Assessing the lifetime performance index of products from progressively type II right censored data using Burr XII model

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Abstract

Process capability analysis has been widely applied in the field of quality control to monitor the performance of industrial processes. In practice, lifetime performance index \( C_L \) is a popular means to assess the performance and potential of their processes, where \( L \) is the lower specification limit. Nevertheless, many processes possess a non-normal lifetime model, the assumption of normality is often erroneous. Progressively censoring scheme is quite useful in many practical situations where budget constraints are in place or there is a demand for rapid testing. The study will apply data transformation technology to constructs a maximum likelihood estimator (MLE) of \( C_L \) under the Burr XII distribution based on the progressively type II right censored sample. The MLE of \( C_L \) is then utilized to develop a new hypothesis testing procedure in the condition of known \( L \). Finally, we give two examples to illustrate the use of the testing procedure under given significance level \( α \).

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1. Introduction

Process capability analysis is an effective means of measuring process performance and potential capability. Process capability analysis has the following benefits: continuously monitoring the process quality through process capability indices (PCIs) in order to assure the products manufactured are conforming to the specifications; supplying information on product design and process quality improvement for engineers and designer; and providing the basis for reducing the cost of product failures (see [18]). In the manufacturing industry, PCIs are utilized to assess whether product quality meets the required level. For instance, Montgomery [15] (or [11]) proposed the process capability index \( C_L \) (or \( C_{PL} \)) for evaluating the lifetime performance of electronic components, where \( L \) is the lower specification limit, since the lifetime of electronic components exhibits the larger-the-better quality characteristic of time orientation. All of the above PCIs have been developed or investigated under normal lifetime model. Nevertheless, in many process which including manufacture process, service process and business operation process, the assumption of normality is common in process capability analysis, and is often not valid. Therefore, the assumption of normality is often erroneous for the

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lifetime model of many businesses and products. And the use of PCIs based on the assumption of normality may yield misleading results.

Several generalizations and modifications of classical PCIs have been proposed to handle non-normal quality data. The first is investigation of the properties of PCIs and their estimators under a specific non-normal distribution. For example, Tong et al. [25] constructed a uniformly minimum variance unbiased estimator (UMVUE) of $C_L$ under an exponential distribution. Moreover, the UMVUE of $C_L$ is then utilized to develop the hypothesis testing procedure. The purchasers can then employ the testing procedure to determine whether the lifetime of electronic components adheres to the required level. Manufacturers can also utilize this procedure to enhance process capability. The second is the data transformation method. The data transformation method is to transform the data using mathematical functions into normally distributed data. For example, Rivera et al. [20] proposed a data transformation method for handling non-normal data by a monotonically increasing function. Box and Cox [4] presented a useful family of power transformations. Johnson [9] built a system of distributions based on the moment method, called the Johnson transformation system. The third is the non-parametric method. The non-parametric method is to use non-normal percentiles to modify classical PCIs. For example, Clements [6] proposed the method of non-normal percentiles to calculate $C_p$ and $C_{pk}$ indices for a distribution of any shape, using the Pearson family of curves. Pearn and Kotz [19] also applied Clements’s method to construct the second-generation index $C_{pm}$ and the third-generation $C_{pmb}$ for non-normal data.

In this paper, process capability analysis is also utilized to assess the non-normal quality data under a specific non-normal distribution. Hence, the lifetime performance index (or larger-the-better process capability index) $C_L$ is also utilized to measure product quality with the Burr XII distribution. The Burr XII distribution was first introduced in the literature by Burr [5], and has gained special attention in the last two decades due to the potential of using it in practical situations. The probability density function (p.d.f.) and the failure rate function of the Burr XII distribution are given, respectively, by

$$f_X(x, c, k) = ckx^{c-1}(1 + x^c)^{-(k+1)}, \quad x > 0, \quad c > 0, \quad k > 0, \quad (1)$$

and

$$h_X(x, c, k) = ckx^{c-1}(1 + x^c)^{-(k+1)}, \quad x > 0, \quad c > 0, \quad k > 0, \quad (2)$$

where $k>0$ is the parameter but it does not affect the shape of failure rate function $h_X(x,c,k)$ and $c>0$ is the shape parameter. Its capacity to assume various shapes often permits a good fit when used to describe biological, clinical, or other experimental data. It has also been applied in areas of quality control, reliability studies, duration, and failure time modeling. The shape parameter plays an important role for the Burr XII distribution (see [28]). Hence, this study proposes a “shape-first” fitting approach that is to fit the shape parameter $c$ before fitting the parameter $k$. Inferences for Burr XII distribution were discussed by many authors. Rodriguez [21] constructed a moment ratio diagram for the coverage area in the skewness and kurtosis corresponding to the Burr XII family. Wang and Keats [27] gave the maximum likelihood method for obtaining point and interval estimators of the parameters of the Burr XII distribution. Abdel-Ghaly et al. [1] applied the Burr XII distribution to measure software reliability. Zimmer et al. [31] also presented statistical and probabilistic properties of the Burr XII distribution and described its relationship to other distributions used in reliability analyses. Moore and Papadopoulos [16] derived Bayesian estimators of the parameter $k$ and the reliability function for the Burr XII distribution under three different loss functions. Soliman [23] told us that the Burr XII distribution has been applied in areas of quality control, duration and failure time modeling. Wu and Yu [28] proposed $m$ pivotal quantities to test the shape parameter and establish confidence interval of the shape parameter of the two-parameter Burr type XII distribution under the failure-censored plan. Liu and Chen [13] proposed a novel modification of Clements’s method (see [6]) using the Burr XII distribution to improve the accuracy of estimates of indices associated with one-sided specification limits for non-normal process data. Li et al. [12] proposed the empirical estimators of reliability performances for Burr XII distribution under LINEX error loss. Wu et al. [29] use the method of maximum likelihood to derive the point estimators of the parameters for Burr XII distribution. And their main purpose is to construct the exact confidence interval and region for the parameters. It is apparent that the Burr XII distribution can cover the curve shape characteristics for the normal, Weibull, logistic, lognormal, and Extreme Value type I distributions. The Burr XII distribution has been recognized as a useful model for the analysis of lifetime data.

In life testing experiments, the experimenter may not always be in a position to observe the life times of all the products (or items) put on test. This may be because of time limitation and/or other restrictions (such as money, material resources, mechanical or experimental difficulties) on data collection. Therefore, censored samples may arise
in practice. In this paper, we consider the case of the progressively type II right censoring. Progressively type II right censoring is a useful scheme in which a specific fraction of individuals at risk may be removed from the experiment at each of several ordered failure times (see [7]). The experimenter can remove units from a life test at various stages during the experiments, possibly resulting in a saving of costs and time (see [22]). A schematic illustration is depicted in Fig. 1, where $x_{1,n}, x_{2,n}, \ldots, x_{m,n}$ denote the failure times and $R_1, R_2, \ldots, R_m$ denote the corresponding numbers of units removed (withdrawn) from the test. Let $m$ be the number of failures observed before termination and $x_{1,n} \leq x_{2,n} \leq \cdots \leq x_{m,n}$ be the observed ordered lifetimes. Let $R_i$ denote the number of units removed at the time of the $i$th failure, $0 \leq R_i \leq n - \sum_{j=1}^{i-1} R_j - i$, $i = 2, 3, \ldots, m - 1$, with $0 \leq R_1 \leq n - 1$ and $R_m = n - \sum_{j=1}^{m-1} R_j - m$, where $R_i$’s and $m$ are pre-specified integers (see [26,3,14,24,12]). Note that if $R_1 = R_2 = \cdots = R_{m-1} = 0$, so that $R_m = n - m$, this scheme reduces to the conventional type II right censoring scheme. Also note that if $R_1 = R_2 = \cdots = R_m = 0$, so that $m = n$, the progressively type II right censoring scheme reduces to the case of no censoring scheme (complete sample case).

The main aim of this study will apply data transformation technology to constructs a maximum likelihood estimator (MLE) of $CL$ under the Burr XII distribution with the progressively type II right censored sample. The MLE of $CL$ is then utilized to develop a new hypothesis testing procedure in the condition of known $L$. The new testing procedure can be employed by managers to assess whether the lifetime of products adheres to the required level in the condition of known $L$. The new proposed testing procedure can handle non-normal quality data.

The rest of this paper is organized as follows. Section 2 introduces some properties of the lifetime performance index for lifetime of product with the Burr XII distribution based on the progressively type II right censored sample. The MLE of $CL$ is then utilized to develop a new hypothesis testing procedure in the condition of known $L$. The new testing procedure can be employed by managers to assess whether the lifetime of products adheres to the required level in the condition of known $L$. The new proposed testing procedure can handle non-normal quality data.

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2. The lifetime performance index

Suppose that the lifetime of products may be modeled by a Burr XII distribution. Let $X$ denote the lifetime of such a product and $X$ has the Burr XII distribution with the p.d.f. is given as (1). Clearly, a longer lifetime implies a better product quality. Hence, the lifetime is a larger-the-better type quality characteristic. The lifetime is generally required to exceed $L$ unit times to both be economically profitable and satisfy customers. Montgomery [15] developed a capability index $C_L$ for properly measuring the larger-the-better quality characteristic. $C_L$ is defined as follows:

$$C_L = \frac{\mu - L}{\sigma},$$

where the process mean $\mu$, the process standard deviation $\sigma$, and $L$ is the lower specification limit.
To assess the lifetime performance of products, \( C_L \) can be defined as the lifetime performance index. Under \( X \) has the Burr XII distribution and the data transformation \( Y=\ln(1+X^c), \ c>0 \), the distribution of \( Y \) is an exponential distribution. Hence, the p.d.f. of \( Y \) is \( f_Y(y,k)=k\exp(-ky), \ y>0, k>0 \) (also see [10]). Moreover, there are several important properties, as follows:

- The lifetime performance index \( C_L \) can be rewritten as

\[
C_L = \frac{\mu - L}{\sigma} = \frac{1/k - L}{1/k} = 1 - kL, \quad C_L < 1, \tag{4}
\]

where the process mean \( \mu = E(X) = 1/k \), the process standard deviation \( \sigma = \sqrt{\text{Var}(X)} = 1/k \), and \( L \) is the lower specification limit.

- The cumulative distribution function of \( Y \) is given by

\[
F_Y(y, k) = 1 - \exp(-ky), \quad y > 0. \tag{5}
\]

- The failure rate function \( r_Y(y,k) \) is defined by

\[
r_Y(y, k) = \frac{f_Y(y, k)}{1 - F_Y(y, k)} = \frac{k\exp(-ky)}{1 - [1 - \exp(-ky)]} = k, \quad k > 0. \tag{6}
\]

The important properties can be determined by using data transformation \( Y=\ln(1+X^c), \ c>0 \). Since, the data transformation \( Y=\ln(1+X^c), \ c>0 \) for \( X>0 \) is one-to-one and strictly increasing, so data set of \( X \) and transformed data set of \( Y \) have the same effect in assessing the lifetime performance of products. Moreover, the data transformation \( Y=\ln(1+X^c), \ c>0 \) enables the calculation of important properties to be easy. When the mean \( 1/k(L) \), then the lifetime performance index \( C_L \) is defined by

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\]

where the process mean \( \mu = E(X) = 1/k \), the process standard deviation \( \sigma = \sqrt{\text{Var}(X)} = 1/k \), and \( L \) is the lower specification limit.

3. The conforming rate

If the lifetime of a product \( X \) which \( Y=\ln(1+X^c), \ c>0 \) exceeds the lower specification limit \( L \), then the product is defined as a conforming product. The ratio of conforming products is known as the conforming rate, and can be defined as

\[
P_r = P(Y \geq L) = \int_L^\infty k e^{-ky} \, dy = e^{-kL} = e^{C_L-1}, \quad -\infty < C_L < 1. \tag{7}
\]

Obviously, a strictly increasing relationship exists between conforming rate \( P_r \) and the lifetime performance index \( C_L \). Table 1 lists various \( C_L \) values and the corresponding conforming rates \( P_r \).

For the \( C_L \) values which are not listed in Table 1, the conforming rate \( P_r \) can be obtained through interpolation. The conforming rate can be calculated by dividing the number of conforming products by the total number of products sampled. To accurately estimate the conforming rate, Montgomery [15] suggested that the sample size must be large. However, a large sample size is usually not practical from the perspective of cost, since collecting the lifetime data of new products need many monies. In addition, a complete sample is also not practical due to time limitation and/or other restrictions (such as money, material resources, mechanical or experimental difficulties) on data collection. Since an one-to-one mathematical relationship exists between the conforming rate \( P_r \) and the lifetime performance index \( C_L \). Therefore, utilizing the one-to-one relationship between \( P_r \) and \( C_L \), lifetime performance index can be a flexible and effective tool, not only evaluating product quality, but also for estimating the conforming rate \( P_r \).

4. Maximum likelihood estimator of lifetime performance index

In lifetime testing experiments of products, the experimenter may not always be in a position to observe the lifetimes of all the items on test due to time limitation and/or other restrictions (such as money, material resources, mechanical or experimental difficulties) on data collection. Therefore, censored samples may arise in practice. In this paper, we
Table 1
The lifetime performance index $C_L$ vs. the conforming rate $P_r$.

<table>
<thead>
<tr>
<th>$C_L$</th>
<th>$P_r$</th>
<th>$C_L$</th>
<th>$P_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\infty$</td>
<td>0.00000</td>
<td>0.15</td>
<td>0.42741</td>
</tr>
<tr>
<td>$-9.00$</td>
<td>0.00004</td>
<td>0.20</td>
<td>0.44933</td>
</tr>
<tr>
<td>$-8.00$</td>
<td>0.00012</td>
<td>0.25</td>
<td>0.47237</td>
</tr>
<tr>
<td>$-7.00$</td>
<td>0.00033</td>
<td>0.30</td>
<td>0.49659</td>
</tr>
<tr>
<td>$-6.00$</td>
<td>0.00091</td>
<td>0.35</td>
<td>0.52205</td>
</tr>
<tr>
<td>$-5.00$</td>
<td>0.00248</td>
<td>0.40</td>
<td>0.54881</td>
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<td>$-4.50$</td>
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<td>0.45</td>
<td>0.57695</td>
</tr>
<tr>
<td>$-4.00$</td>
<td>0.00673</td>
<td>0.50</td>
<td>0.60653</td>
</tr>
<tr>
<td>$-3.50$</td>
<td>0.01111</td>
<td>0.55</td>
<td>0.63763</td>
</tr>
<tr>
<td>$-3.00$</td>
<td>0.01832</td>
<td>0.60</td>
<td>0.67032</td>
</tr>
<tr>
<td>$-2.50$</td>
<td>0.03019</td>
<td>0.65</td>
<td>0.70469</td>
</tr>
<tr>
<td>$-2.00$</td>
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</tr>
<tr>
<td>$-1.50$</td>
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<td>0.80</td>
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</tr>
<tr>
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<td>0.22313</td>
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<td>0.86071</td>
</tr>
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<td>0.00</td>
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<td>0.90484</td>
</tr>
<tr>
<td>0.05</td>
<td>0.38674</td>
<td>0.95</td>
<td>0.95123</td>
</tr>
<tr>
<td>0.10</td>
<td>0.40657</td>
<td>1.00</td>
<td>1.00000</td>
</tr>
</tbody>
</table>

Consider the case of progressively type II right censoring scheme. Progressively type II right censoring scheme is quite useful in many practical situations where budget constraints are in place or there is a demand for rapid testing.

Let $X$ denote the lifetime of such a product and $X$ has the Burr XII distribution with the p.d.f. as (1). With progressively type II right censoring, $n$ units are placed on test. Consider that $X_1, n \leq X_2, n \leq \cdots \leq X_m, n$ is the corresponding progressively type II right censored sample, with censoring scheme $R = (R_1, R_2, \ldots, R_m)$. Since the joint p.d.f. of $X_1, n, X_2, n, \ldots, X_m, n$ is given by

$$A \prod_{i=1}^{m} f_X(x_{i,n}, c, k)[1 - F_X(x_{i,n}, c, k)]^{R_i},$$

where $A = n(n - R_1 - 1) \cdots (n - R_1 - R_2 - \cdots - R_{m-1} - m + 1)$, $f_X(x,c,k)$ is the p.d.f. of $X$ and $F_X(x,c,k)$ is the cumulative distribution function of $X$.

So, the likelihood function is given by

$$L(k) = A\left(ck\right)^m \prod_{i=1}^{m} \frac{x_{i,n}^{c-1}}{1 + x_{i,n}^{c}} \exp[-k(1 + R_i) \ln(1 + x_{i,n}^{c})].$$

Assuming that $c$ is given (or known), the maximum likelihood estimate of $k$ can be derived by solving the equation:

$$\frac{d}{dk} \ln(L(k)) = \frac{m}{k} - \sum_{i=1}^{m} (1 + R_i) \ln(1 + x_{i,n}^{c}) = 0.\tag{10}$$

Hence, we show that the MLE $\hat{k}$ of $k$ is given by

$$\hat{k} = \frac{m}{\sum_{i=1}^{m} (1 + R_i) \ln(1 + X_{i,n}^{c})},\tag{11}$$

where $c$ is given (or known). By using the invariance of MLE (see [30]), the MLE of $C_L$ can be written as

$$\hat{C}_L = 1 - \hat{k}L = 1 - \frac{mL}{\sum_{i=1}^{m} (1 + R_i) \ln(1 + X_{i,n}^{c})}.\tag{12}$$
Let $W = \sum_{i=1}^{m} (1 + R_i) \ln(1 + X_{i,n}^c)$, then the MLE of $C_L$ can be rewritten as $\hat{C}_L = 1 - (mL/W)$ and we show that $2kW \sim \chi^2_{(2m)}$. Hence, the expectation of $\hat{C}_L$ can be derived as follows:

$$
E(\hat{C}_L) = E \left( 1 - \frac{mL}{\sum_{i=1}^{m} (1 + R_i) \ln(1 + X_{i,n}^c)} \right) = 1 - \frac{mkL}{2kW}.
$$

But $E(\hat{C}_L) \neq C_L$, where $C_L = 1 - kL$. Hence, the MLE $\hat{C}_L$ is not an unbiased estimator of $C_L$. But when $m \to \infty$, $E(\hat{C}_L) \to C_L$, so the MLE $\hat{C}_L$ is asymptotically unbiased estimator. Moreover, we also show that $\hat{C}_L$ is consistent.

5. Testing procedure for the lifetime performance index

Construct a statistical testing procedure to assess whether the lifetime performance index adheres to the required level. The one-sided hypothesis testing and one-sided confidence interval for $C_L$ are obtained using the pivotal quantity $2kW$. Assuming that the required index value of lifetime performance is larger than $c^*$, where $c^*$ denotes the target value, the null hypothesis $H_0$: $C_L \leq c^*$ and the alternative hypothesis $H_1$: $C_L > c^*$ are constructed. By using $\hat{C}_L$, the MLE of $C_L$ as the test statistic, the rejection region can be expressed as $\{\hat{C}_L > C_0^*\}$.

Given the specified significance level $\alpha$, the critical value $C_0^*$ can be calculated as follows:

$$
\begin{align*}
\sup_{C_L \leq c^*} P(\hat{C}_L > C_0^*) &\leq \alpha \\
 \Rightarrow P(\hat{C}_L > C_0^* | C_L = c^*) &= \alpha, \\
 \Rightarrow P \left( 1 - \frac{mL}{W} > C_0^* | k = \frac{1 - c^*}{L} \right) &= \alpha, \\
 \Rightarrow P \left( 2kW > \frac{2mkL}{1 - C_0^*} | k = \frac{1 - c^*}{L} \right) &= \alpha, \\
 \Rightarrow P \left( 2kW \leq \frac{2mkL}{1 - C_0^*} | k = \frac{1 - c^*}{L} \right) &= 1 - \alpha,
\end{align*}
$$

(13)

where $2kW \sim \chi^2_{(2m)}$. From Eq. (13), utilizing $\text{CHIINV}(1 - \alpha, 2m)$ function which represents the lower 100$(1 - \alpha)$th percentile of $\chi^2_{(2m)}$, then

$$
\frac{2m(1 - c^*)}{1 - C_0^*} = \text{CHIINV}(1 - \alpha, 2m)
$$

is obtained. Thus, the following critical value can be derived:

$$
C_0^* = 1 - \frac{2m(1 - c^*)}{\text{CHIINV}(1 - \alpha, 2m)},
$$

(14)
where $c^*$, $\alpha$ and $m$ denote the target value, the specified significance level and the number of observed failures before termination, respectively. The critical value $C_0^*$ can be computed by the Microsoft Excel software. Moreover, we also find that $C_0^*$ is independent of $n$ and $R_i$, $i = 1, 2, \ldots, m$.

Given the specified significance level $\alpha$, the level $(1 - \alpha)$ one-sided confidence interval for $C_L$ can be derived as follows:

With the pivotal quantity $2kW$, where $2kW \sim \chi^2_{(2m)}$ and $CHIINV(1 - \alpha, 2m)$ which represents the lower 100$(1 - \alpha)$th percentile of $\chi^2_{(2m)}$.

$$ P(2kW \leq CHIINV(1 - \alpha, 2m)) = 1 - \alpha, \quad \text{where } C_L = 1 - kL \text{ and } \hat{C}_L = 1 - \frac{mL}{W}. $$

$$ \Rightarrow P \left( 1 - kL \geq 1 - \frac{\hat{k}L CHIINV(1 - \alpha, 2m)}{2m} \right) = 1 - \alpha, \quad \text{where } \hat{k} = \frac{m}{W}. $$

$$ \Rightarrow P \left( C_L \geq 1 - \frac{(1 - \hat{C}_L)CHIINV(1 - \alpha, 2m)}{2m} \right) = 1 - \alpha, \quad (15) $$

From Eq. (15), then

$$ C_L \geq 1 - \frac{(1 - \hat{C}_L)CHIINV(1 - \alpha, 2m)}{2m} \quad (16) $$

is the level 100$(1 - \alpha)$% one-sided confidence interval for $C_L$. Thus, the level 100$(1 - \alpha)$% lower confidence bound for $C_L$ can be written as

$$ LB = 1 - \frac{(1 - \hat{C}_L)CHIINV(1 - \alpha, 2m)}{2m}, \quad (17) $$

where $\hat{C}_L$, $\alpha$ and $m$ denote the MLE of $C_L$, the specified significance level and the number of observed failures before termination, respectively.

The managers can then employ the one-sided hypothesis testing to determine whether the lifetime performance index adheres to the required level. The proposed testing procedure about $C_L$ can be organized as follows:

Step 1. By using the “shape-first” approach and the least squares estimation based on the minimum Error Sum of Squares ($SSE$) to fit the optimal value of $c$ for progressively type II right censored data. For various values of $c$, the least squares estimation is utilized to estimate $\hat{k}$. The optimal value of $c$ is selected by minimizing $SSE$ for various values of $c$. Then, $c$ is defined as known.

Step 2. The goodness of fit test based on the Gini statistic (see [8]) for the progressively type II right censored data $X_{1,n}, X_{2,n}, \ldots, X_{m,n}$.

Step 3. Let the data transformation $Y_i = \ln(1 + X_{i,n}^c)$, $c > 0$, $i = 1, 2, \ldots, m$ for the progressively type II right censored data $X_{1,n}, X_{2,n}, \ldots, X_{m,n}$.

Step 4. Determine the lower lifetime limit $L$ for products and performance index value $c^*$, then the testing null hypothesis $H_0$: $C_L \leq c^*$ and the alternative hypothesis $H_1: C_L > c^*$ is constructed.

Step 5. Specify a significance level $\alpha$.

Step 6. Calculate the value of test statistic $\hat{C}_L = 1 - (mL/\sum_{i=1}^m (1 + R_i) \ln(1 + X_{i,n}^c))$.

Step 7. Obtain the critical value $C_0^*$ from Eq. (14), according to the target value $c^*$, the significance level $\alpha$ and the number of observed failures before termination $m$.

Step 8. The decision rule of statistical test is provided as follows:

If $\hat{C}_L > C_0^*$, it is concluded that the lifetime performance index of product meets the required level.

Based on the proposed testing procedure, the lifetime performance of products is easy to assess. Two numerical examples of the proposed testing procedure is given in Section 6, and these numerical examples illustrate the use of the testing procedure. In addition, the proposed testing procedure can be constructed with the one-sided confidence.
interval too. The decision rule of statistical test is “If performance index value $c^* \notin [L, \infty)$, it is concluded that the lifetime performance index of product meets the required level.”

6. Numerical examples

In this section, we propose the new hypothesis testing procedure to a practical data set and a simulation data.

Example 1 considered is the failure data of $n=19$, $m=8$ electrical insulating fluids from Nelson [17]. In Example 2, we discuss a simulated data, the simulated data is generated from the Burr XII distribution with $c=3$, $k=0.5$, $n=20$, $m=10$ (see [24]). These numerical examples illustrate the use of the testing procedure. The proposed testing procedure not only can handle non-normal lifetime data, but also can handle the progressively type II right censored sample.

Example 1. Nelson ([17], p. 105, Table 1.1) reported data on times to breakdown of an insulating fluid between electrodes at a voltage of 34 kV (min). The 19 times to breakdown are 0.19, 0.78, 0.96, 1.31, 2.78, 3.16, 4.15, 4.67, 4.85, 6.50, 7.35, 8.01, 8.27, 12.06, 31.75, 32.52, 33.91, 36.71, and 72.89. The 19 observations are appropriate for a Burr XII model (see [24]). In the numerical example, a progressively type II censored sample of size $m=8$ is generated randomly from the $n=19$ observations recorded at 34 kV. The observed failure times and the progressive censoring scheme are given as follows:

$$\{x_{i,19}, i=1, \ldots, 8\} = \{0.19, 0.78, 0.96, 1.31, 2.78, 4.85, 6.50, 7.35\}$$

Then, we also state the proposed testing procedure about $CL$ as follows:

Step 1. Let $X$ have the Burr XII distribution with the p.d.f. is

$$f_X(x, c, k) = ckx^{c-1}(1+x^c)^{-(k+1)}, \quad x > 0, \quad c > 0, \quad k > 0. \quad (18)$$

then the p.d.f. of $X$ is given as

$$F_X(x, c, k) = 1 - (1+x^c)^{-k}, \quad x > 0, \quad c > 0, \quad k > 0. \quad (19)$$

and $F_X(x, c, k)$ satisfies

$$\ln(1 - F_X(x, c, k)) = -k \ln(1+x^c), \quad x > 0, \quad c > 0, \quad k > 0. \quad (20)$$

Consider that $X_{1,n} \leq X_{2,n} \leq \cdots \leq X_{m,n}$ is the corresponding progressively type II right censored sample, with censoring scheme $R=(R_1, R_2, \ldots, R_m)$. The expectation of $F_X(x_{i,n})$ is $1 - \prod_{j=m-i+1}^{m}(aj/(1+aj)), i=1, \ldots, m$, where $a_j = j + \sum_{i=m-j+1}^{m}R_i$ (see [3]).

By using the approximate equation $\ln[1 - (1 - \prod_{j=m-i+1}^{m}(aj/(1+aj)))] \approx -k \ln(1+X_{i,n}^c), i=1, 2, \ldots, m$, and the least squares estimation method for various values of $c$ and the “shape-first” approach to fit the optimal value of $c$ and estimate $k$ such that $SSE$ is minimized.

Table 2

<table>
<thead>
<tr>
<th>$c$</th>
<th>$SSE$</th>
<th>$k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.30276</td>
<td>0.46766</td>
</tr>
<tr>
<td>0.2</td>
<td>0.23226</td>
<td>0.45774</td>
</tr>
<tr>
<td>0.3</td>
<td>0.17520</td>
<td>0.44091</td>
</tr>
<tr>
<td>0.4</td>
<td>0.13066</td>
<td>0.41962</td>
</tr>
<tr>
<td>0.5</td>
<td>0.09692</td>
<td>0.39598</td>
</tr>
<tr>
<td>0.6</td>
<td>0.07198</td>
<td>0.37160</td>
</tr>
<tr>
<td>0.7</td>
<td>0.05397</td>
<td>0.34761</td>
</tr>
<tr>
<td>0.8</td>
<td>0.04126</td>
<td>0.32470</td>
</tr>
<tr>
<td>0.9</td>
<td>0.03253</td>
<td>0.30225</td>
</tr>
<tr>
<td>1.0</td>
<td>0.02674</td>
<td>0.28343</td>
</tr>
</tbody>
</table>
The values of SSE and the estimation of $k$ ($\hat{k}$) for various values of $k$ are shown in Table 2 and Fig. 2. This display indicates that $c = 1.4$ is very close to the optimum value and $\hat{k} = 0.21999$. Then, $c$ is defined as known.

Step 2. The goodness of fit test based on the Gini statistic for the progressively type II right censored data $X_{1,n}, X_{2,n}, \ldots, X_{m,n}$. The observed failure times $\{x_{i,19}, i = 1, \ldots, 8\} = \{0.19, 0.78, 0.96, 1.31, 2.78, 4.85, 6.50, 7.35\}$, and apply the Gini statistics (see [8]) to the hypothesis that the data from the Burr XII distribution with the p.d.f. is

$$f_X(x, 1.4, k) = 1.4kx^{0.4}(1 + x^{1.4})^{-(k+1)}, \quad x > 0, \; k > 0.$$  \hspace{1cm} (21)

at level $\alpha = 0.05$.

The null hypothesis is $H_0: X \sim$ the Burr XII distribution with the p.d.f. as (21).

The Gini statistic given as follows:

$$G_m = \frac{\sum_{i=1}^{m-1} iW_{i+1}}{(m-1)\sum_{i=1}^{m} W_i}$$

where $W_i = (m - i + 1)(Z_i - Z_{i-1})$, $Z_0 = 0$, $i = 1, 2, \ldots, m$, $Z_1 = nY_1$, $Z_i = [n - \sum_{j=1}^{i-1}(R_j + 1)](Y_i - Y_{i-1})$, $i = 1, 2, \ldots, m$, and the data transformation $Y_i = \ln(1 + X_{i,n}^{1.4})$.

For $m = 3, \ldots, 20$, the rejection region $\{G_m > \xi_{1 - \alpha/2} \text{ or } G_m < \xi_{\alpha/2}\}$, where the critical value $\xi_{\alpha/2}$ is the $100(\alpha/2)$th percentile of the $G_m$ statistic. The Gini statistic

$$G_8 = \frac{\sum_{i=1}^{7} iW_{i+1}}{(8 - 1)\sum_{i=1}^{8} W_i} = 0.41920.$$ \hspace{1cm} \xi_{0.025} = 0.28748 < G_8 = 0.41920 < \xi_{0.975} = 0.71252 \text{ (see [8])}, \text{ so we cannot reject } H_0 \text{ at the 0.05 level of significance.}$$

We can conclude the observed failure times from the Burr XII distribution with the p.d.f. is $f_X(x, 1.4, k) = 1.4kx^{0.4}(1 + x^{1.4})^{-(k+1)}, \; x > 0, \; k > 0$, at level $\alpha = 0.05$.

Step 3. The observed failure times $\{x_{i,19}, i = 1, \ldots, 8\}$ and we take the data transformation of $y_i = \ln(1 + x_{i,19}^{1.4})$, $i = 1, 2, \ldots, 8$. 

![Fig. 2. Plot of residual sum of squares (SSE) vs. c.](image-url)
Step 4. The lower lifetime limit \( L \) is assumed to be 0.4546, i.e., if the lifetime of an insulating fluid exceeds 0.6739 min, then the insulating fluid is defined as a conforming product. To deal with the product managers’ concerns regarding operational performance, the conforming rate \( P_C \) of operational performances is required to exceed 80%. Referring to Table 1, the \( C_L \) value operational performances are required to exceed 0.80. Thus, the performance index value is set at \( c^* = 0.80 \). The testing hypothesis \( H_0: C_L \leq 0.80 \) vs. \( H_1: C_L > 0.80 \) is constructed.

Step 5. Specify a significance level \( \alpha = 0.05 \).

Step 6. Calculate the value of test statistic:

\[
\hat{C}_L = 1 - \frac{8 \times 0.4546}{32.8879} = 0.889
\]

Step 7. Obtain the critical value \( C_0^* = 0.878 \) from Eq. (14), according to \( c^* = 0.80, m = 8 \) and the significance level \( \alpha = 0.05 \).

Step 8. Because of \( \hat{C}_L = 0.889 > C_0^* = 0.878 \), so we clearly reject the null hypothesis \( H_0: C_L \leq 0.80 \). Thus, we can conclude that the lifetime performance index of insulating fluids does meet the required level.

Example 2. Soliman [24] proposed a simulated data, the simulated data is a progressively type II censored sample from the Burr XII distribution with \((c = 3, k = 0.5)\). The progressively type II right censored sample and the progressive censoring scheme are given as follows:

\[
\{x_{i,20}, i = 1, \ldots, 10\} = \{0.265236, 0.771365, 0.815063, 0.843791, 0.858741, 0.917901, 0.919019, 0.948285, 1.0703, 1.57634\}, R = (0, 0, 2, 0, 0, 2, 0, 2, 0, 4), m = 10 and n = 20.
\]

Step 1. Let \( X \) have the Burr XII distribution with the p.d.f.

\[
f_X(x, c, k) = c k x^{c-1}(1 + x^c)^{-(k+1)}, \quad x > 0, c > 0, k > 0.
\]  

(22)

then the p.d.f. of \( X \) is given as

\[
F_X(x, c, k) = 1 - (1 + x^c)^{-k}, \quad x > 0, c > 0, k > 0.
\]  

(23)

and \( F_X(x, c, k) \) satisfies

\[
\ln(1 - F_X(x, c, k)) = -k \ln(1 + x^c), \quad x > 0, c > 0, k > 0.
\]  

(24)

Consider that \( X_{1,n} \leq X_{2,n} \leq \cdots \leq X_{n,n} \) is the corresponding progressively type II right censored sample, with censoring scheme \( R = (R_1, R_2, \ldots, R_m) \).

The expectation of \( F_X(x_{i,n}) \) is

\[
1 - \prod_{j=m-i+1}^m (a_j/(1 + a_j)), \quad i = 1, \ldots, m, \text{ where } a_j = j + \sum_{i=m-j+1}^m R_i
\]

(see [3]).

By using the approximate equation \( \ln[1 - (1 - \prod_{j=m-i+1}^m (a_j/(1 + a_j)))] \approx -k \ln(1 + X_{i,n}^c), i = 1, 2, \ldots, m, \) and the least squares estimation method for various values of \( c \) and the “shape-first” approach to fit the optimal value of \( c \) and estimate \( k \) such that \( SSE \) is minimized.

The values of \( SSE \) and the estimation of \( k (\hat{k}) \) for various values of \( k \) are shown in Table 3 and Fig. 3. This display indicates that \( c = 2.8 \) is very close to the optimum value and \( \hat{k} = 0.64402 \). Then, \( c \) is defined as known.

Step 2. The goodness of fit test based on the Gini statistic for the progressively type II right censored data \( X_{1,n}, X_{2,n}, \ldots, X_{n,n} \). The observed failure times \( \{x_{i,20}, i = 1, \ldots, 10\} = \{0.265236, 0.771365, 0.815063, 0.843791, 0.858741, 0.917901, 0.919019, 0.948285, 1.0703, 1.57634\}, \) and apply the Gini statistics (see [8]) to the hypothesis that the data from the Burr XII distribution with the p.d.f. is

\[
f_X(x, 2.8, k) = 2.8 k x^{1.8}(1 + x^{2.8})^{-(k+1)}, \quad x > 0, k > 0.
\]  

(25)

at level \( \alpha = 0.05 \).

The null hypothesis is \( H_0: X \sim \text{the Burr XII distribution with the p.d.f. as (25)} \).
Table 3
Values of $SSE$ and $\hat{k}$ for various values of $c$ and the optimal value (bold value) of $c$.

<table>
<thead>
<tr>
<th>$c$</th>
<th>$SSE$</th>
<th>$\hat{k}$</th>
<th>$c$</th>
<th>$SSE$</th>
<th>$\hat{k}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>0.16575</td>
<td>0.66908</td>
<td>3.1</td>
<td>0.13852</td>
<td>0.62743</td>
</tr>
<tr>
<td>2.2</td>
<td>0.15761</td>
<td>0.66689</td>
<td>3.2</td>
<td>0.14586</td>
<td>0.61493</td>
</tr>
<tr>
<td>2.3</td>
<td>0.15076</td>
<td>0.66421</td>
<td>3.3</td>
<td>0.15084</td>
<td>0.60833</td>
</tr>
<tr>
<td>2.4</td>
<td>0.14516</td>
<td>0.66104</td>
<td>3.4</td>
<td>0.15661</td>
<td>0.60154</td>
</tr>
<tr>
<td>2.5</td>
<td>0.14079</td>
<td>0.65741</td>
<td>3.5</td>
<td>0.16312</td>
<td>0.59457</td>
</tr>
<tr>
<td>2.6</td>
<td>0.13762</td>
<td>0.65335</td>
<td>3.6</td>
<td>0.17033</td>
<td>0.58744</td>
</tr>
<tr>
<td>2.7</td>
<td>0.13562</td>
<td>0.64888</td>
<td>3.7</td>
<td>0.17817</td>
<td>0.58020</td>
</tr>
<tr>
<td><strong>2.8</strong></td>
<td><strong>0.13475</strong></td>
<td><strong>0.64402</strong></td>
<td>3.8</td>
<td>0.18660</td>
<td>0.57284</td>
</tr>
<tr>
<td>2.9</td>
<td>0.13498</td>
<td>0.63881</td>
<td>3.9</td>
<td>0.19556</td>
<td>0.56540</td>
</tr>
<tr>
<td>3.0</td>
<td>0.13625</td>
<td>0.63327</td>
<td>4.0</td>
<td>0.19556</td>
<td>0.56540</td>
</tr>
</tbody>
</table>

The Gini statistic given as follows:

$$G_m = \frac{\sum_{i=1}^{m-1} i W_{i+1}}{(m-1)\sum_{i=1}^{m} W_i}$$

where $W_i = (m-i+1)(Z_0 Z_{i-1})$, $Z_0 = 0$, $i = 1, 2, \ldots, m$, $Z_1 = nY_1$, $Z_i = [n - \sum_{j=1}^{i-1} (R_j + 1)](Y_i - Y_{i-1})$, $i = 1, 2, \ldots, m$, and the data transformation $Y_i = \ln(1 + X_i^{2.8})$.

For $m=3, \ldots, 20$, the rejection region $\{G_m > \xi_{1-\alpha/2}\}$ or $G_m < \xi_{\alpha/2}\}$, where the critical value $\xi_{\alpha/2}$ is the 100($\alpha/2$)th percentile of the $G_m$ statistic. The Gini statistics

$$G_{10} = \frac{\sum_{i=1}^{9} i W_{10}}{(10-1)\sum_{i=1}^{10} W_i} = 0.66788.$$  

$\therefore \xi_{0.025} = 0.31232 < G_{10} = 0.66788 < \xi_{0.975} = 0.68768$ (see [8]), so we cannot reject $H_0$ at the 0.05 level of significance.

We can conclude the observed failure times from the Burr XII distribution with the p.d.f. is $f_X(x, 2.8, k) = 2.8kx^{1.8}(1 + x^{2.8})^{-(k+1)}$, $x > 0, k > 0$, at level $\alpha = 0.05$.

Fig. 3. Plot of residual sum of squares ($SSE$) vs. $c$.  

Indicates that $c=2.8$ is very close to the optimum value
Step 3. The observed failure times \( \{x_{i,20}, i = 1, \ldots, 10\} \) and we take the data transformation of \( y_i = \ln(1 + x_{i,20}^{0.8}) \), \( i = 1, 2, \ldots, 10 \).

Step 4. The lower lifetime limit \( L \) is assumed to be 0.155275, i.e. if the lifetime of a product exceeds 0.528817, then the product is defined as a conforming product. To deal with the product managers’ concerns regarding operational performance, the conforming rate \( P_r \) of operational performances is required to exceed 80%. Referring to Table 1, the \( C_L \) value operational performances are required to exceed 0.80. Thus, the performance index value is set at \( c^* = 0.80 \). The testing hypothesis \( H_0: C_L \leq 0.80 \) vs. \( H_1: C_L > 0.80 \) is constructed.

Step 4. Specify a significance level \( \alpha = 0.05 \).

Step 5. Calculate the value of test statistic:

\[
\hat{C}_L = 1 - \frac{10 \times 0.155275}{15.33101} = 0.899
\]

Step 6. Obtain the critical value \( C_0^* = 0.873 \) from Eq. (14), according to \( c^* = 0.80 \), \( m = 10 \) and the significance level \( \alpha = 0.05 \).

Step 7. Because of \( \hat{C}_L = 0.899 > C_0^* = 0.873 \), so we clearly reject the null hypothesis \( H_0: C_L \leq 0.80 \). Thus, we can conclude that the lifetime performance index of products does meet the required level.

7. Conclusions

In many process which including manufacture process, service process and business operation process, the assumption of normality is common in process capability analysis, and is often not valid. Therefore, the assumption of normality is often erroneous for the lifetime model of many products. Moreover, in life testing experiments, the experimenter may not always be in a position to observe the life times of all products (or items) put on test. This may be because of time limitation and/or other restrictions (such as money, material resources, mechanical or experimental difficulties) on data collection. Therefore, censored samples may arise in practice. Progressive censoring scheme is quite useful in many practical situations where budget constraints are in place or there is a demand for rapid testing. This study purposes to utilize the lifetime performance index \( C_L \) in assessing the lifetime performance of businesses and products more generally and accurately. A new approach of analyzing non-normal quality data is proposed in this study. Under the assumption of Burr XII distribution, this study constructs a MLE of \( C_L \) with the progressively type II right censored sample. The MLE of \( C_L \) is then utilized to develop the new hypothesis testing procedure in the condition of known \( L \).

The proposed testing procedure is easily applied and can effectively evaluate whether the lifetime of products meets requirements. Moreover, the proposed testing procedure can handle non-normal lifetime data. The managers can utilize this procedure to enhance product process capability.

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References