Improved FVM for two-layer shallow-water models:
Application to the Strait of Gibraltar

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Available online 7 November 2006

Abstract

This paper deals with the numerical simulation of flows of stratified fluids through channels with irregular geometry. Channel cross-sections are supposed to be symmetric but not necessarily rectangular. The fluid is supposed to be composed of two shallow layers of immiscible fluids of constant densities, and the flow is assumed to be one-dimensional. Therefore, the equations to be solved are a coupled system composed of two Shallow Water models with source terms involving depth and breadth functions. Extensions of Roe’s Q-scheme are proposed where a suitable treatment of the coupling and source terms is performed by adapting the techniques developed in [Vázquez-Cendón ME. Improved treatment of source terms in upwind schemes for the shallow water equations in channels with irregular geometry. J Comp Phys 1999;148:497–526; García-Navarro P, Vázquez-Cendón ME. On numerical treatment of the source terms in the shallow water equations. Comput Fluids 2000;29(8):17–45; Castro MJ, Macías J, Parés C. A Q-Scheme for a class of systems of coupled conservation laws with source term. Application to a two-layer 1-D shallow water system. Math Model Numer An 2001;35(1):107–27]. Finally we apply the numerical scheme to the simulation of the flow through the Strait of Gibraltar. Real bathymetric and coast-line data are considered to include in the model the main features of the abrupt geometry of this natural strait connecting the Atlantic Ocean and the Mediterranean Sea. A steady state solution is obtained from lock-exchange initial conditions. This solution is then used as initial condition to simulate the main semidiurnal and diurnal tidal waves in the Strait of Gibraltar through the imposition of suitable boundary conditions obtained from observed tidal data. Comparisons between numerical results and observed data and some tests on friction sensitivity are also presented.

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Keywords: Q-schemes; Coupled conservation laws; Source terms; 1D shallow water equations; Two-layer exchange flows; Hyperbolic systems; Strait of Gibraltar; Internal tides; Fortnightly and monthly signal; Friction effects

1. Introduction

Two layer flows, or flows which can be idealized as such, occur naturally in estuary flows, ocean currents, and atmospheric flows. Two layer flows also occur as a result of man’s interaction with natural flows by the addition of different density pollutants. This kind of flows are interesting from a theoretical and practical point of view. In this work, our interest focused on the computation of maximal and tidally induced two-layer exchange flows through the Strait of Gibraltar. In this narrows, connecting the Atlantic Ocean with the Mediterranean Sea, two layers of different waters can be distinguished: the colder and less saline Atlantic water flowing at surface and penetrating into the Mediterranean, and the deeper, denser Mediterranean water flowing into the Atlantic. The observation of this simplified picture shows that the simpler model to be used to simulate the flow in this region must unavoidably consider this two-layer structure.
In the first part of this paper, the 1D two layer shallow water equations for channels with irregular geometry both in breadth and depth are presented. This general formulation extends the one-layer shallow water equations and takes into account all the sources and coupling terms due to the density difference between layers and also the terms due to the geometry. This kind of models has been widely used in oceanographical applications but mainly for channels with rectangular cross-sections. In this work a model taking into account more general cross-sections is considered.

The formulation presented is developed in the framework of hyperbolic conservation laws with source terms. A numerical method recently developed (see [9,8]) for this type of problems is considered. In particular, the numerical scheme is based on the use of a generalized Q-scheme to discretize the flux terms and an upwinding technique to discretize the source terms.

In previous works (see [9,8]) we assessed model performance by comparing model results against the approximate stationary analytical solutions provided [2,12] in channels with simplified geometries. In these works, Armii and Farmer characterize the concept of maximal solution for two-layer exchange flows through these simplified channels (a maximal solution is that allowing the maximal flux to be exchanged through the channel fixed the ratio between the two layers fluxes). In the present contribution, a two-layer maximal exchange solution is computed for a realistic representation of the Strait of Gibraltar geometry and where an appropriate density ratio between the Mediterranean and Atlantic waters is chosen. The one-dimensional geometry considered for the Strait of Gibraltar is based on the consideration of a 1D symmetric channel with local sections equivalent to the actual sections of the Strait in a sense to be defined.

Then the main semi-diurnal and diurnal tidal waves in the Strait of Gibraltar are simulated. This is done by taking the steady state solution of the previous experiment as initial condition and imposing the effect of tides through the boundary conditions at the open boundaries. The boundary conditions are constructed from tidal data obtained from [13]. The numerical results are validated against the observed data provided by the same authors and also compared with the synopsis of the essential elements of the time-dependent response of the flow in the Strait of Gibraltar made in [3] from observed data collected in April 1986.

Finally, some tests have been performed in order to assess model sensibility to friction and to determine how friction affects the amplitude of monthly and fortnightly signal appearing in model variable time series. The final aim is to compare these with the observed signals. This may be of interest for adjusting model friction coefficients. We conclude with some final remarks.

2. Governing equations

In [10,8] the general equations governing the one-dimensional flow of two shallow layers of immiscible fluids along a straight channel with symmetric cross-sections of arbitrary shape were deduced. We address the reader to the former reference for further details. The system of equations is written under the form of two coupled systems of conservation laws with source terms in the sense introduced in [9].

Let us introduce some notation. In general, index 1 makes reference to the upper layer and index 2 to the lower one. The coordinate $x$ refers to the axis of the channel; $y$ is the horizontal coordinate normal to the axis; $z$, the vertical coordinate; and $t$, the time; $g$ is the gravity; $\rho_i$ is the density of the $i$th layer ($\rho_1 < \rho_2$), and $r = \frac{\rho_1}{\rho_2}$, their ratio. Variables $b(x)$ and $\sigma(x,z)$ are, respectively, bottom and breadth functions, i.e., channel bottom is defined by the surface of equation $z = b(x)$ and channel walls by the equations: $y = \pm \frac{1}{2} \sigma(x,z)$. The variables $A_i(x,t)$ and $h_i(x,t)$ represent the wetted cross-section and the thickness of the $i$th layer at the section of coordinate $x$ at time $t$ (see Fig. 1), respectively. Therefore $A_i$ and $h_i$, $i=1,2$ are related through the equations:

\begin{align}
A_1 &= \int_{b(x)+h_1}^{b(x)+h_2} \sigma(x,z)dz, \\
A_2 &= \int_{b(x)+h_2}^{b(x)+h_1} \sigma(x,z)dz.
\end{align}

Finally, $v_i(x,t)$ and $Q_i(x,t) = v_i(x,t)A_i(x,t)$ represent the velocity and the discharge of the $i$th layer.

The general equations governing the one-dimensional flow of two shallow layers of immiscible fluids along a straight channel with symmetric cross-sections of arbitrary shape are written as follows (see [8] for details):

\begin{equation}
\frac{\partial W}{\partial t} + \frac{\partial F}{\partial x}(\sigma, W) = B(\sigma, W)\frac{\partial W}{\partial x} + V(\sigma, W) + S(x, \sigma, W),
\end{equation}

where

\begin{align}
W(x,t) &= [A_1(x,t), Q_1(x,t), A_2(x,t), Q_2(x,t)]^T, \\
\sigma(x,t) &= [\sigma_1(x,t), \sigma_2(x,t), \sigma_3(x,t)]^T,
\end{align}

Fig. 1. Notations for channel cross-sections.
being \( \sigma_1(x, t) = \sigma(x, b(x) + h_2(x, t) + h_1(x, t)) \) at the free surface, \( \sigma_2(x, t) = \sigma(x, b(x) + h_2(x, t)) \) the channel breadth at the interface and

\[
\frac{1}{\sigma_2} = 1 - \frac{r}{\sigma_3} + \frac{r}{\sigma_4},
\]

\[
\mathbf{F}(\sigma, \mathbf{W}) = \begin{bmatrix}
Q_1 \\
Q_2 \\
0 \\
0
\end{bmatrix},
\]

\[
\mathbf{B}(\sigma, \mathbf{W}) = \begin{bmatrix}
-\sigma_1 g_1 \\
0 \\
0 \\
0
\end{bmatrix},
\]

\[
\mathbf{V}(\sigma, \mathbf{W}) = \begin{bmatrix}
\frac{1}{\sigma_1} A_1^2 \\
0 \\
\frac{1}{\sigma_1} A_2^2
\end{bmatrix},
\]

\[
\mathbf{S}(x, \sigma, \mathbf{W}) = \begin{bmatrix}
g_1 \sigma_1 \int_h^{b+h_1+k_2} \frac{\sigma_1}{\sigma_2} dz - g_2 \sigma_1 \frac{\sigma_2}{\sigma_1} dz \\
-\sigma_1 g_1 A_1 + g_2 A_2 \left( \frac{\sigma_1}{\sigma_2} \int_h^{b+h_1+k_2} \frac{\sigma_1}{\sigma_2} dz + \frac{1-\sigma_2}{\sigma_1} \int_h^{b+h_1+k_2} \frac{\sigma_1}{\sigma_2} dz \right)
\end{bmatrix},
\]

where \( \sigma_0(x, t) = \sigma(x, b(x)) \) is the channel breadth at the bottom.

3. Numerical scheme

In [9] a numerical scheme based on Approximate Riemann Solvers for general coupled systems of conservation laws with source terms is developed. In this work, the former scheme is applied to the system (3) in the particular case where \( \sigma \) is constant, i.e., for channels with constant breadth rectangular cross-sections.

Nevertheless, the general case (3) does not fit into the abstract framework developed in the cited work due to the flux dependence on \( x \) via the breadth \( \sigma \). For the case of a single shallow water layer this difficulty has been studied in [15]. In this section these techniques are applied to construct a numerical scheme for (3).

First, let us consider the following system, where the space domain \( S(x, \sigma, \mathbf{W}) \) has been dropped:

\[
\frac{\partial \mathbf{W}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} (\sigma, \mathbf{W}) = \mathbf{B}(\sigma, \mathbf{W}) \frac{\partial \mathbf{W}}{\partial x} + \mathbf{V}(\sigma, \mathbf{W}).
\]

It is easy to verify that, in matrix form, this system reads as follows:

\[
\mathbf{W}_t + \mathbf{A}(\sigma, \mathbf{W}) \mathbf{W}_x = 0,
\]

where \( \mathbf{A} \) is the \( 4 \times 4 \) matrix whose expression is given by

\[
\mathbf{A}(\sigma, \mathbf{W}) = \mathbf{J}(\sigma, \mathbf{W}) - \mathbf{B}(\sigma, \mathbf{W}),
\]

with

\[
\mathbf{J}(\sigma, \mathbf{W}) = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
-\frac{g_1}{\sigma_1} + \frac{\sigma_2}{\sigma_1} A_1 & 2 \frac{g_1}{\sigma_1} & 0 & 0 \\
0 & 0 & -\frac{g_1}{\sigma_1} + \frac{\sigma_2}{\sigma_1} A_2 & 2 \frac{g_1}{\sigma_1}
\end{bmatrix},
\]

Observe that the term \( \mathbf{V}(\sigma, \mathbf{W}) \) is contained in the flux term when the system is expressed in the form (11), as

\[
\frac{\partial \mathbf{F}}{\partial x} = \frac{\partial \mathbf{F}}{\partial \mathbf{W}} \mathbf{W}_x + \frac{\partial \mathbf{F}}{\partial \sigma} \mathbf{J}(\sigma, \mathbf{W}) \mathbf{W}_x + \mathbf{V}(\sigma, \mathbf{W}).
\]

The eigenvalues of \( \mathbf{A} \) can be classified in two external and two internal eigenvalues. The external eigenvalues, \( \lambda_{\text{ext}} \), \( j = 1, 2 \) are related to the propagation speed of barotropic perturbations and the internal eigenvalues \( \lambda_{\text{int}} \), \( j = 1, 2 \) to the propagation of baroclinic perturbations. Unfortunately, analytical expressions of the eigenvalues are not available. Nevertheless, in the case \( r \approx 1 \), a first-order approximation was given in [20]. Attending to the nature of the internal eigenvalues, the flow is said to be supercritical if the sign of the internal eigenvalues differs, critical if one of them takes the value zero, otherwise the flow is called subcritical. It can be deduced from the expression of the matrix \( \mathbf{A} \) that critical, supercritical and subcritical sections can be characterized by sections where \( G_1 = 1, G_2 > 1, G_2 < 1 \), respectively, where

\[
G_2 = F_2^2 + (1 - r) \sigma_2 F_1^2 F_2^2,
\]

and

\[
F_1^2 = \frac{\sigma_2^2}{g' \left( \frac{\sigma_1}{\sigma_2} \right) \left( \frac{d_1}{\sigma_1} \right)},
\]

In these definitions, \( g' = g(1-r) \) is the reduced gravity, and \( G \) and \( F_i \), \( i = 1, 2 \) are the appropriate definitions of the composite Froude number and the internal Froude numbers, respectively, when arbitrary geometries are considered.

The internal eigenvalues may become complex corresponding to the development of shear instabilities. Nevertheless, in this work only the case when the matrix \( \mathbf{A} \) has four different real eigenvalues is considered, i.e., the flow is supposed to be stable and the system hyperbolic. With the numerical scheme presented here unstable flows cannot be simulated. To do so, friction and/or mixing has to be included in the model.

To obtain a numerical scheme for (10) we apply Roe’s method [19] as it was done in [9] for a particular case. To begin with, the space domain is decomposed into \( M \) computing cells \( I_i = [x_{i-1/2}, x_{i+1/2}] \) for \( i = 1 \) to \( M \). Although the numerical scheme is effectively thought and used for irregular meshes, for the sake of simplicity, we assume that these cells have a constant size \( \Delta x \) and that \( x_{i+1/2} = i \Delta x \). That means that \( x_i = (i - 1/2) \Delta x \) is the center of the cell.
Let $\Delta t$ be the time step and $\rho^* = n\Delta t$. The approximation of $W(x_i, n\Delta t)$ given by the numerical scheme will be represented by

$$W^n_i = [A^n_{i,1}, Q^n_{i,1}, A^n_{i,2}, Q^n_{i,2}]^T.$$  

The same notation is used for the approximated velocities:

$$u^n_{ij} = \frac{Q^n_{ij}}{A^n_{ij}}, \quad j = 1, 2.$$  

Once the solution has been computed at the time $t^n$, the calculation of $W^n_{i+1}$ requires the choice of an ‘intermediate state’ between $W^n_i$ and $W^n_{i+1}$ at the intercell $x_{i+1/2}$, that is used to linearize the system. This intermediate state will be denoted by

$$\tilde{W}^n_{i+1/2} = \left[\tilde{A}^n_{i+1/2,1}, \tilde{Q}^n_{i+1/2,1}, \tilde{A}^n_{i+1/2,2}, \tilde{Q}^n_{i+1/2,2}\right]^T.$$  

$\tilde{h}^n_{i+1/2,j}$, $j = 1, 2$, are the thickness related to $\tilde{A}^n_{i+1/2,j}$, $j = 1, 2$ through (1) and (2), and $\tilde{A}^n_{i+1/2}$ represents the value of $A$ related to $\tilde{h}^n_{i+1/2,j}$, $j = 1, 2$ at $x_{i+1/2}$. Finally, $\tilde{A}^n_{i+1/2}$ and $\tilde{B}^n_{i+1/2}$ will denote the value of the matrices $A$ and $B$, respectively, corresponding to $\tilde{A}^n_{i+1/2}$ and $\tilde{W}^n_{i+1/2}$.

In the description of the numerical scheme we will use the following matrices:

$$A^n_{i+1/2} = \left[\begin{array}{cc} \lambda_{i+1/2,1} & 0 \\ \vdots & \ddots \\ 0 & \lambda_{i+1/2,4} \end{array}\right],$$

whose diagonal coefficients are the eigenvalues of $\tilde{A}^n_{i+1/2}$. By $K^n_{i+1/2}$ we denote a matrix whose columns are eigenvectors corresponding to these eigenvalues. The following matrices will also be used:

$$\text{sgn}(A^n_{i+1/2}) = \left[\begin{array}{cccc} \text{sgn}(\lambda_{i+1/2,1}) & 0 \\ \vdots & \ddots \\ 0 & \text{sgn}(\lambda_{i+1/2,4}) \end{array}\right],$$

$$A^\pm_{i+1/2} = \left[\begin{array}{cccc} (\lambda_{i+1/2,1})^\pm & 0 \\ \vdots & \ddots \\ 0 & (\lambda_{i+1/2,4})^\pm \end{array}\right],$$

$$\tilde{A}^\pm_{i+1/2} = K^n_{i+1/2} A^\pm_{i+1/2} K^{-1}_{i+1/2}, \quad |\tilde{A}^n_{i+1/2}| = \tilde{A}^+_n - \tilde{A}^-_{i+1/2}.$$  

Once the intermediate states are chosen, Roe’s method is applied in the usual way by solving the linearized Riemann Problems at each interface and integrating on the cells. The numerical scheme obtained if the Roe’s averages are chosen (see [8] for details) can be written as follows:

$$W^n_{i+1} = W^n_i + \frac{\Delta t}{\Delta x} (F^n_{i+1/2} - F^n_{i/2})$$

$$+ \frac{\Delta t}{2\Delta x} \left( \tilde{B}^n_{i+1/2} \cdot (W^n_i - W^n_{i+1}) + \tilde{B}^n_{i+1/2} \cdot (W^n_{i+1} - W^n_i) \right)$$

$$+ \frac{\Delta t}{2\Delta x} \left( \tilde{V}^n_{i+1/2} + \tilde{V}^n_{i+2/2} \right),$$

with

$$F^n_{i+1/2} = \frac{1}{2} \left( (F(\sigma^n_i, W^n_i) + F(\sigma^n_{i+1}, W^n_{i+1})) - \frac{1}{2} \left( \tilde{A}^n_{i+1/2} \cdot (W^n_{i+1} - W^n_i) \right),$$

and

$$\tilde{V}^n_{i+1/2} = (0, \tilde{V}^n_{i+1/2,1}, 0, \tilde{V}^n_{i+1/2,2})^T,$$

$$\tilde{V}^n_{i+1/2,j} = \frac{g}{2} \left( \frac{1}{\sigma^n_{i+1,j} - \sigma^n_{i,j}} (A^n_{i+1,j})^2 \right)$$

$$+ \frac{g}{2} \left( \frac{1}{\sigma^n_{i+1,j}} (A^n_{i,j})^2 \right), \quad j = 1, 2.$$  

This is in fact the general writing of a $Q$-scheme for solving (10). The particular cases are defined by the specific choice of the matrices $\tilde{A}^n_{i+1/2}$.

### 3.1. Source terms

It is known that, when solving shallow water systems, numerical schemes based on centered discretizations can fail in approximating steady flows. In [24,4] a suitable upwinding discretization was introduced to avoid this undesirable behaviour. Following these authors, we propose the following numerical scheme to solve (3):

$$W^n_{i+1} = W^n_i + \frac{\Delta t}{\Delta x} (F^n_{i+1/2} - F^n_{i/2})$$

$$+ \frac{\Delta t}{2\Delta x} \left( \tilde{B}^n_{i+1/2} \cdot (W^n_i - W^n_{i+1}) + \tilde{B}^n_{i+1/2} \cdot (W^n_{i+1} - W^n_i) \right)$$

$$+ \frac{\Delta t}{2\Delta x} \left( \tilde{V}^n_{i+1/2} + \tilde{V}^n_{i+1/2} \right),$$

where

$$P^n_{i+1/2} = \frac{1}{2} K^n_{i+1/2} (I\text{d} \pm \text{sgn}(A^n_{i+1/2})) K^{-1}_{i+1/2},$$

and $\tilde{S}^n_{i+1/2}$ is an approximation of the integral of the source term $S$ at $[x_i, x_{i+1}]$. A suitable expression for the numerical treatment of the source terms $S^n_{i+1/2}$ is as follows (see [8] for details):

$$\tilde{S}^n_{i+1/2} = (0, \tilde{S}^n_{i+1/2,2})^T,$$

where

$$\tilde{S}^n_{i+1/2,2} = g \tilde{A}^n_{i+1/2} \left( \frac{1}{\sigma^n_{i+1/2}} (A^n_{i+1,1} + A^n_{i+1,2} - (A^n_{i,1} + A^n_{i,2})) \right.$$

$$\left. - (b^n_{i+1} + h_{i+1,1} + h_{i+1,2} - (b^n_i + h_{i,1} + h_{i,2})) \right),$$

and
\[
\tilde{S}_{i+1/2} = g A_{i+1/2} \left( \frac{1}{\delta_{i+1/2}} (A_{i+1,2} - A_{i,2}) + \frac{r}{\delta_{i+1/2}} (A_{i+1,1} - A_{i,1}) - (b_{i+1} + r h_{i+1,1} + h_{i+1,2} - (b_i + r h_{i,1} + h_{i,2})) \right). 
\]

(21)

In the deduction of the schemes CFL-like requirements have to be imposed (see [19]). In practice, we propose the following condition:

\[
\max \{|\lambda_{i+1/2}|, 1 \leq i \leq 4, 1 \leq i \leq M \} \frac{\Delta t}{\Delta x} \leq \gamma,
\]

where \(0 < \gamma \leq 1\) is the CFL number.

Finally, in order to prevent the numerical viscosity of the \(Q\)-schemes from vanishing when any of the eigenvalues of the matrices \(\tilde{A}_{i+1/2}\) are zero, we apply Harten regularization [16].

In [4] an enhanced consistency condition called conservation property or \(\mathcal{C}\)-property was introduced in order to characterize how exactly a numerical scheme approximates a steady solution representing water at rest. In [8], the extension of that enhanced consistency condition to the proposed numerical schemes was done, and a result that gives some conditions to ensure that a generalized \(Q\)-scheme satisfies this property was proved.

4. Numerical results: application to Gibraltar Narrows

In this section the model is applied to the simulation of the exchange flow through the Strait of Gibraltar. Two numerical experiments are presented. The aim of the first one is to obtain a steady state solution representing the secular exchange flow through the Strait of Gibraltar when no forcing is applied. This is done by performing a lock-exchange experiment. The goal of the second experiment is to study tidal effects on the exchange flow and on the behaviour of the interface. This tidal experiment is performed imposing the steady state solution already computed as initial condition and imposing the tidal forcing through an adequate choice of boundary conditions. Animations corresponding to both experiments can be found at http://www.damflow.org.

To perform these experiments the real geometry of the Strait is approached by constructing appropriate breadth and bottom functions. To do so, we use bathymetric and coast line data and construct from them an “equivalent symmetric channel” approximating the actual topographic characteristics of the Strait of Gibraltar. This is done as follows: first an axis defining the Strait orientation is settled. Along this axis we consider \(M = 200\) transversal cross-sections, \(S_i, i = 1, \ldots, M\), whose areas are numerically computed. This spatial discretization corresponds to a \(\Delta x = 600\) m. The bottom depth, at each of these sections, is taken as the maximal depth, \(b_i\), found in the bathymetric data. Finally, at each cross-section, channel breadth is approximated by a continuous piecewise linear function constructed in such a way that the cross-section areas are preserved. Fig. 2(a) and (b) shows the bottom function and the breadth at the top of the channel obtained by this procedure. In all the figures, the Atlantic Ocean is located to the left and the Mediterranean Sea to the right.

4.1. The secular exchange: maximal steady state solution

The objective of this experiment is the obtaining of a steady state solution that represents the secular exchange in the sense of Armi and Farmer theory ([1,2,12,3]). This solution will also provide an initial condition for further numerical experiments, here for the tidal forced simulation. To obtain this solution a lock-exchange experiment is newly performed. Fig. 3(a) shows the initial condition: The Atlantic waters, on the left, are separated from the Mediterranean denser waters by an artificial dam. The ratio of densities is equal to 0.99805, this ratio corresponds to densities \(\rho_1 = 1.027 \text{ kg/m}^3\) for the Atlantic water and \(\rho_2 = 1.029 \text{ kg/m}^3\) for the Mediterranean water. The relationship \(Q_1 = -Q_2\) is imposed at the boundaries and CFL = 0.5. Fig. 3(b) depicts the interface and free surface of the steady state reached. The discharge is \(Q_1 = \ldots\)
For analysis of the nature of the flow, we can look at the eigenvalues of the computed solution. Fig. 3(c) and (d) shows, respectively, the four eigenvalues and the two internal eigenvalues corresponding to the steady state solution along the Strait. They represent the internal and external wave speeds. It can be observed the presence of a subcritical region bounded by two sections where one of the two internal eigenvalues takes the value zero. One of these sections is located near Camarinal Sill and the other one at Tarifa Narrows. Out of this subcritical region, the two internal eigenvalues have the same sign and therefore the flow is supercritical. In the framework of the hydraulic theory developed by Armi and Farmer, this subcritical region is called a “control region” and the two critical sections bounding it are called “control sections”. In the case of a channel with a simple geometry composed by a combination of an offset sill and narrows, when a control region is present, the exchange rate is fully defined without reference to reservoirs conditions. When this occurs, the flow regimen established is referred to as maximal exchange. When a maximal flow occurs, in the sense defined above, the exchange rate is greatest, which motivates its name. As pointed out above, in the numerical experiment presented here the same configuration with the presence of a control region is found for the more complex geometry of a channel that has been considered to represent the Strait of Gibraltar. In this sense, the simulated flow represents a maximal exchange.

If we compare Fig. 3(b) with the hypothetical steady state proposed in Armi and Farmer [3], it can be observed that both figures agree for the location of the interface to the east of Camarinal Sill and to the west of Spartel Sill, being the flow supercritical in both regions, but they differ in the region between these two locations. In [3], the steady state proposed exhibits a control section placed at Spartel Sill and an internal hydraulic jump which are not present in the computed solution. Later on, we will come back to this discussion.

4.2. Tidal simulation

In order to study the essential elements of the time dependent response in the Strait of Gibraltar to tidal forcing, the next numerical experiment consists in a simulation
of the main semi-diurnal ($M_2$ and $S_2$) and diurnal ($O_1$ and $K_1$) tidal waves in this narrows. The numerical experiment has been designed as follows: using the steady state solution reached at the previous lock-exchange experiment as initial condition, the model is integrated over 30 semi-diurnal tidal cycles to achieve a stable quasi-time-periodic solution. The model is forced at the open boundaries with boundary conditions that simulate the four main tidal components to be considered ($M_2$, $S_2$, $O_1$ and $K_1$):

$$h_1(x_B,t) + h_2(x_B,t) = \bar{h}_B - \sum_{n=1}^{4} Z_n(x_B) \cos(\omega_n t - \phi_n(x_B)).$$

Here $x_B$ represents a point of the open boundaries (left or right); $Z_n(x_B)$ and $\phi_n(x_B)$ are the prescribed surface elevation amplitudes and phases of the $n$th tidal constituent at the boundary sections; $\omega_n$ its frequency; $\bar{h}_B$ the total depth of the water column corresponding to the steady state solution at this boundary. The $M_2$, $S_2$, $O_1$ and $K_1$ tidal elevation amplitudes, phases and frequencies used here were obtained by interpolating the measured data at the coastal stations of Trafalgar and Spartel (on the western end) and the coastal stations of Punta Carnero and Ceuta (on the eastern end) obtained by García-Lafuente et al. [13]. Once the quasi-time-periodic regime is established, the model is integrated for another 29-day period. Thereafter an harmonic analysis is performed on the tidal elevation over this 29-day period in order to compare the results with the measured data provided by [13] at some relevant points. The comparison between observed and numerical data are presented in Tables 1 and 2, where phases are related to the $M_2$ tide.

As it can be observed, the maximum differences do not exceed 3.0 cm (that is, about 5% in relative units) for the $M_2$ and $S_2$ amplitudes and 2° for their phases. The agreement between observed and predicted tidal constants for the $O_1$ and $K_1$ tidal components is not as good as for the $M_2$ and $S_2$ constituents. This is due, on the one hand, to the fact that 2D effects are stronger on the $O_1$ and $K_1$ tidal waves, as it can be seen in their phases chart (see [13]). Therefore, a 2D model including Coriolis effects must be used for a better simulation of these effects. On the other hand, the diurnal tides ($O_1$ and $K_1$) are very sensitive to small variations of the bottom morphology (see [21]). As a consequence, a more precise representation of the topography would also be necessary.

Fig. 4 shows the evolution of the free surface elevation at Tarifa Narrows given by the model during the 29-day tidal forcing experiment. Notice that, as expected, the model reproduces at this location two periods of spring tides and two other periods of neap tides, in agreement with the boundary conditions imposed.

In Figs. 5 and 6 the free surface and the interface at twelve different stages (separated by one hour time) of a semi-diurnal tidal cycle corresponding to a spring tide are depicted. The starting and final points of this time-series, located at the day 18 of the simulated period, are marked with stars in Fig. 4. The sequence of figures shown here can be compared with the essential elements of the time dependent response in the Strait of Gibraltar summarized by [3]. A detailed description can be found in [8]. Here we limit ourselves to sketch the overall model behaviour.

Through the simulation it can be observed that Tangier Basin acts as a reservoir that fills and drains on each tidal cycle. The brief description of the process is as follows: at spring tides about one hour after High Water, when Tangier Basin has been filling and the interface there has raised, the moving control close to Camarinal Sill is lost. This happens for about one hour. For neap tides this occurs closer or at High Water conditions and the loss control situation

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**Table 1**

<table>
<thead>
<tr>
<th>$M_2$</th>
<th>$S_2$</th>
<th>$K_1$</th>
<th>$O_1$</th>
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<td>Pred</td>
<td>Obs</td>
<td>Pred</td>
</tr>
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<td>39.88</td>
<td>14.13</td>
<td>14.85</td>
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<tr>
<td>2.73</td>
<td>2.16</td>
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</table>

**Table 2**

<table>
<thead>
<tr>
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<th>$S_2$</th>
<th>$K_1$</th>
<th>$O_1$</th>
</tr>
</thead>
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</tr>
<tr>
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<td>55.15</td>
<td>21.5</td>
<td>20.08</td>
</tr>
<tr>
<td>3.8</td>
<td>1.22</td>
<td>2.25</td>
<td>3.11</td>
</tr>
</tbody>
</table>

Fig. 4. Simulated free surface elevation (m) at Tarifa Narrows during the tidal forcing (29 days).
lasts longer, for about three hours. Then, the Tangier Basin starts draining, first due to the reversal of the Mediterranean flow at Camarinal Sill (Figs. 5(f), 6(a) and (b)) and later on for a larger Mediterranean outflowing flow through Spartel Sill than the incoming flow through Camarinal Sill (Fig. 6(c)) this makes the interface to drop below the interface level east of Camarinal Sill. The large amount of Mediterranean water incoming through Spartel Sill makes that the moving control close to Camarinal Sill is re-established. Meanwhile, the interface behaves in a complex manner (this can be observed plotting more intermediate snapshots or making an animation of the simulation). About one hour before Low Water, the increasingly larger and progressively more supercritical flow of Mediterranean water entering through Camarinal Sill into the Tangier Basin initiates the formation of a bore west of Camarinal Sill. The bore initially grows up to occupy half of the Tangier Basin, remaining there for about half of the tidal cycle when a new refilling of the basin makes the bore move eastwards and later, when the control is again lost in a new tidal cycle, makes it disappear propagating east of the sill.

East of Camarinal Sill, between the sill and the contraction, there is another fluctuating reservoir of water. In this case it is the Atlantic water that fills or drains from this reservoir of the Strait. Just as Tangier Basin can drain from both ends on a falling tide, so does the pool of Atlantic water east of Camarinal Sill from both ends, west over Camarinal Sill and east through Tarifa Narrows, on a rising tide.

The mean values of the discharges averaged over the 29-day tidal forcing simulation are of 1.0923 Sv for the first layer and of $1.0327 \text{ Sv}$ for the second layer. Observe that these values are larger than the ones obtained for the maximal exchange solution with $q_r = 1$ that were $Q_1 = -Q_2 = 0.855 \text{ Sv}$. Even if the tidal simulation were initialized from a maximal solution with $q_r = 1.0$, through the tidal cycle the mean Atlantic flow becomes larger than the mean Mediterranean flow. All along the tidal simulation the supercritical nature of the flow at both bounding ends of the Strait is always preserved, but the flow that connects these two supercritical regions is highly time-dependent and the behaviour of the interface is complex. Bores develop and the control at Camarinal Sill is periodically lost. From a qualitative point of view, the model solution agrees well with the observations and analysis performed by [3].

5. Fortnightly and monthly signals in model variables

Some authors [6,7,14,22,23] have detected the presence of fortnightly and monthly signals in currents at different
locations in the Strait of Gibraltar. Oceanographers hypothesize on the origin of these fortnightly and monthly signals. Their existence seems to be related to mixing effects. We try to further investigate on this hypothesis using the model presented here. In particular we are interested in shedding some light on which are the producing mechanisms for these signals and also to investigate their sensibility to model friction. If mixing processes were the sole producing mechanism for these signals, they should not appear in a two-layer model, as the one presented here, where mixing is not parameterized and, therefore, neglected.

In order to perform a sensibility analysis to friction, drag forces between layers and with sea bed have parameterized and included into model equations using the following expressions:

\[ \tau_{\text{int}} = -\rho_1 \tau_{\text{int}} |u_1 - u_2| |(u_1 - u_2)\sigma_3|, \]

for interfacial friction and

\[ \tau_{b,1} = -\rho_1 \tau_{\text{bot}} |u_1| |(u_1 - u_2)\sigma_3|, \quad \tau_{b,2} = -\rho_2 \tau_{\text{bot}} |u_2| |(u_1 - u_2)\sigma_3|, \]

for the first and second layer bottom friction, where \( \tau_{\text{int}} \) and \( \tau_{\text{bot}} \) are the interfacial and bottom friction coefficients, that are usually given by a Manning’s law [11].

In order to investigate the presence for fortnightly and monthly signals in model variables (free surface elevation, interface location, and fluxes at each layer) several numerical experiments have been performed. They have been grouped in two series: The first one consisted on simulating the two main semidiurnal (\( M_2 \) and \( S_2 \)) and diurnal (\( O_1 \) and \( K_1 \)) tidal waves in the Strait of Gibraltar for several values of the friction coefficients ranging, in the upper limit, from \( 5 \times 10^{-3} \) for bottom friction (\( 5 \times 10^{-4} \) for interface friction) and decreasing values down to the frictionless case, i.e., \( \tau_{\text{bot}} = 0 \). The upper limits considered for these two coefficients were taken to be equal or slightly higher than the values found in the bibliography (for example, [18] takes \( \tau_{\text{bot}} = 10^{-3} \), [5], take \( \tau_{\text{bot}} = 0, 3 \times 10^{-3}, \) and \( 6 \times 10^{-3} \) and \( \tau_{\text{int}} = 10^{-4}, \) and [21] \( \tau_{\text{bot}} = 5 \times 10^{-3} \) and \( \tau_{\text{int}} = 5 \times 10^{-4} \)). A power spectrum analysis of model variables time series at several locations in the Strait of Gibraltar was performed. Here we will focus on the time series at Camarinal Sill, as they can be compared with the analysis of available observed data (for example in [23]). The spectrum frequency diagrams for this first group of experiments in Fig. 7 shows the presence of a fortnightly signal in model variables at Camarinal Sill, but no significant monthly signal appears.

Fig. 6. Simulated free surface and interface during a semidiurnal tidal cycle at the Strait of Gibraltar. (Free surface with dots. Interface with stars. Bottom with dashed lines). (a) \( t = 445 \) h, (b) two hours before Low Water (\( t = 446 \) h), (c) \( t = 447 \) h, (d) Low Water (\( t = 448 \) h), (e) \( t = 449 \) h, (f) \( t = 450 \) h.
It is shown that the amplitude of this signal for the fluxes increases when the friction is reduced, having as limit the amplitude for the frictionless experiment (Fig. 8). The opposite occurs for the free surface elevation and the interface, i.e., signal amplitude reduces as friction is reduced (Fig. 9).

In the second group of experiments the $N_2$ tidal constituent is also included. In this case, besides the fortnightly signal, a monthly signal also appears at free surface elevation, interface location, and upper layer flow time series while there is no significant monthly signal for lower layer flow. Fig. 10 shows the $h_1$ and $Q_1$ normalized power spectrum diagrams for the experiment with friction ($r_{bot} = 5 \times 10^{-3}$ and $r_{int} = 5 \times 10^{-4}$). The amplitude of the fortnightly signal for the flows is slightly smaller but
roughly the same than in the previous set of experiments (see Fig. 8) while it is slightly larger for free surface elevation (see Fig. 9). The monthly signals that now appear (Figs. 11 and 12) show a behaviour analogous to fortnightly signals when friction is progressively reduced.

From these results we can conclude that the origin of the existence of the fortnightly and monthly signals at different locations in the Strait of Gibraltar is due not only to mixing effects (as model does not include them and some fortnightly and monthly modulation is found in model variables) but they also may be produced but the interaction, through model non-linearities, between the several tidal constituents that have been imposed as boundary conditions and the abrupt bottom topography of the Strait of Gibraltar.

Finally we have compared the behaviour and location of the interface for two numerical experiments, one with no friction imposed and the second one with bottom and interface friction. The main features observed are: (1) in Low Water (Fig. 13(a)) the bore presents a larger amplitude and penetrates further into the Tangier Basin in the frictionless case; (2) when tide rise (Fig. 13(b)), the internal
waves formed at the interface that moves eastward produced by the release of the internal bore travel behind in the frictionless case; [3] in High Water (Fig. 13(c)) these waves have passed through the Sill maintaining the delay for the frictionless case; [4] at Ebb tide (Fig. 13(d)) the interface for both experiments is essentially again in phase.

6. Final remarks

In this work, a formulation of the two layer Shallow Water Equations for channels with irregular geometry under the form of two coupled systems of conservation laws with source terms is presented and a generalized $Q$-scheme for solving the system with a suitable treatment of the coupling and source terms has been proposed.

The model has been used to simulate the flow exchange through the Strait of Gibraltar. To do so, an equivalent symmetric channel approaching the real geometry of the Strait has been constructed and a steady state solution has been computed starting from lock-exchange initial conditions. This solution, that represents the secular exchange through the Strait, presents the main features predicted by the hydraulic theory developed by Armi and Farmer. Moreover, the discharge obtained agrees quite closely with experimental measurements. Next, the main semidiurnal and diurnal tidal waves in the Strait of Gibraltar have been simulated by imposing the effect of the tides through the boundary conditions at the two open boundaries. Despite of model simplifications (mainly due to its one-dimensional character), the agreement with measurements (at least for the semidiurnal tides) is quite good. The development of internal bores travelling westward and eastward are well simulated. The numerical experiment presented here has revealed a complicated pattern of time dependent hydraulics fluctuations involving changing interfacial levels and moving control points at different stages of the tide, being these patterns in good agreement with the analysis of observed data performed in [3]. Therefore, these results seems to confirm that this approach is well suited for oceanographical purposes and, in particular, to understand some of the mechanisms that drive the exchange flows through natural channels as the Strait of Gibraltar.

Finally the model has been used to address a key question dealing with the causes producing fortnightly and monthly signals in currents in the Strait of Gibraltar and their sensibility to model friction, trying to give some insight on this subject. Some partial conclusions have been found among them that mixing is not the sole producing mechanism.

References


