of preference in which they rank them may induce a kind of psychological paralysis in the voter”.

To help decision makers arrange the alternatives, we propose that they sort alternatives within a given finite scale in which each term is identified with a linguistic category—e.g., very good, good, regular, bad, very bad—. We assign a score to each linguistic category, then associate a collective score with each alternative by using an aggregation operator that works on individual scores. Alternatives are thus ordered based on collective scores.

After this first stage, we introduce a distance between the collective weak order obtained and each individual weak order. Through these distances, we propose an index for measuring the overall contribution of each decision maker to the agreement. Opinions of decision makers with not positive indices are excluded, then we reinitiate the process with only the opinions of individuals contributing positively to the agreement. We repeat this procedure by recalculating new collective orders and overall indices until we obtain a final subset of decision makers who all contribute positively to the agreement, then weigh the scores that decision makers indirectly assign to alternatives by their overall contribution to the agreement indices, to obtain the final collective ranking of alternatives.

When all individual assessments have the same importance, the collective decision could be distorted by extravagant or spiteful opinions. Some individuals, in fact, could have reasons to conceal sincere opinions to favor or penalize certain alternatives. Consensus, however, implies some effort for moving individual opinions toward a majority position. It thus makes sense to weigh individual assessments in order of their contribution to the agreement. Weighting individual opinions with the above indices, then decision makers have an incentive not to declare opinions very divergent from the majority opinion to avoid being penalized by reducing their influence over the collective ranking or being expelled from the group.

2. **Preliminaries**

Let \( V = \{v_1, \ldots, v_m\} \) a set of decision makers (or voters) who show their preferences on the pairs of a set of alternatives \( X = \{x_1, \ldots, x_p\} \), with \( m, n \geq 3 \). \( \mathcal{P}(V) \) denotes the power set of \( V \) \( (I \in \mathcal{P}(V) \iff I \subseteq V) \). Linear orders are complete, antisymmetric and transitive binary relations, and weak orders (or complete preorders) are complete and transitive binary relations. With \(|I|\), we denote the cardinal of any set \( I \).

We consider that each decision maker classifies alternatives within a fixed finite scale given by a set of linguistic categories \( \mathcal{L} = \{l_1, \ldots, l_p\} \), with \( p \geq 2 \), equipped with a linear order \( l_1 > \cdots > l_p \). Under this assumption decision makers are not totally free for declaring their preferences. However, the common language they use enables us to aggregate their opinions robustly (see Balinski and Laraki [1]).
The *individual assignment* of decision maker $v_i$ is a mapping $C_i: X \rightarrow \mathcal{L}$ that assigns a linguistic category $C_i(x_a) \in \mathcal{L}$ to each alternative $x_a \in X$. Associated with $C_i$, we consider the weak order $R_i$ defined by $x_a R_i x_b$ if $C_i(x_a) \geq C_i(x_b)$.

A *profile* is a vector $C = (C_1, \ldots, C_m)$ of individual assignments. We use $\mathcal{C}$ to denote the set of profiles. We assume that every linguistic category $I_k \in \mathcal{L}$ has an associated score $s_k \in [0, \infty)$ in such a way that $s_1 \geq s_2 \geq \cdots \geq s_p$ and $s_1 > s_p = 0$.

For the decision maker $v_i$, let $S_i: X \rightarrow [0, \infty)$ be the *scoring mapping* that assigns the score to each alternative:

$$S_i(x_a) = s_k \Leftrightarrow C_i(x_a) = I_k.$$

The *scoring vector* of $v_i$ is $(S_i(x_1), \ldots, S_i(x_n))$.

An aggregation operator

$$F : [0, \infty]^m \rightarrow [0, \infty]$$

that satisfies the following two conditions:

1. *Idempotency*: $F(a, \ldots, a) = a$.
2. *Monotonicity*: if $a_i \leq b_i$ for every $i \in \{1, \ldots, m\}$, then $F(a_1, \ldots, a_m) \leq F(b_1, \ldots, b_m)$.

Note that from monotonicity, idempotency is equivalent to *compensativeness*:

$$\min\{a_1, \ldots, a_m\} \leq F(a_1, \ldots, a_m) \leq \max\{a_1, \ldots, a_m\}.$$

Interesting examples of aggregation operators belong to the very general class of OWA (Ordered Weighted Averaging) operators introduced by Yager [8], e.g., average, median and oligarchic aggregators –the average of intermediate values after excluding highest and lowest values—see also Yager and Kacprzyk [9], Fodor and Roubens [7, Chapter 5], and Calvo, Kolesárová, Komorníková and Mesiar [3, 4, 2].

Given an aggregation operator $F$, we assign a *collective score* to each alternative by means of the *scoring mapping*

$$S^F: X \rightarrow [0, \infty]$$

defined by

$$S^F(x_a) = F(S_1(x_a), \ldots, S_m(x_a)),$$

for every alternative $x_a \in X$.

For each alternative $x_a \in X$, we associate a *collective scoring vector* $(S^F(x_1), \ldots, S^F(x_n))$. The *collective weak order* on $X$, $R^F$, is given by $x_a R^F x_b$ if $S^F(x_a) \geq S^F(x_b)$.

### 3. Agreement measures

Once decision makers show their opinions on alternatives and we generate a collective weak order by using an aggregation operator, it is interesting to know the distance of each decision maker to the collective opinion. After this, we introduce an agreement measure that shows us what is the agreement in each subset of decision makers.

### Definition 1.

An agreement measure $\mathcal{M}: \mathcal{C} \times \mathcal{P}(V) \rightarrow [0, 1]$ is a mapping that assigns a number in the unit interval to each profile and each subset of decision makers, and satisfies the following conditions:

1. **Unanimity**: $\mathcal{M}(C, V) = 1 \Leftrightarrow C_1 = \cdots = C_m$.
2. **Anonymity**: For every permutation $\pi$ on $\{1, \ldots, m\}$ and $(C, I) \in \mathcal{C} \times \mathcal{P}(V)$ it holds

$$\mathcal{M}(C, I) = \mathcal{M}(C_\pi, I_\pi),$$

where $C_\pi = (C_{\pi(1)}, \ldots, C_{\pi(m)})$ and $v_i \in I_\pi$ if and only if $v_{\pi(i)} \in I$, i.e., $I_\pi = \{ v_{\pi(i)} \mid v_i \in I \}$.
3. **Neutrality**: For every permutation $\sigma$ on $\{1, \ldots, n\}$ and $(C, I) \in \mathcal{C} \times \mathcal{P}(V)$ it holds

$$\mathcal{M}(C, I) = \mathcal{M}(C^\sigma, I),$$

where $C^\sigma = (C^\sigma_1, \ldots, C^\sigma_m)$ is the profile obtained from $C$ by relabeling the alternatives according to $\sigma$, i.e., $C^\sigma_i(x_{\sigma(u)}) = C_i(x_u)$ for all $i \in \{1, \ldots, m\}$ and $u \in \{1, \ldots, n\}$.

Unanimity means that the maximum agreement in the set of all decision makers is only achieved when all opinions are the same. Anonymity requires symmetry with respect to decision makers, and neutrality means symmetry with respect to alternatives.

Given a metric $d : [0, \infty]^n \times [0, \infty]^n \rightarrow [0, \infty]$ and $S, S'$ individual or collective scoring mappings, we denote

$$d(S, S') = d ((S(x_1), \ldots, S(x_n)), (S'(x_1), \ldots, S'(x_n))).$$

### Definition 2.

Given a metric $d$ and an aggregation operator $F$, the mapping $\mathcal{M}^F_d: \mathcal{C} \times \mathcal{P}(V) \rightarrow [0, 1]$ defined by

$$\mathcal{M}^F_d(C, I) = \begin{cases} 1 - \frac{\sum_{i \in I} d(S_i, S^F)}{|I| \cdot \Delta}, & \text{if } I \neq \emptyset, \\ 0, & \text{if } I = \emptyset, \end{cases}$$

where $\Delta = d((s_1, \ldots, s_1), (0, \ldots, 0))$, is the overall agreement measure based on $d$ and $F$.

Note that $\Delta$ is the maximum distance among scoring vectors. Consequently, $\mathcal{M}^F_d(C, I) \in [0, 1]$ for every $(C, I) \in \mathcal{C} \times \mathcal{P}(V)$. It is easy to see that $\mathcal{M}^F_d$ satisfies the properties required to be an agreement measure.

We now introduce indices that measure the overall contribution to the agreement of each decision maker. Using these indices, we modify the initial group decision procedure for prioritizing consensus.

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1. Aggregation operators are usually defined in the unit interval. See Fodor and Roubens [7, Chapter 5], and Calvo, Kolesárová, Komorníková and Mesiar [3, 4, 2].

2. This notion was widely explored by Bosch [2] for linear orders.

3. Consensus has different meanings, one of which is related to an interactive and sequential procedure where some decision makers must change their preferences to improve agreement. Usually, a moderator advises decision makers to modify opinions. See, for instance, Eklund, Rusinowska and de Swart [6].
Definition 3. Given an agreement measure $\mathcal{A}$, we define the overall contribution to the agreement of the decision maker $v_i$ as

$$w_i(C) = \sum_{I \subseteq V} \left( \mathcal{A}(C, I) - \mathcal{A}(C, I \setminus \{v_i\}) \right),$$

for every profile $C \in \mathcal{C}$.

Clearly, if $v_i \notin I$, then $\mathcal{A}(C, I) - \mathcal{A}(C, I \setminus \{v_i\}) = 0$. If $w_i(C) > 0$, we say that decision maker $v_i$ contributes positively to the agreement; and if $w_i(C) < 0$, we say that decision maker $v_i$ contributes negatively to the agreement.

We now introduce a new collective preference by weighting scores that decision makers assign indirectly to alternatives with the corresponding overall contribution to the agreement indices.

Definition 4. Given an aggregation operator $F$, the collective weak order $R^w$ is defined by

$$x_u R^w x_v \iff S^w(x_u) \geq S^w(x_v),$$

where

$$S^w(x_u) = F\left(w_1 \cdot S_1(x_u), \ldots, w_m \cdot S_m(x_u)\right).$$

Note that if we use the vector of the corresponding overall contribution to the agreement indices, we prioritize decision makers in order of their contribution to the agreement, as suggested by Cook, Kress and Seiford [4].

4. The multi-stage group decision procedure

Given an aggregation operator $F$ and an agreement measure $\mathcal{A}$, we propose the following multi-stage decision making procedure.

1. Decision makers $V = \{v_1, \ldots, v_m\}$ sort alternatives of $X = \{x_1, \ldots, x_n\}$ based on the linguistic categories $\mathcal{L} = \{l_1, \ldots, l_n\}$. So, we have individual weak orders $R_1, \ldots, R_m$ which rank alternatives within the fixed set of linguistic categories.

2. Taking into account scores $s_1, \ldots, s_n$ associated with $l_1, \ldots, l_n$, a score is assigned to each alternative for every decision maker: $S_i(x_u)$, for $i = 1, \ldots, m$ and $u = 1, \ldots, n$.

3. A collective score is obtained for each alternative:

$$S^F(x_u) = F\left(S_1(x_u), \ldots, S_m(x_u)\right)$$

and we rank alternatives through the collective weak order $R^F$:

$$x_u R^F x_v \iff S^F(x_u) \geq S^F(x_v).$$

4. We calculate overall contributions to the agreement (Definition 3) for all decision makers: $w_1(C), \ldots, w_m(C)$.

a) If $w_i(C) \geq 0$ for every $i \in \{1, \ldots, m\}$, then we obtain new collective scores by:

$$S^w(x_u) = F\left(w_1 \cdot S_1(x_u), \ldots, w_m \cdot S_m(x_u)\right),$$

for $u = 1, \ldots, n$, and rank alternatives by using the collective weak order $R^w$:

$$x_u R^w x_v \iff S^w(x_u) \geq S^w(x_v).$$

This process is equivalent to that for with normalized overall contributions to the agreement:

$$w'_i(C) = \frac{w_i(C)}{w_1(C) + \cdots + w_m(C)}, \quad i = 1, \ldots, m.$$

Notice that now $w'_1(C) + \cdots + w'_m(C) = 1$.

b) Otherwise, we exclude opinions of decision makers whose overall contributions to the agreement are negative and initiate the decision procedure for remaining decision makers

$$V' = \{v_i \in V \mid w_i(C) \geq 0\}.$$

The process continues until all decision makers contribute positively to the agreement. At this stage, we apply the previous procedure (4a) to obtain the final ranking on the set of alternatives.

4.1. An illustrative example

Consider five decision makers who sort alternatives of $X = \{x_1, \ldots, x_7\}$ based on the set of linguistic categories $\mathcal{L} = \{l_1, \ldots, l_6\}$ and associated scores given in Table 1. Table 2 shows assessments of decision makers and corresponding weak orders.

Table 1. Linguistic categories.

<table>
<thead>
<tr>
<th>$\mathcal{L}$</th>
<th>Meaning</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l_1$</td>
<td>Excellent</td>
<td>$s_1 = 10$</td>
</tr>
<tr>
<td>$l_2$</td>
<td>Very good</td>
<td>$s_2 = 7$</td>
</tr>
<tr>
<td>$l_3$</td>
<td>Good</td>
<td>$s_3 = 4$</td>
</tr>
<tr>
<td>$l_4$</td>
<td>Regular</td>
<td>$s_4 = 2$</td>
</tr>
<tr>
<td>$l_5$</td>
<td>Bad</td>
<td>$s_5 = 1$</td>
</tr>
<tr>
<td>$l_6$</td>
<td>Very bad</td>
<td>$s_6 = 0$</td>
</tr>
</tbody>
</table>

Table 3 gives individual scores and two collective scores: $S^d$ for the average and $S^o$ for the olympic aggregator. Table 4 shows collective weak orders provided by the average $R^d$ and the olympic aggregator $R^o$.

Case 1. First we use the average as an aggregation operator, the Euclidean (or $L_2$) metric,

$$d((a_1, \ldots, a_n), (b_1, \ldots, b_n)) = \sqrt{\sum_{i=1}^{n}(a_i - b_i)^2},$$

and its associated agreement measure.

Taking into account distance between individual and collective scores, we obtain the following overall contributions to the agreement:
We now obtain the following normalized indices:

\[ w_1(C) = 0.735548827 \]
\[ w_2(C) = 0.414899002 \]
\[ w_3(C) = 1.132602066 \]
\[ w_4(C) = -0.225015149 \]
\[ w_5(C) = -0.511082118 \]

Then, \( w_3(C) > w_1(C) > w_2(C) > 0 > w_4(C) > w_5(C) \).

Since decision makers \( v_4 \) and \( v_5 \) contribute negatively to the agreement, they are expelled from the group and we restart the process with the first three decision makers. The new collective scores are: \( S^A(x_1) = 7, S^A(x_2) = 8, S^A(x_3) = 6, S^A(x_4) = 2, S^A(x_5) = 3.333, S^A(x_6) = 1.666, S^A(x_7) = 1.333 \), quite different from those obtained in the first iteration (Table 3).

The new overall contributions to the agreement are:

\[ w_1(C) = 1.0043468 \]
\[ w_2(C) = 0.7587926 \]
\[ w_3(C) = 0.82545947 \]

Now all decision makers contribute positively to the agreement, but now the first decision maker contributes more to the agreement than the third one, just the opposite of that in the initial stage where opinions of all decision makers were taken into account.

We now obtain the following normalized indices:

\[ w'_1(C) = 0.38798858 \]
\[ w'_2(C) = 0.29312869 \]
\[ w'_3(C) = 0.31888273 \]

Taking into account these normalized indices, we generate the collective scores \( S^{w'}(x_u) \), for \( u = 1, \ldots, n \), by using

\[ w'_1(C) \cdot S_1(x_u) + w'_2(C) \cdot S_2(x_u) + w'_3(C) \cdot S_3(x_u) \]

Multiplying these collective scores by 3, yields normalized collective scores within the interval of original scores associated with linguistic categories, \([0, 10]\). We thus obtain the normalized collective scores in Table 5.

After this normalization, if we take into account scores associated with linguistic categories, we obtain both a ranking and a classification of the alternatives. In this sense, \( x_1 \) and \( x_2 \) are located between very good and excellent, \( x_3 \) between good and very good, \( x_4 \) and \( x_5 \) between regular and good, and \( x_6 \) and \( x_7 \) between bad and regular.

Table 6 shows initial and final orders. Note that the final outcome differs from the initial one due to the exclusion of decision makers \( v_4 \) and \( v_5 \) and by weights we apply to scores given by the first three decision makers.

<table>
<thead>
<tr>
<th>Initial (weak) order</th>
<th>Final (linear) order</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_3 )</td>
<td>( x_2 )</td>
</tr>
<tr>
<td>( x_2 \ x_5 )</td>
<td>( x_1 )</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>( x_3 )</td>
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<tr>
<td>( x_6 )</td>
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<td>( x_4 )</td>
<td>( x_4 )</td>
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<tr>
<td>( x_7 )</td>
<td>( x_6 )</td>
</tr>
<tr>
<td>( x_7 )</td>
<td>( x_7 )</td>
</tr>
</tbody>
</table>

Case 2. We now use the olympic aggregation operator,
the Manhattan (or $L_1$) metric,
\[
d((a_1,\ldots,a_n),(b_1,\ldots,b_n)) = \sum_{i=1}^{n} |a_i-b_i|,
\]
and its associated agreement measure.

Taking into account distances between individual and collective scores, we obtain the following overall contributions to the agreement:
\[
\begin{align*}
    w_1(C) &= 1.40666667 \\
    w_2(C) &= 1.22809524 \\
    w_3(C) &= 1.19238095 \\
    w_4(C) &= -0.45047619 \\
    w_5(C) &= -1.1115873
\end{align*}
\]
Then, $w_1(C) > w_2(C) > w_3(C) > 0 > w_4(C) > w_5(C)$, quite different from Case 1. As in Case 1, $v_4$ and $v_5$ contribute negatively to the agreement, so they are expelled from the group, and we reiniate the process with the first three decision makers. The new collective scores are: $S^O(x_1) = 7$, $S^O(x_2) = 7$, $S^O(x_3) = 7$, $S^O(x_4) = 2$, $S^O(x_5) = 4$, $S^O(x_6) = 3$, $S^O(x_7) = 1$.

New overall contributions to the agreement are:
\[
\begin{align*}
    w'_1(C) &= 0.39122807 \\
    w'_2(C) &= 0.24912281 \\
    w'_3(C) &= 0.35964912
\end{align*}
\]
All decision makers now contribute positively to the agreement, yielding the following normalized indices
\[
\begin{align*}
    w'_1(C) &= 1.06190476 \\
    w'_2(C) &= 0.67619048 \\
    w'_3(C) &= 0.97619048
\end{align*}
\]
We now weight individual scores by using these indices and obtain the outcome given by the olympic aggregator, multiplied by 3 for normalization, as in the previous case. Table 7 shows normalized collective scores.

| $x_1$ | 7.33157895 |
| $x_2$ | 8.17368421 |
| $x_3$ | 6.25263158 |
| $x_4$ | 2.22105263 |
| $x_5$ | 3.50175439 |
| $x_6$ | 1.75087719 |
| $x_7$ | 1.24912281 |

As in Case 1, $x_1$ and $x_2$ are located between very good and excellent, $x_3$ between good and very good, $x_4$ and $x_5$ between regular and good, and $x_6$ and $x_7$ between bad and regular.

Table 8 shows initial and final orders. Note that the final order is the same in Cases 1 and 2. The final and initial outcomes thus differ. Again, this is due to the exclusion of decision makers $v_4$ and $v_5$, and by the weights we apply to scores given by the first three decision makers.

### Table 8. Case 2. Collective orders.

<table>
<thead>
<tr>
<th>Initial (weak) order</th>
<th>Final (linear) order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_2$ $x_3$ $x_5$</td>
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<tr>
<td>$x_1$</td>
<td>$x_1$</td>
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<td>$x_4$</td>
<td>$x_3$</td>
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<tr>
<td>$x_6$</td>
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</tbody>
</table>

Note that in Cases 1 and 2 we have used different metrics and aggregation operators, yet in both cases, decision makers $v_4$ and $v_5$ are expelled from the group in the first iteration, and the final linear order and classification of alternatives coincide in analyzed cases. This may not, however, necessarily hold for every metric and aggregation operator.

### 5. Conclusions

To facilitate decision makers the task of ranking a high number of alternatives, we have proposed that decision makers sort alternatives through a small set of linguistic categories. We associate a score with each linguistic category, then decision makers indirectly assign a score to each alternative.

Given an aggregation operator, we obtain a collective score for each alternative and then a collective weak order on the set of alternatives. Given a metric, we calculate distances between individual and collective weak orders, using these distances to assign an index to decision makers that measures their overall contribution to the agreement. Decision makers who contribute negatively to the agreement are expelled from the group and we iterate the decision procedure until all decision makers contribute positively to the agreement.

Taking into account the indices we obtain in the final step, we weight individual scores and obtain final collective ranking of alternatives. After normalization, we sort the alternatives by using the linguistic categories that individuals use in their assessments. The final collective ranking we obtain after implementing the proposed multi-stage decision procedure yields valuable information about alternatives.

Note the following regarding our multi-stage decision making procedure:

- The procedure is flexible: different linguistic categories with different associated scores can be used. We can also consider different aggregation operators and metrics, and agreement measures different from that in Definition 2 can be also used.
Due to the overall contribution to the agreement indices, which multiply individual scores, are usually irrational numbers, so when the number of decision makers is high, it is unlikely that the decision procedure provides ties among alternatives.

Our proposal requires only a single judgment for each individual about alternatives.

Since the proposed decision procedure penalizes individuals who are far from majority positions, this provides incentives for decision makers to moderate their opinions which, otherwise, may be excluded or underestimated.

All decision makers use a common language, which avoids problems arising when purely ordinal methods are used (see Balinski and Laraki [1]).

It is easy to implement computer programs for generating the final ranking on the set of alternatives.

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