Asymptotics of the Modified Bessel and the Incomplete Gamma Matrix Functions

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Abstract—in this paper, an asymptotic expression of the incomplete gamma matrix function and integral expressions of Bessel matrix functions are given. Results are applied to study the asymptotic behavior of the modified Bessel function. © 2003 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

The connection between the modified Bessel matrix function and Laguerre matrix polynomials has been established in [1, pp. 44-46]. The aim of this paper is to study the asymptotics of the modified Bessel matrix function $I_A(z)$. These results are fundamental in a further study of developments in series of Laguerre matrix polynomials.

This paper is organized as follows. Section 2 deals with the introduction and asymptotics of the incomplete gamma matrix function. Integral expressions of Bessel matrix functions are given in Section 3. Results of Sections 2 and 3 are used in Section 4 to study the asymptotics of the modified Bessel matrix function.

Throughout this paper, for a matrix $A$ in $\mathbb{C}^{r \times r}$, its 2-norm will be denoted by

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2},$$

where for any vector $y$ in $\mathbb{C}^r$, $\|y\|_2 = (y^Hy)^{1/2}$ is the Euclidean norm of $y$. We denote by $\mu(A)$ the logarithmic norm of $A$, defined by [2,3]

$$\mu(A) = \max \left\{ z ; z \text{ eigenvalue of } \frac{A + A^H}{2} \right\}.$$
We denote by $\bar{\mu}$ the number

$$\bar{\mu}(A) = \min \left\{ \lambda; \lambda \text{ eigenvalue of } \frac{A + A^H}{2} \right\}.$$ 

By [2, p. 647], it follows that $\|e^{At}\| \leq e^{\mu(A)}$ for $t \geq 0$. Hence, $t^A = \exp(A \ln t)$ satisfies

$$\|t^A\| \leq \begin{cases} t^{\mu(A)}, & \text{if } t \geq 1, \\ t^\bar{\mu}(A), & \text{if } 0 \leq t \leq 1. \end{cases}$$

(1.1)

If $A$ is a matrix such that $A + nI$ is invertible for all integer $n \geq 0$, then $\Gamma(A)$ is invertible and by expressions (4) and (5) of [1], it follows that

$$(A)_n = A(A + I) \cdots (A + (n - 1)I) = \Gamma(A + nI)\Gamma^{-1}(A), \quad n \geq 1. \quad (1.2)$$

Let $A \in \mathbb{C}^{r \times r}$, $F(z)$ a matrix function, and let $g(z)$ be a positive scalar function. We say that $F(z)$ behaves $O(g(z), A)$ in a domain $\Omega$, if $\|F(z)\| \leq M(A)g(z)$, for some positive constant $M(A), z \in \Omega$, and $F(z)$ commutes with $A$.

2. THE INCOMPLETE GAMMA MATRIX FUNCTION OF REAL ARGUMENT

The gamma matrix function has recently been treated in [4]. In this section, the incomplete gamma matrix functions are introduced. Expressions of the incomplete gamma function $\gamma(z, x)$ and its complement $\Gamma(z, x)$ given in [5, p. 151] suggest the following definition.

**DEFINITION 2.1.** Let $B$ be a matrix in $\mathbb{C}^{r \times r}$ such that

$$\bar{\mu}(B) > 0,$$

and let $x$ be a positive real number. Then the incomplete gamma matrix function $\gamma(B, x)$ and its complement are defined by

$$\gamma(B, x) = \int_0^x t^{B-l} dt, \quad \Gamma(B, x) = \int_x^\infty e^{-t} t^{B-l} dt. \quad (2.2)$$

By inequality (1.1) and [5, pp. 1,151, the expressions of $\gamma(B, x)$ and $\Gamma(B, x)$ are well defined and

$$\Gamma(B) = \gamma(B, x) + \Gamma(B, x). \quad (2.3)$$

The next result provides an asymptotic expression of $\Gamma(B, x)$.

**THEOREM 2.1.** Let $B$ be a matrix in $\mathbb{C}^{r \times r}$, let $x$ be a positive real number, and let $p$ be a nonnegative integer. Then one gets the asymptotic expression

$$\Gamma(B, x) = e^{-x} x^{B-l} \left[ \sum_{k=0}^{p} \frac{(-1)^k (I - B)_k}{x^k} + O \left( x^{-p-1}, B \right) \right]. \quad (2.4)$$

**PROOF.** Given $x > 0$, let us consider the substitution $u = t - x$ into expression (2.2) of $\Gamma(B, x)$ and the remainder after $p + 1$ terms in a Taylor series of the function $(1 + s)^{B-l}$ given in [6, p. 49]. Then one gets

$$\Gamma(B, x) = e^{-x} x^{B-l} \int_0^\infty e^{-u} \left[ \sum_{k=0}^{p} \frac{(I - B)_k}{k!} \left( \frac{-u}{x} \right)^k + \frac{(-1)^{p+1}}{p!} \left( \frac{u}{x} \right)^{p+1} \left( \int_0^1 \left( 1 - \frac{w}{x} \right)^{B-(p+2)l} dw \right) \right] du = e^{-x} x^{B-l} \left[ \sum_{k=0}^{p} \frac{(-1)^k (I - B)_k}{x^k} + R_{p+1} \right], \quad (2.5)$$

where $R_{p+1} = O \left( x^{-p-1}, B \right)$. \hfill \Box
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\[ R_{p+1} = \frac{(-1)^{p+1} (I - B)_{p+1}}{p!} \int_0^\infty \frac{u^{p+1} e^{-u}}{x^{p+1}} \left( \int_0^1 (1 - v)^p \left( 1 + \frac{uv}{x} \right)^{B-(p+2)} dv \right) du. \]  

(2.6)

Taking into account (1.1), from (2.6) it follows that

\[ \|R_{p+1}\| \leq \frac{\|(I - B)_{p+1}\|}{p! x^{p+1}} \int_0^\infty u^{p+1} e^{-u} \left( \int_0^1 (1 - v)^p \left( 1 + \frac{uv}{x} \right)^{\mu(B)-(p+2)} dv \right) du. \]  

(2.7)

If \( \mu(B) \leq p + 2 \), by (2.7), one gets

\[ \|R_{p+1}\| \leq \frac{\|(I - B)_{p+1}\|}{p! x^{p+1}} \int_0^1 (1 - v)^p dv \int_0^\infty u^{p+1} e^{-u} du = \frac{\|(I - B)_{p+1}\|}{x^{p+1}} = O(x^{-p-1}, B). \]  

(2.8)

If \( \mu(B) > p + 2 \), then by (2.7) and (1.1), one gets

\[ \|R_{p+1}\| \leq \frac{\|(I - B)_{p+1}\|}{p! x^{p+1}} \int_0^1 (1 - v)^p dv \int_0^\infty u^{p+1} e^{-u}(1 + u)^{\mu(B)-(p+2)} du = O(x^{-p-1}, B). \]  

(2.9)

By (2.5), (2.8), and (2.9) the result is established.

3. INTEGRAL EXPRESSIONS OF BESSEL MATRIX FUNCTIONS

In this section, we provide integral expressions of Bessel matrix functions \( J_A(z) \) and \( I_A(z) \) where \( A \) is a matrix in \( \mathbb{C}^{r \times r} \) such that \( \mu(A) > -1/2 \) and \( z \) is a complex number in \( \mathbb{D}_0 = \mathbb{C} \setminus [-\infty, 0] \).

By Lemma 2 of [4, p. 209], if \( k > -1/2 \), one gets

\[ \Gamma^{-1}(A + (k + 1)I) = \frac{\Gamma^{-1}(A + I/2)}{\Gamma(k + 1/2)} \int_{-1}^1 t^{2k} \left( 1 - t^2 \right)^{A-I/2} dt, \]

and substituting the above expression into the series expression of the Bessel matrix function given in [7, p. 137], it follows that

\[ J_A(z) = \sum_{k \geq 0} \frac{(-1)^k (z/2)^{2k}}{\Gamma(k + 1)} \Gamma^{-1}(A + (k + 1)I) (z/2)^{A+2kI} \]

(3.1)

\[ = \left( \frac{z}{2} \right)^A \frac{\Gamma^{-1}(A + I/2)}{\Gamma(k + 1/2)} \sum_{k \geq 0} \frac{(-1)^k (z/2)^{2k}}{\Gamma(k + 1)
\Gamma(k + 1/2)} \int_{-1}^1 t^{2k} \left( 1 - t^2 \right)^{A-I/2} dt. \]

Note that by (1.1) and [5, p. 114], one gets

\[ \sum_{k \geq 0} \frac{|z/2|^{2k}}{\Gamma(k + 1)
\Gamma(k + 1/2)} \int_{-1}^1 t^{2k} \left( 1 - t^2 \right)^{A-I/2} dt \leq \frac{|z|}{2} \left( \frac{\mu(A)}{\mu(A) + 1/2} \right)^{1/2} \]

(3.2)

By the dominated convergence theorem [8, p. 84], (3.1),(3.2) and taking into account the identity \( 2^{2k} \Gamma(k + 1) \Gamma(k + 1/2) = \sqrt{\pi} (2k) [5, p. 114] \), it follows that

\[ J_A(z) = \left( \frac{z}{2} \right)^A \frac{\Gamma^{-1}(A + I/2)}{\sqrt{\pi}} \int_{-1}^1 (1 - t^2)^{A-I/2} \sum_{k \geq 0} \frac{(-1)^k (zt)^{2k}}{(2k)!} dt \]

(3.3)

\[ = \left( \frac{z}{2} \right)^A \frac{\Gamma^{-1}(A + I/2)}{\sqrt{\pi}} \int_{-1}^1 (1 - t^2)^{A-I/2} \cos(zt) dt, \quad |\arg(z)| < \pi, \quad \mu(A) > -\frac{1}{2}. \]
Starting from the scalar analytic equations of the parameter $\nu$ [5, p. 109]

$$I_\nu(z) = e^{-\nu \pi i / 2} J_\nu \left( z e^{\pi i / 2} \right), \quad -\pi < \arg(z) < \frac{\pi}{2},$$

$$I_\nu(z) = e^{\nu \pi i / 2} J_\nu \left( z e^{-\pi i / 2} \right), \quad -\pi < \arg(z) < \pi,$$

and by applying the matrix functional calculus [9, Chapter 11], one gets

$$I_A(z) = e^{-A \pi i / 2} J_A \left( z e^{\pi i / 2} \right), \quad -\pi < \arg(z) < \frac{\pi}{2},$$

$$I_A(z) = e^{A \pi i / 2} J_A \left( z e^{-\pi i / 2} \right), \quad -\pi < \arg(z) < \pi.$$  \hfill (3.4)

By (3.3) and (3.4), if $-\pi < \arg(z) < \pi / 2$, it follows that

$$I_A(z) = \left( \frac{z}{2} \right)^A I_\nu^{-1}(A + I/2) \sqrt{\pi} \int_{-1}^{1} (1 - t^2)^{A-1/2} \cosh(zt) dt.$$ \hfill (3.6)

In an analogous way, by (3.3) and (3.5) the same result is obtained for $-\pi / 2 < \arg(z) < \pi$. Thus, (3.6) holds for $|\arg(z)| < \pi$, $\mu(A) > -1/2$. Making the substitution $t = 1 - 2u^2$ in (3.6), it follows that

$$I_A(z) = \left( \frac{z}{2} \right)^A I_\nu^{-1}(A + I/2) \sqrt{\pi} \left[ \int_0^1 u^2 A (1 - u^2)^{A-1/2} e^{z(1-2u^2)} du + \int_0^1 u^2 A (1 - u^2)^{A-1/2} e^{z(1-2u^2)} du \right] = \left( \frac{2z}{2} \right)^{A-1/2} \pi \int_0^1 u^2 A (1 - u^2)^{A-1/2} e^{-2u^2} \, du.$$ \hfill (3.7)

Summarizing, the following result has been established.

**Theorem 3.1.** Let $A$ be a matrix in $\mathbb{C}^{n \times n}$ such that $\mu(A) > -1/2$. Then for complex number $z$ satisfying $|\arg(z)| < \pi$, expressions (3.3), (3.6), and (3.7) hold true.

4. THE MODIFIED BESSEL MATRIX FUNCTION OF REAL ARGUMENT

In this section, we provide asymptotic expression of the modified Bessel matrix function $I_A(z)$ where $A$ is a matrix in $\mathbb{C}^{m \times r}$ such that $\mu(A) > -1/2$ and $x$ is a real number. Making the substitution $t = 2u^2x$ in (3.7), it follows that

$$I_A(z) = \left( \frac{2z}{2} \right)^{A-1/2} \pi \left[ \int_0^{2x} u^2 A (1 - u^2)^{A-1/2} e^{z(1-2u^2)} du + \int_0^{2x} u^2 A (1 - u^2)^{A-1/2} e^{z(1-2u^2)} du \right] = \left( \frac{2z}{2} \right)^{A-1/2} \pi \int_0^{2x} u^2 A (1 - u^2)^{A-1/2} e^{-2u^2} \, du.$$ \hfill (4.1)

**Remark 4.1.** Note that function $(1 - t/2x)^{A-1/2}$ presents a singularity at point $t = 2x$, and thus, Taylor’s theorem cannot be applied directly in all the interval $[0, 2x]$. However, if $0 < \delta < 2x$ is small enough, if we denote by $I(\delta)$

$$I(\delta) = \int_{2x - \delta}^{2x} t^{A-1/2} \left( 1 - \frac{t}{2x} \right)^{A-1/2} e^{-t} dt,$$

by (1.1) one gets

$$\|I(\delta)\| \leq e^{-2x+\delta} \int_{2x - \delta}^{2x} \left\{ \left( 2x \right)^{\mu(A)-1/2} \left( 1 - \frac{t}{2x} \right)^{\mu(A)-1/2} \right\} \left( 1 - \frac{t}{2x} \right)^{\mu(A)-1/2} e^{-t} dt$$

$$\leq e^{-2x+\delta} \max \left\{ \left( 2x \right)^{\mu(A)-1/2} (2x - \delta)^{\mu(A)-1/2} \right\} \int_{2x - \delta}^{2x} \left( 1 - \frac{t}{2x} \right)^{\mu(A)-1/2} e^{-t} dt$$

$$= e^{-2x+\delta} \max \left\{ \left( 2x \right)^{\mu(A)-1/2} (2x - \delta)^{\mu(A)-1/2} \right\} \frac{(2x)^{1/2 - \mu(A)} \hat{\mu}(A) + 1/2}{\mu(A) + 1/2 \hat{\mu}(A)}$$

$$\leq e^{-2x} \left( \left( p^{\mu(A)} - \hat{\mu}(A) \right) e^{(\delta \hat{\mu}(A)+1/2)} \right) \to 0,$$ as $\delta \to 0$, $x \to \infty.$
THEOREM 4.1. Let \( A \) be a matrix in \( \mathbb{C}^{n \times r} \) satisfying \( \mu(A) > -1/2 \), let \( p \) be a nonnegative integer, and let \( x \) be a positive real number. Then the following asymptotic expansion holds:

\[
I_A(x) = \frac{e^{x^2}}{\sqrt{2\pi x}} \left[ \sum_{k=0}^{\infty} \frac{(I/2 - A)k (A + I/2)_k}{k! (2x)^k} + O \left( x^{-p-1}, A \right) \right].
\] (4.3)

PROOF. With the notation of Remark 4.1, let \( x > 0 \) and \( x' = x - 6/2 \). By (4.1) and Taylor's theorem applied to \( f(u) = (1 - u)^{-1/2} \), one gets

\[
I_A(x) = \frac{e^{x^2}}{\sqrt{2\pi x}} \left[ \sum_{k=0}^{\infty} \frac{(I/2 - A)_k}{k! (2x)^k} \int_0^{2x'} t^{A+(k-1/2)} e^{-t} dt + S_{p+1} (0, 2x', A) + I(\delta) \right]
\] (4.4)

where for \( 0 \leq a < b \),

\[
S_{p+1}(a, b, A) = \int_a^b t^{A-1/2} e^{-t} r_{p+1}(t, x, A) dt,
\] (4.5)

where \( r_{p+1}(t, x, A) \) is Taylor's remainder. Taking into account (2.4) with \( p = 0 \) and \( B = A + (k + 1/2)I \), together (2.3) and (4.4), it follows that

\[
I_A(x) = \frac{e^{x^2}}{\sqrt{2\pi x}} \left[ \Gamma \left( A + \left( k + \frac{1}{2} \right) I \right) - e^{-2x'} (2x')^{A+(k-1/2)} I + O \left( \frac{1}{x'}, A \right) \right] + S_{p+1} (0, 2x', A) + I(\delta).
\] (4.6)

Considering (4.5) with \( a = \sqrt{2x}, b = 2x' \), Taylor's remainder expression [6, p. 49], and (1.1), it follows that

\[
\|S_{p+1} (\sqrt{2x}, 2x', A)\| \leq \frac{\|e^x - 1\| p!}{\sqrt{2x}} \int_0^{2x'} t^{\mu(A)-1/2} e^{-t} dt \times \left[ \int_0^{\sqrt{2x}} \left( \frac{t}{2x} - v \right)^p (1 - v)^{\mu(A) - p - 3/2} dv \right] dt.
\] (4.7)

Note that by Taylor's remainder expression [6, p. 49], one gets

\[
\frac{1}{p!} \int_0^{\sqrt{2x}} \left( \frac{t}{2x} - v \right)^p (1 - v)^{\mu(A) - p - 3/2} dv = \frac{1}{(1/2 - \mu(A))_{p+1}} \left[ (1 - t)^{\mu(A) - 1/2} - \sum_{k=0}^{p} \frac{(1/2 - \mu(A))_k}{k!} \left( \frac{t}{2x} \right)^k \right]
\] (4.8)

By (4.7), (4.8), and taking into account the substitution \( u = t/2x \), it follows that

\[
\|S_{p+1} (\sqrt{2x}, 2x', A)\| \leq \frac{\|e^x - 1\| p!}{\sqrt{2x}} \frac{e^{-\sqrt{2x}(2x)^{\mu(A)+1/2}}}{(1/2 - \mu(A))_{p+1}} \int_0^1 u^{\mu(A)-1/2} \left[ (1 - u)^{\mu(A) - 1} - \sum_{k=0}^{p} \frac{(1/2 - \mu(A))_k}{k!} \right] du \times O \left( x^{-p-1}, A \right).
\] (4.9)
Note that by (4.5) with $a = 1 < b = \sqrt{2x}$, using Lagrange's remainder in (4.8), and (1.1), one gets

$$
\left\| S_{p+1} \left(1, \sqrt{2x}, A \right) \right\| \leq \frac{\left\| (I/2 - A)^{p+1/2} \right\| (2x)^{-p-1} \left\| (1/2 - \mu(A))_{p+1}(p+1)! \right\|}{\left\| (1/2 - \mu(A))_{p+1}(p+1)! \right\|} 
\times \int_1^{\sqrt{2x}} t^{\mu(A)+p+1/2} e^{-t} \left( 1 - \frac{t}{2x} \right)^{\frac{\mu(A)-p-3/2}{2}} dt, \tag{4.10}
$$

where $0 < \xi(t/2x) < 1/\sqrt{2x}$. Since $(1 - \xi(t/2x))^{\mu(A)-p-3/2} < (1 - 1/\sqrt{2x})^{\mu(A)-p-3/2} = 1 + O(1/\sqrt{2x})$ when $\mu(A) < p + 3/2$, by (4.10), it follows that

$$
\left\| S_{p+1} \left(1, \sqrt{2x}, A \right) \right\| \leq \frac{\left\| (I/2 - A)^{p+1} \right\| (2x)^{-p-1} \left\| (1/2 - \mu(A))_{p+1}(p+1)! \right\|}{\left\| (1/2 - \mu(A))_{p+1}(p+1)! \right\|} \left( 1 + O\left( \frac{1}{\sqrt{2x}} \right) \right) 
\times \int_1^{\infty} t^{\mu(A)+p+1/2} e^{-t} dt = O\left( x^{-p-1}, A \right). \tag{4.11}
$$

In an analogous way, one gets

$$
\left\| S_{p+1} (0, 0, A) \right\| \leq O\left( x^{-p-1}, A \right). \tag{4.12}
$$

By (1.2), (4.2), (4.9), (4.11), (4.12), and taking limits in expressions (4.6) and

$$
S_{p+1} (0, 2x', A) = S_{p+1} (0, 0, A) + S_{p+1} \left(1, \sqrt{2x}, A \right) + S_{p+1} \left(2x', \sqrt{2x}, A \right), \tag{4.13}
$$

as $\delta \to 0$, the result is established.

**Remark 4.2.** Note that using the notation of Remark 4.1, by (4.1) and Taylor's theorem it follows that

$$
I_A(x) = \frac{e^{\delta}T^{-1}(A + I/2)}{\sqrt{2\pi x}} \left\{ \int_0^{2x'} \left[ \sum_{k \geq 0} \frac{(I/2 - A)_k}{k! (2x)_k} \left( \frac{t}{2x} \right)^k \right] t^{A-I/2} e^{-t} dt + I(\delta) \right\},
$$

and as the series converges uniformly in the interval $[0, 2x']$, one gets

$$
I_A(x) = \frac{e^{\delta}}{\sqrt{2\pi x}} \left\{ \sum_{k \geq 0} \frac{T^{-1}(A + I/2) (I/2 - A)_k \gamma(A + (k + 1/2)I, 2x')}{k! (2x)_k} + T^{-1} \left( A + \frac{I}{2} \right) I(\delta) \right\}. \tag{4.14}
$$

Taking limits in the above expression as $\delta \to 0$, by (4.4), (4.9), (4.11)-(4.13), and Remark 4.1, one gets the asymptotic expansion

$$
I_A(x) = \frac{e^{\delta}}{\sqrt{2\pi x}} \sum_{k \geq 0} \frac{T^{-1}(A + I/2) (I/2 - A)_k \gamma(A + (k + 1/2)I, 2x)}{k! (2x)_k} + O\left( x^{-p-1}, A \right), \tag{4.14}
$$

where the series is convergent and $\mu(A) > -1/2$. Expression (4.14) is a matrix version of the scalar expression (1) of [10, Section 7.25, p. 204].

**REFERENCES**