On retarded canards: 
Complex oscillations in delayed slow-fast systems

Maciej Krupa∗ Jonathan Touboul†§

May 6, 2014

Abstract We analyze canard explosions in delayed differential equations with a one-dimensional slow manifold. This study is applied to explore the dynamics of the van der Pol slow-fast system with delayed self-coupling. In the absence of delays, this system provides a canonical example of a canard explosion. The presence of delays significantly enriches the dynamics, and varying the delay induces canard explosion, mixed mode oscillations as well as transitions to complex bursting periodic orbits. We show that as the delay is increased a family of ‘classical’ canard explosions ends as a Bogdanov-Takens bifurcation occurs at the folds points of the S-shaped critical manifold. Canard explosion and mixed-mode oscillations are investigated by means of geometric perturbation analysis, and bursting by means of slow-fast periodic averaging.

Keywords Delayed Differential Equations Slow-Fast systems Canards Explosions Mixed-Mode Oscillations

Contents

1 Motivations and Model 4

2 Canard explosions in delayed differential equations 5
2.1 Fenichel theorem for delayed equation 5
2.2 Center manifold near singularities of the fast system 7
2.3 Canard explosion 8

3 Oscillatory dynamics of the delayed van der Pol system 13
3.1 Analysis of the full system 14
3.2 The nature of periodic orbits: numerical explorations 16
3.3 Analysis of the fast system 18
3.4 Canard explosion 28

∗MYCENAE Laboratory, Inria Paris-Rocquencourt, fistname.lastname@inria.fr
†The Mathematical Neurosciences Laboratory, Center for Interdisciplinary Research in Biology (CNRS UMR 7241, INSERM U1050, UPMC ED 158, MEMOLIFE PSL*)
Introduction

Nonlinear dynamical systems with multiple timescales and delays are essential in applications. For instance, realistic models of neuronal dynamics accounting for the dynamics of neuronal areas involve several excitable elements, whose dynamics occur on very different timescales, interacting after delays due to the transmission of information through synapses. Similar problems arise in different domains, including mechanical systems [3], macroscopic phenomena arising in chemistry, physics or social science. Such nonlinear systems involving multiple timescale dynamics and delays generally display a rich phenomenology, and particularly a wide repertoire of complex periodic behaviors. Slow-fast systems have been chiefly analyzed in finite-dimensional contexts. The topic of the present paper is to analyze the role of delays in dynamics of slow-fast systems, and more specifically in the emergence of complex oscillations.

Slow-fast system have attracted a lot of attention from theoreticians and applied mathematicians. One phenomenon of particular interest in such systems is the so-called canard explosion, that describes a very fast transition, upon variation of a parameter, from a small amplitude limit cycle to a relaxation oscillation, type of periodic solution consisting of long periods of quasi static behaviors interspersed with short periods of rapid transitions. These oscillations are ubiquitous in systems modeling chemical or biological phenomena [20, 28]. Canards were first studied about thirty years ago [1] in the context the van der Pol (vdP) equation with constant forcing. The authors showed that close to a Hopf bifurcation in this system, a small change of the forcing parameter leads to such a fast transition from small amplitude limit cycles to large amplitude relaxation cycles. This canard explosion happens within an exponentially small range of the control parameter. These phenomena generically arise in two-dimensional dynamical systems [20]. In higher dimensional systems with multiple timescales, more complex oscillatory patterns may arise. Two examples are given by the so-called Mixed Mode Oscillations (MMO) and bursting. MMOs are periodic orbits of a dynamical system characterized by an alternation between oscillations of very distinct large and small amplitude (see [2] for a recent review on the subject). Bursting correspond to periodic orbits composed
of periods of fast oscillations interspersed with periods of silence, and are a
typical regime of neuronal behavior (see e.g. [24] for a classification of different
possible geometries of bursting). Such transitions and dynamics have classi-
cally dealt with finite-dimensional systems. Here, we shall extend these analysis
to an infinite-dimensional setting by considering retarded slow-fast differential
equations.

This problem was first addressed by Campbell et al. [3] for the analysis
of a model of controlled drilling. Using a small delay approximations and the
property that such system present a two-dimensional inertial manifold [7], they
propose a two-dimensional ODE representation of the infinite-dimensional de-
layed system. This allows them to use the standard analysis of canards ex-
plusions in two dimensions and obtain a picture consistent with simulations of
the original delayed system. Here, we do not restrict our analysis to small de-
lays, and therefore reduction to a small dimensional ODE is no more possible.
In this general case, canard explosions persist. Indeed, as shown in [26], the
generic mechanism of canard explosions in two dimensions relies upon Fenichel
theory [17], the existence of connections in the fast subsystem, and the analysis
of trajectories near non-hyperbolic points (fold points and canard points). For
higher dimensional problem, center manifold reduction near fold points and ca-
nard points is necessary. Such elements are found in general delayed systems
with one-dimensional slow manifold, as is the case of our system of interest,
the self-coupled delayed van der Pol oscillator. In particular, Fenichel theorem
for advance and delayed equations has been developed in [23], center manifold
theory is classical [10, 15, 4]. We show that these elements lead to the presence
of canard explosions in delayed equations.

Interestingly, the self-coupled delayed vdP oscillator presents richer dynam-
ics than the sole canard explosion. Indeed, the fast dynamics is described by a
one-dimensional delayed equation manifesting highly non-trivial (and certainly
not reducible to a one-dimensional) dynamics. Similar to the non-delayed vdP
equation, it presents two fold bifurcations. But the presence of a sufficiently
large delay induces two additional Hopf curves, that connect to the curves of
fold bifurcations at a Bodganov-Takens point of the fast subsystem. This struc-
ture yields, in the slow-fast system, a curve of Hopf bifurcations connecting the
(supercritical) singular Hopf bifurcation arising from the canard explosion to
the curve of subcritical Hopf bifurcations, and therefore presenting a Bautin
point (Generalized Hopf). This dynamical picture yields complex oscillatory
patterns including small cycles, relaxation cycles, MMOs and bursts.

The paper is organized as follows. In section 1, we discuss neuroscience
motivations for the analysis of canard explosions in delayed systems. Theoretical
section 2 gives extensions of the classical results on canard explosion [26, 25]
to delayed systems. These results are applied to the delayed vdP system in
sections 3 and 4. Section 3 focuses on the oscillatory dynamics of the system and
provides analysis of the full system and of the fast system, while section 4 shows
how these elements are the source of complex oscillatory patterns including
MMOs, bursting and chaotic behaviors.
1 Motivations and Model

In this article, we will prove some general results on functional differential equations with multiple timescales and apply these to characterize the dynamics of the delayed self-coupled van der Pol system. This is a problem of general mathematical interest, and with wide applications. For instance, in neuroscience, the mean-field analysis of a large network of interacting neurons described by Fitzhugh-Nagumo equations [18] naturally lead to the analysis of such phenomena. The Fitzhugh-Nagumo (FhN) model is very similar to the vdP system. This model, initially introduced as a simplification of the very versatile but extremely complex Hodgkin-Huxley model of spike generation in squid giant axons, describes the excitable dynamics of the membrane potential of the neuron, \( v \), is coupled to a slow recovery variable \( w \), through the equations:

\[
\begin{aligned}
    v' &= v - \frac{v^3}{3} + w \\
    w' &= \varepsilon(a + b v - \gamma w)
\end{aligned}
\]  

where \( a \) represents the input current the neurons is subjected to and \( b \) is a interaction strength between the voltage and the recovery variable. The small parameter \( \varepsilon \) represents the timescale ratio between the voltage and the recovery variables. Note that the classical vdP oscillator corresponds simply to the case \( \gamma = 0 \). We denote \( v_t \) to indicate the value of the map \( v \) at time \( t \) for compactness of notations (instead of the more usual notation \( v(t) \)).

The model describes the behavior of an isolated neuron. However, relevant cortical behaviors generally arise at macroscopic scales involving a large number of neurons. In networks, the transport of information through the axon and the dynamics of release and binding of neurotransmitter at a synapse induce delays in the transmission of information. The coupling between neurons is maintained by specific structures called synapses. In [30], a linear feedback was introduced to model recurrent self-coupling. Here, for biological considerations, we shall examine the limit of networks of FhN neurons with electrical synapses (also called diffusive coupling). Such coupling between cells proceeds through direct contact of the intracellular domain of the two cells across so-called gap-junctions.

In this model, a \( N \)-neurons network with stochastic input satisfies the equations:

\[
\begin{aligned}
    dv_i^t &= (v_i^t - \frac{(v_i^t)^3}{3} + w_i^t + \frac{1}{N} \sum_{j=1}^{N}(v_j^t - v_{i-\tau}^t)) \, dt + \sigma dW_i^t \\
    dw_i^t &= \varepsilon(a + b v_i^t - \gamma w_i^t) \, dt
\end{aligned}
\]  

where \( W_i^t \) are independent Brownian motions. The presence of noise allows to prove (see [33]) that in the limit \( N \to \infty \), the behavior of a given neuron in the network satisfies the implicit stochastic equation:

\[
\begin{aligned}
    dv_t &= (v_t - \frac{(v_t)^3}{3} + w_t + J(v_t - E[v_{t-\tau}])) \, dt + \sigma dW_t \\
    dw_t &= \varepsilon(a + b v_t - \gamma w_t)
\end{aligned}
\]  

where \( E[v_t] \) denotes the statistical expectation of the process \( v_t \). Note that the presence of noise is essential to derive this limit. In specific cases [34, 35], such
mean-field equations can be exactly reduced to ordinary differential equations in aggregate variables (e.g., mean and standard deviation). Here, the nonlinear dynamics of the cells prevents from such a reduction. However, in the zero noise limit, one obtains the following self-coupled delayed Fitzhugh-Nagumo equation:

\[
\begin{align*}
  v'_t &= v_t - \frac{v_t^3}{3} + w_t + J(v_t - v_{t-\tau}) \\
  w'_t &= \varepsilon (a + b v_t - \gamma w_t).
\end{align*}
\]

(4)

This is precisely the equation we choose to analyze in the present manuscript. When \( \tau = 0 \), this equation is simply the original FhN model since the interaction term vanishes.

The parameters \( (a, b, \gamma) \) only affect the dynamics of the slow variable, and for any choice of parameters, the fast equation remains identical. Among these models, the simplest case happens to be the delayed vdP system corresponding to \( \gamma = 0 \) and \( b = -1 \):

\[
\begin{align*}
  x'_t &= x_t - \frac{x_t^3}{3} + y_t + J(x_t - x_{t-\tau}) \\
  y'_t &= \varepsilon (a - x_t),
\end{align*}
\]

(5)

because its equilibria have a very simple expression as a function of the parameters. This simplicity will allow to perform analytical characterization of its dynamics in the main text, and the FhN model can be understood qualitatively, and its analysis will be performed in appendix C.

But as a first step in order to uncover the behavior of these systems, we will develop a general analysis of slow-fast delayed differential equations.

2 Canard explosions in delayed differential equations

General theory of canard explosion developed for finite-dimensional ordinary differential equations, relies on three main ingredients: the existence and persistence of slow manifolds (Fenichel theory), a center manifold theory (generally near a fold point or a more degenerate point, e.g. Bogdanov-Takens point) and the existence and persistence of connections in the fast system. In this section, we show that all these elements persist in the infinite-dimensional setting of delayed differential equations, and that together, they yield canard transitions.

2.1 Fenichel theorem for delayed equation

The Fenichel theorem \[17\] proves the existence of slow manifolds and their persistence in slow-fast dynamical systems. While the original result holds for finite-dimensional equations, much effort has been devoted to extend this theory to infinite dimensional systems. In particular, the theorem has recently been extended to the context of advanced-retarded equations (more general than studied here) by Hupkes and Sandstede \[23\]. For such equations, the absence of
a semiflow due to the advance term prevents from using classical analysis based on semi-groups [10]. Their result of applies to the case of delayed differential equations. However, in that simpler case, we can use usual methods of spectral decomposition of the system and the semiflow of the equation, therefore readily extending the more classical techniques, developed in [11, 10], in order to prove Fenichel theory. This is why we state here a result which is convenient for our purposes, and include a sketch of the proof based on these methods in appendix A.

Let \( X = C([-h, 0], \mathbb{R}^n) \). We consider the following delay equation:

\[
\begin{align*}
    x'_t &= \int_0^h d\zeta(y_t, \tau)x_{t-\tau} + f(x_t, y_t, \varepsilon) \\
y'_t &= \varepsilon g(x_t, y_t),
\end{align*}
\]

with \( x \in \mathbb{R}^n, y \in \mathbb{R}, f : X \times \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^n, f(0, y, 0) = 0 \) and \( D_1 f(0, y, 0) = 0 \). We denote by \( x^t \) the element of \( X \) corresponding to the map \( x^t(\theta) = x_{t+\theta} \) for \( \theta \in [-\tau, 0] \). The above equation is classically written as a dynamical system in terms of the variable \( x^t \) taking values in \( X \). The fast subsystem, obtained by setting \( \varepsilon = 0 \) in (6), is given by

\[
x' = \int_0^h d\zeta(y, \lambda) x_{t-\tau} + f(x_t, y, \lambda),
\]

with \( y \) playing the role of a parameter. The set of equilibria of (6) parametrized by \( y \) is known as the critical manifold. We denote this set by \( S \). Suppose that a segment of \( S \) can be represented as a graph of a function \( \phi : [y_1, y_2] \to \mathbb{R}^n \). Then we can translate this segment of \( S \) to the origin. Hence we can assume that \( x = 0 \) is a solution of (7), referred to as the trivial equilibrium. Moreover, we can include the linear part of \( f \) at 0 in the term containing the integral. As a result of this rearrangement we have \( f = O(|x|^2) \). The associated dispersion relationship at the trivial equilibrium (obtained by linearization of (6) and evaluation on exponential functions with parameter \( \lambda \)) reads:

\[
\Delta(y, \lambda) = \lambda I - \int_0^h d\zeta(y, \tau)e^{-\lambda \tau}.
\]

Characteristic exponents governing the stability of the trivial solution are the values of \( \lambda \) such that \( \Delta(y, \lambda) = 0 \).

Now that these elements have been introduced, we state a generalization of the Fenichel theorem [17] to the context of delay equations of the form (6). As mentioned this result has been proved in much larger generality in [23].

Our proof may be easy to follow due to the relatively simple setting and more ‘classical’ approach. Moreover, the proof is used further in the paper as a basis for the proof of the theorem on canard explosion (more specifically, to prove the forthcoming Lemma 1).

**Theorem 1.** Suppose that there exist \( y_1 < y_2 \) such that the characteristic roots for \( y \in [y_1, y_2] \) (i.e., solutions of \( \Delta(y, \lambda) = 0 \)) are not on the imaginary axis. Then, for \( \varepsilon > 0 \) sufficiently small, there exist:
(i) a slow manifold $S_\varepsilon$ of the form $x = \phi(y)$, $y \in [y_1, y_2]$, satisfying $\phi = O(\varepsilon)$,
(ii) a finite dimensional unstable manifold $W^u_\varepsilon$ consisting of all the solutions that are exponentially repelled from $S_\varepsilon$. Any solution starting close to $S_\varepsilon$ becomes $O(e^{-c/\varepsilon})$ close to $W^u_\varepsilon$ before leaving a small neighborhood of $S_\varepsilon$.

The proof of this theorem is provided in appendix A.

2.2 Center manifold near singularities of the fast system

In this section we turn our attention to the behavior of the system close to singularities of the fast system; we discuss center manifold reductions around fold, canard and Bogdanov-Takens points. We consider a system of the form (6), and assume that there exists a fold point. This means that there exists $y_0$ such that the trivial solution of (6) has a simple 0 eigenvalue, i.e. the equation $\Delta(y_0, \lambda) = 0$ has a simple root $\lambda = 0$. For simplicity of notation we assume that $y_0 = 0$. Let $L$ denote the linear operator on $X$ given by

$$L(\Phi) = \int_0^h d\zeta(0, \tau) \Phi(-\tau)d\tau.$$ 

It follows that $L$ has a simple 0 eigenvalue. We consider the extended system

$$\begin{align*}
x' &= \int_0^h d\zeta(y, \tau)x_{t-\tau} + f(x_t, y, \varepsilon) \\
y' &= \varepsilon g(x, y) \\
\varepsilon' &= 0.
\end{align*}$$

Note that the point $(0, 0, 0)$ is a non-hyperbolic equilibrium with three eigenvalues equal to 0. It follows from center manifold theory for delay equations [15, 11] that there exists a three dimensional center manifold containing this point. Let $\Phi$ be the eigenfunction of the 0 eigenvalue and let $\Psi$ the the eigenfunction of the 0 eigenvalue of a suitably chosen adjoint operator, which are both constant functions in the case of $\lambda = 0$. Let $P$ be the projection with $\text{Im}(P) = \text{span}(\Phi)$ and the kernel given by the direct sum of the remaining eigenspaces. We have the following result.

**Proposition 1.** There exists a function $h : \mathbb{R}^3 \rightarrow \text{ker} P$ such the center manifold is given by

$$\{(x, \Phi + h(x_c, y, \varepsilon), y, \varepsilon) : (x_c, y, \varepsilon) \text{ are in a small neighborhood of } (0, 0, 0).\}$$

The reduction of (9) to the center manifold has the form

$$\begin{align*}
x_c' &= f_c(x_c, y, \varepsilon) \\
y' &= \varepsilon g_c(x_c, y, \varepsilon) \\
\varepsilon' &= 0,
\end{align*}$$

7
with
\[
g_c(x_c, y, \varepsilon) = g(x_c \Phi + r(x_c, y, \varepsilon), y, \varepsilon),
\]
\[
f_c(x_c, y, z) = \Psi r(f(x_c \Phi + h(x_c, y, \varepsilon), y, \varepsilon)),
\]
where \( r \) is the operator assigning to an element \( x \in X \) its value at \( 0 \). \( x(0) \).

\textbf{Proof.} We refer to [15] for details on center manifold reduction. Here we just point out that the eigenspace of 0 consists of a vector \( \phi \in X \) and the vectors \( (0, 1, 0)^T \) and \( (0, 0, 1)^T \) in the \( y \) and \( \varepsilon \) directions. Hence, the reduction in the \( y \) and \( \varepsilon \) directions is the same as in the ODE case.

We say that the point \((0, 0, 0)\) is a non-degenerate fold point if \( f_{c,xx}(0, 0, 0) \neq 0 \), \( f_{c,y}(0, 0, 0) \neq 0 \) and \( g_c(0, 0, 0) \neq 0 \). If \( g_c(0, 0, 0) = 0 \) then \((0, 0, 0)\) is a canard point. Non-degeneracy conditions for a canard point are complicated in general. These are recalled in the course of appendix D. We will not restate them here but rather refer the reader to [25] (the presence of delays does not modify these conditions since we have reduced the problem on a finite-dimensional manifolds). In the specific example we consider in the sequel these conditions are simpler and will be verified. The dynamics of a fold point or a canard point restricted to the center manifold is now as described in [25].

Finally, consider a system of the form (6) at a fold point with an additional degeneracy of Bogdanov-Takens type. This means that there exists \( y_0 \) such that the trivial solution of (6) has a double 0 eigenvalue with one eigenvector and one generalized eigenvector. In the extended system (9), this point is thus a non-hyperbolic equilibrium with four eigenvalues equal to 0. It follows from center manifold theory for delay equations [15, 5] that there exists a four dimensional center manifold containing this point. Reduction of the system around this point is similar to that of the previous section, and detailed calculations are provided in the particular case of the delayed van der Pol system in section 3.

\section{Canard explosion}

We consider a system of the form (9), depending on a regular parameter \( \mu \) with an S-shaped critical manifold, as shown in Fig. 1. Fix \( \mu = \mu_0 \). Let \( S = S_- \cup \{(x_m, y_m)\} \cup S_r \cup \{(x_M, y_M)\} \cup S_+ \).

The following hypothesis are necessary for a canard explosion result:

\textbf{(H1)} \((x_m, y_m)\) and \((x_M, y_M)\) are a non-degenerate canard point and a non-degenerate fold point\(^\dagger\).

\textbf{(H2)} \( S_- \cup S_+ \) consists of sinks of \( \Phi \) and \( S_r \) consists of saddle points with one dimensional unstable direction.

\textbf{(H3)} There exist connections from \( S_r \rightarrow S_\pm \) as shown in Fig. 1.

\(^\dagger\)The case of Bogdanov-Takens degeneracy at the fold is analyzed in depth in the van der Pol context in section 3.
We will verify hypotheses (H1) and (H2) in the context of (5) and present numerical evidence that (H3) is also satisfied. For the remainder of this section we assume that (H1)-(H3) hold. We begin with a result on extending a center manifold $C_{\varepsilon}$ existing near the canard point.

**Lemma 1.** Let $S_{r,\varepsilon}$ be a Fenichel slow manifold obtained by Theorem 4 and let $C_{\varepsilon}$ be the center manifold constructed near the canard point. There exists a choice of $W^u(S_{r,\varepsilon})$ (including a choice of $S_{r,\varepsilon}$ itself) such that $W^u(S_{r,\varepsilon})$ extends the center manifold $C_{\varepsilon}$. More specifically, $W^u(S_{r,\varepsilon})$ can be such that it agrees with $C_{\varepsilon}$ on their overlap.

This lemma is illustrated in Fig. 2.

**Proof.** First note that the manifolds $C_{\varepsilon}$ and $W^u(S_{r,\varepsilon})$ can be chosen so that their regions of existence overlap (see Fig. 2). On the overlap, the tangent spaces to $C_{\varepsilon}$ and $W^u(S_{r,\varepsilon})$ are close to each other, by construction. Hence, on the overlap, the two manifolds are close to each other. Note further that near the canard point the stable part of the spectrum is bounded away from 0, and hence, one can choose a neighborhood $V \subset X$ of $W^u(S_{r,\varepsilon})$ such that the trajectories in $V$ are exponentially attracted to $W^u(S_{r,\varepsilon})$ as long as they stay in $V$. By choosing the domain of existence of $W^u(S_{r,\varepsilon})$ so that it extends sufficiently close to the fold we can guarantee that $C_{\varepsilon}$ has a non-empty intersection with $V$. More specifically, we can choose a subset of $C_{\varepsilon}$ containing an interval $I_0$ defined by $y = y_0$ bounded by two points $(x_{c1}, y_0)$ and $(x_{c2}, y_0)$ such that $\Phi_{c,t}(x_{c1}, y_0)$ escapes from $C_{\varepsilon}$ towards $S_{-,\varepsilon}$ and $\Phi_{c,t}(x_{c2}, y_0)$ escapes from $C$ towards $S_{+,\varepsilon}$.
Figure 2: Center manifold $C_\varepsilon$ and the unstable manifold of the saddle-type slow manifold $S_{r,\varepsilon}$ (see Lemma 1).

($\Phi_{c,t}$ denotes the flow of (10)). We consider an interval $I_1 \subset \{ y = y_1 \} \subset C_\varepsilon$ given by a transition map from $I_0$ to $I_1$ by the flow $\Phi_{c,t}$. By exponential attraction of $W^u(S_{r,\varepsilon})$ the interval $I_1$ is exponentially close to $W^u(S_{r,\varepsilon})$. We now extend the manifold $C_\varepsilon$ by applying the semi-flow $\Phi_t$ to initial conditions in $I_1$ and intersecting with $V$. This gives a $C^0$ manifold $\tilde{C}_\varepsilon$ exponentially close to $W^u(S_{r,\varepsilon})$. To see that this manifold is smooth notice that each trajectory on $\tilde{C}_\varepsilon$ can be obtained as a fixed point of an operator analogous to the operator defined by (31) with $-\infty$ (respectively $\infty$) replaced by $-k_1/\varepsilon$, where $k_1 > 0$ is such that the transition time from $I_1$ to $I_2$ equals $k_1/\varepsilon + O(1)$ (respectively $k_2/\varepsilon$, where $k_2 > 0$ is such that the transition time from $I_2$ to the vicinity of the upper fold equals $k_2/\varepsilon + O(1)$). It is now possible to prove, using the same approach as in the proof of the center manifold theorem that these trajectories depend smoothly on initial conditions in $I_1$ and on parameters, including $\varepsilon$. It follows that $C_\varepsilon$ is smooth.

Lemma 2. There exists a choice of the stable slow manifold $S_{+,\varepsilon}$ and the center manifold $C_\varepsilon$ such that a segment of $S_{+,\varepsilon}$ is included in $C_\varepsilon$. Moreover, there exists a smooth curve in the parameter space of the form $(\mu_{c}(\varepsilon), \varepsilon)$ such that if $\mu = \mu_{c}(\varepsilon)$ then $S_{+,\varepsilon}$ connects to $S_{r,\varepsilon}$. The connection from $S_{+,\varepsilon}$ to $S_{r,\varepsilon}$ is called a canard solution.

**Proof.** We first modify the construction of $C_\varepsilon$ and $S_{+,\varepsilon}$ to ensure that a segment of $S_{+,\varepsilon}$ is included in $C_\varepsilon$. Note that $S_{+,\varepsilon}$ is defined as a graph of a function $\Phi_{\varepsilon} : \mathbb{R} \to X$. We define $\varphi_{\varepsilon} = r(\Phi_{\varepsilon}(y))$, where $r$ is the restriction operator.
introduced in Appendix A. The invariance of $S_{+,\varepsilon}$ now implies
\[
\varepsilon \varphi'_\varepsilon(y)g(\varphi_\varepsilon(y), y) = \int_0^h d\zeta(y, \tau)\Phi_\varepsilon(y)(-\tau) + F(\varphi_\varepsilon(y), y).
\] (12)

Note that $\varphi_\varepsilon$ is not defined on the neighborhood of $(x_m, y_m)$. We extend it by an arbitrary function, just making sure the extension has the same degree of regularity. We fix $y_1$ and $y_2$ satisfying $y_m < y_1 < y_2 < y_M$ and let $\kappa : \mathbb{R} \to \mathbb{R}$ be a non-negative function equal to 1 on $[y_1, y_2]$ and 0 on a neighborhood of $y_m$. Let $\psi_\varepsilon(y) = \kappa(y)\varphi_\varepsilon(y)$. We define a new variable
\[
\tilde{x} = x - \psi_\varepsilon(y)
\] (13)
and transform (6) to the new variables. It follows from (12) that (6) transforms to
\[
\begin{aligned}
x' &= \int_0^h d\zeta(y, \tau)x_{t-\tau} + \tilde{f}(x_t, y, \varepsilon) \\
y' &= \varepsilon g(x + \varphi_\varepsilon(y), y),
\end{aligned}
\] (14)
where $\tilde{f}(0, y, \varepsilon) = D_x\tilde{f}(0, y, \varepsilon) = 0$. We now pick $y_3$ between $y_2$ and $y_1$ and let $\eta(y)$ be a $C^\infty$ function which satisfies $\eta' > 0$ for $y > y_3$, $\eta(y_2) = 0$, $\eta(y) = 1$ for $y < y_3$. We consider the system
\[
\begin{aligned}
x' &= \int_0^h d\zeta(y, \tau)x_{t-\tau} + \tilde{f}(x_t, y, \varepsilon) \\
y' &= \varepsilon g(x, y)\eta(y).
\end{aligned}
\] (15)

Note that (15) has a saddle type equilibrium point at $(0, y_2)$ with one dimensional unstable manifold. We now construct a center manifold for (14) near the canard point $(x_m, y_m)$. Note that, by choosing $y_2$ sufficiently small, we can ensure that the added saddle point is on the center manifold $\tilde{C}_\varepsilon$, as well as its unstable manifold, which, for $y \in (y_1, y_2)$, coincides with the line $x = 0$. To complete the proof of the claim we make two observations. First, since the dynamics of (14) and (15) coincide on a small neighborhood of the canard point, $\tilde{C}_\varepsilon$ defines also a center manifold of (6). Second, since the dynamics of (15) and (14) are the same for $(y_2, y_3)$, the line segment $\{(0, y), y \in (y_2, y_3)\}$ is both on $\tilde{C}_\varepsilon$ and corresponds to a segment of $S_{+,\varepsilon}$. The first claim of the lemma follows.

To prove the second claim note that using Lemma [1] we can extend the specific choice of the manifold $C_\varepsilon$ whose existence was proved in the preceding paragraph, all the way to the vicinity of $(x_M, y_M)$. The existence of a connecting orbit from $S_{+,\varepsilon}$ to $S_{r,\varepsilon}$ is then a direct conclusion of the arguments in [25] as segments of both $S_{+,\varepsilon}$ and $S_{r,\varepsilon}$ are contained in $\tilde{C}_\varepsilon$. As in [25] we set up a Melnikov integral and observe that its value is determined, up to exponentially small terms, by the restriction of the flow on $C_\varepsilon$ to a small neighborhood of the canard point. $\Box$
We now formulate conditions that guarantee the stability of canard cycles, see [26]. Let \( \lambda_p(y) \) be the unique positive root of \( \Delta(\lambda, y) = 0 \) corresponding to the saddle-type part of \( S_0 \). Let \( \lambda_{n,+}(y) \) be the largest characteristic root corresponding to the right branch of \( S^+ \). We parametrize the branches of \( S \) between the fold points, associated with \( y = y_m \) and \( y = y_M \) as \((y, \phi_-(y))\) and \((y, \phi_+(y))\), with \((y, \phi_-(y))\) and \((y, \phi_+(y))\) corresponding to the stable branches and \((y, \phi_r(y))\) to the saddle type branch. For every \( y_* \) satisfying \( y_m < y_* < y_M \) let

\[
R_{n,+}(y) = \int_{y_m}^{y_*} \frac{\lambda_{n,+}(y)}{g(\phi_+(y), y)} dy \\
R_{n,-}(y) = \int_{y_*}^{y_M} \frac{\lambda_{n,-}(y)}{g(\phi_-(y), y)} dy \\
R_p(y) = \int_{y_*}^{y_m} \frac{\lambda_p(y)}{g(\phi_r(y), y)} dy.
\]

We make the following assumption:

\((H4)\) \(R_{n,+}(y_*) > R_p(y_*)\) for every \( y_* \) satisfying \( y_m < y_* < y_M \).

**Theorem 2.** Suppose \((H4)\) holds, in addition to \((H1)-(H3)\). Then, for every \( \varepsilon \) sufficiently small, there exists a family of canard cycles continuing from small, Hopf type cycles to relaxation cycles, through canards with no head and subsequently canards with head. The transition from small canards to canards with large head takes place in an exponentially small interval of the parameter \( \mu \). The cycles are stable and unique (at most one for each choice of \((\mu, \varepsilon)\)) and depend smoothly on the parameters.

**Proof.** The hypotheses \((H1) - (H4)\) guarantee the local part of canard explosion restricted to the center manifold. By Lemmas 1 and 2 we can choose \( C_\varepsilon \) so that it contains a segments of \( S_{+,\varepsilon} \) and extends to \( W^u(S_r) \). We can now measure the separation between \( S_{+,\varepsilon} \) and \( S_{r,\varepsilon} \) in the 2-dimensional center manifold \( C_\varepsilon \), in which the flow is as described in [26, 25]. When \( S_{+,\varepsilon} \) and \( S_{r,\varepsilon} \) are exponentially close, which corresponds to the parameter region very close to the locus of a connection from \( S_{+,\varepsilon} \) to \( S_{r,\varepsilon} \), then the forward continuation of \( S_{+,\varepsilon} \) follows \( W^u(S_{r,\varepsilon}) \) for a time of order \( O(1/\varepsilon) \) and splits off towards either \( S_{+,\varepsilon} \) or \( S_{-,\varepsilon} \). Either way, it ends up being attracted to \( S_{+,\varepsilon} \) and returning very close to itself in the vicinity of the canard point. We now consider a small neighborhood of a point \( p \in S_{+,\varepsilon} \), near the canard point. Due to \((H4)\) the flow along the forward continuation along \( S_{+,\varepsilon} \) is an exponential contraction, which implies the existence of an asymptotically stable canard cycle. One can now apply standard theory for limit cycles to conclude that canard cycles depend smoothly on parameters if \( \varepsilon > 0 \). Moreover, all canard cycles must be in an exponentially small wedge of the parameter plane around the curve corresponding to a connection from \( S_{+,\varepsilon} \) to \( S_{r,\varepsilon} \). This curve is determined by the flow on \( C_\varepsilon \) and is therefore smooth in the limit \( \varepsilon \to 0 \). \( \square \)
Remark 1. To prove smoothness in $\varepsilon$ in the limit $\varepsilon \to 0$ we can set up an operator on a space of loops, (more specifically a Liapunov-Schmidt operator \[19\]), which would have as fixed points the canard cycles we have found. The Liapunov-Schmidt operator can be expressed using a variation of constants formula similar to \[31\], with the linear flow given by the Floquet flow along the canard. The integration time in the term containing the integral would be the period of the canard cycle. Smooth dependence of canard cycles on parameters and smoothness of curves in the parameter space corresponding to families of canard cycles defined by a specific feature (like the maximal height) may be proved using a similar approach as in the proof of the center manifold theorem \[10\], i.e. by extending the operator to a larger space including the derivative terms and proving the existence of a fixed point in that space.

If (H4) does not hold it is possible to obtain partial results, based on the following.

Theorem 3. Suppose that (H1)-(H3) holds and $R_{n,+}(y_*) > R_p(y_*)$ for some $y_* \in (y_m, y_M)$. Then there exists a smooth curve in the parameter plane of the form $(\mu(\varepsilon), \varepsilon)$ corresponding to the locus of existence of canard cycles with no head passing through the point $(x_m, y_*)$. The canard cycles belonging to this family are asymptotically stable and depend smoothly on $\varepsilon$. Similarly, suppose that $R_{n,-}(y_*) + R_{n,+}(y_M) > R_p(y_*)$ for some $y_* \in (y_m, y_M)$. Then there exists a smooth curve in the parameter plane of the form $(\mu(\varepsilon), \varepsilon)$ corresponding to the locus of existence of canard cycles with head passing through the point $(x_M, y_*)$. The canard cycles belonging to this family are asymptotically stable and depend smoothly on $\varepsilon$.

Proof. The proof is analogous to the proof of Theorem 2. \[\square\]

Remark 2. Note that the condition $R_{n,-}(y_*) + R_{n,+}(y_M) > R_p(y_*)$ must be satisfied for $y_*$ sufficiently small. This implies that canards with ‘large head’ must exist and be stable. Similarly, canard explosion is locally always either subcritical or supercritical if a non degeneracy condition holds (see \[20\]). Finally, all canard cycles whose existence follows from Theorem 3 are exponentially close in the parameter space to a segment of the canard solution and therefore the $\mu(\varepsilon)$ values they correspond to must be exponentially close to $\mu_c(\varepsilon)$. This means that there exists a weak version of a canard explosion even if (H4) does not hold, namely a transition from small cycles to canard cycles with ‘large head’ which occurs in an exponentially small region.

3 Oscillatory dynamics of the delayed van der Pol system

We now use the general canard explosion theory to analyze the dynamics of the delayed van der Pol system \[5\] introduced in section 1. It is trivial to see
that these systems can be written in the form of the general delayed differential
equation studied in section 2, with:

\[
\begin{align*}
  f(x, y, \varepsilon) &= (J + 1)x - \frac{x^3}{3} + y \\
  g(x, y) &= a - bx - \gamma y \\
  \zeta(y, s) &= \mathbb{1}_{s > \tau},
\end{align*}
\]

and a simple change of origin allows modifying the system into an equation such
that the assumptions of section 2 are satisfied.

We start by analyzing the fixed points of the fast system and their stability. In
order to be able to perform a fully analytical study, we will consider the
case of the delayed van der Pol system in section 3.1 because in this case, the
equilibria are much simpler to express than in the Fitzhugh-Nagumo case. The
singular perturbation analysis will follow, and since the fast system is identical
for all models of the class considered, it is not specific to one model. From this
analysis, we account for the complex behavior that are identified numerically in
section 3.1, and specifically discuss behaviors of the Fitzhugh-Nagumo system
in appendix C.

### 3.1 Analysis of the full system

Let us start by analyzing the fixed point of the van der Pol delayed system (4).
The equilibria of the system are given by \((v_a, w_a) = (a, -a + a^3/3)\). The stability
of this fixed point is characterized by the sign of the characteristic roots \(\xi\) of the
system. These are found as solutions of the characteristic equation obtained by
linearizing the system at the fixed point. The linearized equations are:

\[
\begin{align*}
  \dot{x}_t &= (1 - a^2 + J)x_t + y_t - Jx_{t-\tau} \\
  \dot{y}_t &= -\varepsilon x_t
\end{align*}
\]

and the characteristic equation is obtained when looking for solutions of the
form \((x, y)e^{\xi t}\). Substituting this into the linearized equation and denoting by
we readily obtain:

\[
\begin{align*}
  \xi x &= (1 - a^2 + J(1 - e^{-\xi \tau}))x + y \\
  \xi y &= -\varepsilon x
\end{align*}
\]

which has non-trivial solutions when \(\det(\xi I_2 - M) = 0\) with

\[
M = \begin{pmatrix}
  1 - a^2 + J(1 - e^{-\xi \tau}) & 1 \\
  -\varepsilon & 0
\end{pmatrix}
\]

i.e.:

\[
\xi(1 + a^2 - J(1 - e^{-\xi \tau})) + \varepsilon = 0
\]

which is equivalent, since \(\xi = 0\) is never a solution, to the equations:

\[
Je^{-\xi \tau} = -\frac{\varepsilon}{\xi} - \xi + (1 - a^2 + J).
\]
The possible Hopf bifurcations correspond to purely imaginary values of $\xi = i\zeta$:

$$Je^{-i\zeta}\tau = i\zeta (1 - a^2 + J) =: Z(\zeta)$$

which can be easily solved equating modulus and argument. Indeed, one obtains:

$$\begin{cases} J^2 = (\zeta - \bar{\zeta})^2 + (1 - a^2 + J)^2 \\
\tau = \frac{\text{Arg}(Z(\zeta)) + 2k\pi}{\zeta} \end{cases}$$  \hspace{1cm} (17)$$

with $k \in \mathbb{Z}$. These equations hence provide values of the parameters $(a, J, \tau, \varepsilon)$ related to possible Hopf bifurcations. It is easy to see that any possible Hopf bifurcation arise in a bounded region of values of $a \in [1, \sqrt{1 + 2J}]$. Considering $a$ and $\tau$ as free parameters, and can be seen as parametric equations in $\xi$, and the example of $J = 2$ yields the bifurcation diagram displayed in Figure 3. In order to confirm that, at this point, the system does have the universal unfolding of the Hopf bifurcation, and to characterize the sub- or super-critical nature of the Hopf bifurcation, we reduce the system to normal form at this point. This is performed using classical methods [16] and detailed in appendix B. We observe that this is the case for all points except one singular point, for $\tau_s \simeq 0.5$, at which point the first Lyapunov exponent of the system crosses zero. For $\tau < \tau_s$, the Hopf bifurcation is supercritical (hence related to the presence of a family of stable limit cycles), and for $\tau > \tau_s$, the Hopf bifurcation is subcritical. This reduction is valid until $a = \sqrt{1 + 2J}$ where transversality conditions fail, and the Hopf bifurcation disappear. At $\tau = \tau_s$, it is conjectured that the system undergoes a Bautin bifurcation. It is clear that, at this point, the map that associates to $\tau$ the eigenvalues of the Jacobian matrix and the first Lyapunov coefficient is regular. In order to ensure that the system indeed undergoes a Bautin bifurcation, one needs to compute the second Lyapunov coefficient at this point and show that it is not zero. To this end, one needs to extend the analysis performed in appendix B to access terms of order 5 of the reduction of the system on the center manifold, leading to tedious computations. This bifurcation, close from the point $\tau = 1/J$, will be shown to result from the merging of a supercritical singular Hopf bifurcations curve and a subcritical delay-induced bifurcation of the fast system. This hybrid nature of the transition also explains why the Bautin point is located close from a point where the derivative of the Hopf curve is large. We observe that this curve is singular at the folded node $a = 1$ corresponding to the Hopf bifurcation of the non-delayed system. It is easy to see using equations (17) that close to $a = 1$, we have $\zeta = \sqrt{\varepsilon} + \sqrt{J}/\sqrt{a - 1} + O(a - 1)$, and therefore:

$$\tau = 2\sqrt{\frac{a - 1}{J\varepsilon}} + O(a - 1).$$

This behavior explains the singular behavior observed on the Hopf bifurcation curve at $a = 1$: the Hopf bifurcation curve behaves as the square root of $a - 1$, hence shows an infinite differential close to this point. Moreover, the coefficient
Oscillations

Figure 3: Hopf bifurcations in the delayed Van de Pol system for $J = 2$ and $\varepsilon = 0.01$ (the diagram is symmetrical with respect to $\alpha = 0$). Red line: Hopf bifurcations, plain: supercritical, dotter: subcritical, separated by a Bautin point (or Generalized Hopf, GH). Blue line: vertical asymptote $\alpha = \sqrt{1 + 2J}$, green dotted line: $\alpha = 1.01$.

in front of the square root is inversely proportional to the square root of $\varepsilon$, explaining the very sharp evolution away from $\tau = 0$.

The Hopf bifurcation disappear for $\alpha > \sqrt{1 + 2J}$. For $\alpha = \sqrt{1 + 2J} - \eta$ with $0 < \eta \ll 1$, we can show that $\xi = \sqrt{\varepsilon} + (1 + 2J)^{1/4}\sqrt{\eta} + O(\eta)$. Hence the differential of the curve at this point is again infinite at $\eta = 0$, and moreover it is easy to see that the value reached by $\tau$ at this point is $\pi/\sqrt{\varepsilon}$, hence corresponds to very large delays. At this point, transversality conditions fail, and the bifurcation disappears beyond this point.

3.2 The nature of periodic orbits: numerical explorations

The presence of these bifurcations are the hallmark of a transition between fixed point and periodic orbits. Periodic solutions exist in a particular region of the parameter space, painted in yellow in Figure 3. However, this information does not allow distinguishing between the different types of possible oscillatory patterns. In this section, we numerically explore the behavior of the slow-fast delayed vdP system upon variation of the delay.

Throughout the section, we fix $a = 1.01$, $\varepsilon = 0.01$ or $0.05$ and $J = 2$ (green line in the bifurcation diagram Figure 3). We observe distinct qualitative behaviors as $\tau$ is increased. For values of the delay smaller than the value corresponding to the Hopf bifurcation ($\tau \simeq 0.429$), the fixed point $(v_a, w_a)$ remains stable, and precisely when crossing the Hopf bifurcation, a small cycle
appears. In a very small range of delay values, the cycle tends to complexify and display behaviors typical of the presence of period doubling bifurcations. The system also displays chaotic behaviors as delays are further increased (although in a very small neighborhood of the Hopf bifurcation), see Figure 4(b). Complex dynamics of the small cycles will be accounted for in section 3.4.

Figure 4: Simulation of the delayed van der Pol equation for $a = 1.01$, and $\varepsilon$ was increased to 0.05 in order to follow the phenomenon. (a): $\tau = 0.4$, $\tau = 0.401$ and $\tau = 0.408$. The cycle arising from the Hopf bifurcation, after a few period doubling bifurcations, shows a chaotic profile $\tau = 0.41$ (b), as illustrated by the Ruelle plot (left) of the permanent dynamics on a Poincaré section (blue line at $w = -0.66$).

As the delay is further increased suddenly appear large oscillations, hallmark of the presence of canard bifurcations in the system, that will be specifically addressed in section 3.4 using the methods of section 2. These oscillations are more complex than relaxation oscillations emerging in two-dimensional slow-fast systems. They show the typical shape of Mixed-Mode oscillations, with three distinct phases: one small oscillation related to the ghost of the small periodic orbit (highlighted in blue boxes in the figure, ‘classical’ MMO small oscillations), small oscillations related to the properties of the convergence of the fast flow on the critical manifold (red boxes), and large relaxation oscillations corresponding to switches between the two attractive parts of the critical slow manifold. We will study this phenomenon more in details in section 4.1.

As the delay is further increased, these MMOs suddenly disappear in favor of bursting periodic orbits (see Fig. 6), characterized by the presence of very fast oscillations interspersed by periods of slow behavior. The analysis of these orbits will be performed in section 4.2. In the figure, we plotted time series of the bursts for our choice parameters, and for legibility provide a zoom on the trajectories with larger $\varepsilon$ which corresponds to less fast oscillations. This choice allows to display legibly the trajectories in the extended phase space $(x_t, y_t, x_{t-\tau})$.

Moreover, at the transition from MMOs to bursts, the system displays extremely wild chaotic behaviors characterized by irregular switches between MMOs and bursts. These phenomena will be accounted for in section 4.3.
3.3 Analysis of the fast system

The singular perturbation theory analyzes properties of both flows when $\varepsilon \to 0$, by decomposing the dynamics into its slow and fast components. Section 2 develops this singular perturbation theory to delayed differential equations, with a specific focus on canard explosion. Here, we put in good use the results of our theoretical section, together with a separation of the dynamics into slow and fast dynamics, in order to analyze the behaviors of the delayed vdP system numerically exhibited in the previous section.

In order to uncover the precise dynamics and phenomena arising in the delayed FhN or vdP equations, we decompose the analysis by studying first the fast subsystem. The fast dynamics is given by the solution of the singular limit $\varepsilon \to 0$ in equation (4), i.e.:

$$x'_t = x_t - \frac{x_t^3}{3} + y + J(x_t - x_{t-\tau})$$

(18)

where $y$ is a constant value, corresponding to the value of $y_t$ which is constant
Figure 6: Bursting solutions. (a): $\varepsilon = 0.01$ shows alternation of slow dynamics and fast oscillations. For legibility because of the increase in the number of fast oscillations for small $\varepsilon$, Figures (b) and (c) are plotted with $\varepsilon = 0.05$.

in the singular limit. We recall that this equation is identical to the fast system of the FitzHugh Nagumo system. The variable $y$ hence constitutes a parameter of the fast dynamics, which is given by a delayed differential equation. Fixed points are given by the solutions to the algebraic equation:

$$x - \frac{x^3}{3} + y = 0,$$

which has three real solutions (fixed points) when $|y| < \frac{2}{3}$ and one fixed point otherwise. These can be obtained in closed from using Cardano’s method. For
Figure 7: Chaotic trajectories: $\tau = 0.7$. (A): time series shows irregular alternation of bursts (B), MMOs (C) and mixed behaviors with the presence of both MMO and burst (D).

$|y| > 2/3$, we have $\Delta := 9y^2 - 4 > 0$ and hence the unique solution is given by:

$$x_0 = \left( \frac{3y + \sqrt{9y^2 - 4}}{2} \right)^{1/3} + \left( \frac{3y - \sqrt{9y^2 - 4}}{2} \right)^{1/3}$$

and for $|y| < 2/3$, $\Delta < 0$ and the three solutions are given by

$$x_k = 2 \cos \left( \frac{1}{3} \arccos \left( \frac{3y^2}{2} \right) + \frac{2k\pi}{3} \right), \quad k = 0, 1, 2.$$

and for $y = \pm 2/3$, we have a double root $x = \mp 1$ and a simple root $x = \pm 2$.

There are hence three branches of fixed points: $x_+(y) \geq 1$ corresponding to the branch of solutions for $y \geq -2/3$, $x_-(y) \leq -1$ corresponding to $y \leq 2/3$ and $x_0(y) \in [-1, 1]$ defined for $y \in [-2/3, 2/3]$.

**Stability of the equilibria**

The stability of these fixed points is characterized by the sign of the real parts of the characteristic roots $\xi$ of the system. These are found as solutions of the characteristic equation obtained by linearizing the system at the fixed point. It is easy to show that the characteristic roots around a given fixed point $x^*$ are the solutions $\xi$ of the equations:

$$\xi = 1 - (x^*)^2 + J - Je^{-\xi \tau}$$

(19)
This equation may be solved using special functions\(^2\). However, in order to characterize the bifurcations, we only need to access to the points in the parameter space corresponding to the points at which the characteristic roots cross the imaginary axis. Such points can be accessed through elementary algebraic manipulations.

Saddle-node bifurcations correspond to \( \xi = 0 \). Using equation (19), it is clear that such bifurcations occurs if and only if:

\[
x^* = \pm 1
\]

which corresponds, when using the characterization of \( x^* \) as a fixed point of the system, to the point in the parameter space:

\[
y = \pm \frac{2}{3}
\]

and this bifurcation is independent of the delay \( \tau \) and the coupling strength \( J \).

Hopf bifurcations occur when \( \xi = i\zeta \) for \( \zeta > 0 \) (in that case, of course, the complex conjugate root \(-i\zeta\) is also a characteristic root). The characteristic equation can be solved either by equating real and imaginary parts of the equation, or taking the modulus and argument. The later method yields the equations:

\[
J^2 = (1 - (x^*)^2 + J)^2 + \zeta^2
\]

\[
\tau = -\frac{\arctan\left(\frac{\zeta}{1 - (x^*)^2 + J}\right) + 2k\pi}{\zeta}.
\]

Equation (20) fixes the value of \( \zeta \) as a function of the parameters and equation (21) provides the delay associated to the Hopf bifurcation. Such bifurcations exist as soon as (20) has a real solution. One can remark at this point that it is very convenient to consider artificially \( x^* \) as parametrizing the equations. Indeed, as a function of \( x^* \), Hopf bifurcations arise when \( J^2 - (1 - (x^*)^2 + J)^2 \geq 0 \), i.e. \( |x^*| \in [1, \sqrt{1 + 2J}] \). The extremal point corresponding to \( x^* = \sqrt{1 + 2J} \) is associated to the value \( \tau = \frac{1}{J} \). We conclude that the only fixed points which undergo Hopf bifurcations are the two extremal fixed points \( x_\pm(y) \), and that the fixed point \( x_0(y) \), which is unstable for \( \tau = 0 \) (saddle fixed point), remains unstable. Hopf bifurcations correspond to parameters such that:

\[
\begin{cases}
  y = \frac{(x^*)^3}{3} - x^* \\
  \zeta = \sqrt{J^2 - (1 - (x^*)^2 + J)^2} \\
  \tau = -\frac{\arctan\left(\frac{\zeta}{1 - (x^*)^2 + J}\right) + 2k\pi}{\zeta}
\end{cases}
\]

\(2\) Indeed, the solutions to the characteristic equation are given by the Lambert functions \( W_k \) (the different branches of the inverse of \( x \mapsto xe^x \), see e.g. [8]):

\[
\xi = A + \frac{1}{\tau}W_k\left(-\tau Je^{-\tau A}\right)
\]

with \( A = 1 - (x^*)^2 + J \). The stability of \( x^* \) is hence governed by the sign of the real part of the rightmost eigenvalue, given by the real branch \( W_0 \) of the Lambert function, and if the argument of the Lambert function has a real part greater than \( -e^{-1} \) the root is unique. If not, we have two eigenvalues with the same real part corresponding to \( k = 0 \) or \( -1 \).
(a) Fast subsystem

(b) Fast subsystem vs Full system

Figure 8: Hopf bifurcation for the fast subsystem. (a) The branch of solutions $x_{-}(y) (x_{+}(y))$ undergo Hopf bifurcations along the red (blue) curve yielding cycles in the yellow and pink region. Vertical asymptotes of the red (blue) curve correspond to $y = 2/3 (y = -2/3)$ and the Hopf bifurcation disappears for $y \geq \frac{2}{3} \sqrt{1 + 2J(J - 1)} (y \leq \frac{2}{3} \sqrt{1 + 2J(J - 1)})$. Here we chose $J = 2$. (b) Bifurcations for $y = a$ of the fast subsystem superimposed to the bifurcation diagram of the full system.

Considering these equations as parametrized curves, we hence obtain the locus of the possible Hopf bifurcations in the parameter space. For fixed $J = 2$, we obtain the curves displayed on figure Fig. 8 The fast subsystem hence undergoes Hopf bifurcations as a function of the value of the delays $\tau$ and of the slow variable $y$, and are therefore associated to a family of periodic orbits. This curve emerges transversally from the line of fold bifurcations $y = 1$, a point at $\tau = 1/J$, with a slope $\frac{2}{3J^2}$. This point will central in the dynamics: we will show that for $\tau < 1/J$, the system undergoes a supercritical canard explosion, that the system at this point undergoes a Bogdanov-Takens bifurcations, it will account for an angular shape of the curve of Hopf bifurcations for $0 < \varepsilon \ll 1$ and for the presence of a Bautin bifurcation in the neighborhood of this fixed point.

In order to characterize the behavior of the system, we shall now specify the type of Hopf bifurcation obtained. Using classical methods of reduction to normal form similar to those used in appendix for the full system, we can find a closed-form formula for the first Lyapunov coefficient. We find that this coefficient is always positive and tends to zero for $x^* \to 1^+$. Therefore, the Hopf bifurcation is subcritical, and the related cycles are unstable. Calculations being classical, and very similar to those detailed in the full system case, are not displayed, and we will focus on the codimension two bifurcation arising along this curve.
Bogdanov-Takens singularity

The above bifurcation diagram and the computation of the first Lyapunov coefficient of the normal form of the Hopf bifurcation tend to indicate the presence of a singular codimension two point corresponding to \( y = 1 \) and \( \tau = 1/J \). At this point, the saddle-node bifurcation line \( y = 1 \) and the Hopf bifurcation line collide. Characterizing this bifurcation will hence allow understanding the transition for \( y \) close to one, and we shall now reduce the system to normal form around this point. Let us start by noting that at this point the linearized operator corresponds to a the Bogdanov-Takens singularity. Indeed, we verify that \( x_t = t \) is a solution of:

\[
u' = (1 - (x^*)^2)\nu + J(\nu_t - \nu_{t-\tau}).\tag{23}
\]

This is done by the following computation:

\[
u' = 1 \quad \text{and} \quad \nu_t(1 - (x^*)^2) + J(t - (t - \tau)) = J\tau \quad \Rightarrow 1 = J\tau.
\]

Hence the two dimensional space of affine functions \( t \mapsto \alpha + \beta t \) is contained in the solution space of (23) with the action of the RHS of (23) given by \( \alpha + \beta t \mapsto \beta \).

In the basis given by the functions 1 and \( t \) this gives the matrix

\[
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}.
\]

Based on (19) we can draw some conclusions on the bifurcation diagram in the plane \((\tau, y)\). First of all the saddle node bifurcation occurs for \( y = -2/3 \) (or \( y = 2/3 \), case treated in an identical manner). Near such a bifurcation point

\[
y = -2/3 + ((x^* - 1)^2) + O((x^* - 1)^3).
\]

Next, using (19) and expanding the first equation we get

\[
1 - (x^*)^2 + 3J\frac{\tau - 1}{\tau} + O((J\tau - 1)^2) = 0.
\]

It now follows that the resulting curve of Hopf bifurcations is tangent to the \( y = -2/3 \) line, which is consistent with the generic Bogdanov-Takens picture.

In order to ensure that the system at this point indeed enjoys the universal unfolding of the Bogdanov-Takens bifurcation and characterize the local behavior, we reduce the system to normal form at this point. We start by changing variables defining \( x = y + 2/3 \) and \( u(t) = x(\tau t) \), so that equation (18) writes

\[
\dot{u} = \tau J(u_t - u_{t-1}) - \tau u_t^2 + \tau x
\]

or equivalently:

\[
\dot{u} = \delta_u + \nu u_t - \tau u_t^2 + \tau x
\]

with \( \delta_u = (u_t - u_{t-1}) \) and \( \nu = (J\tau - 1) \), which is a convenient notation since the singular point considered corresponds to \( J\tau = 1 \). Normal form reductions for such delayed equations was analyzed in \([15]\), and the methodology exposed
in that article applies to our case. The principle of the reduction follows the classical theory of finite-dimensional normal forms reductions, and in that paper the authors identify the eigenvectors (in our case, the constant function and the identity) and related projectors. Direct reduction of our equation in this system of coordinates provides the equations:

\[
\begin{align*}
\dot{x}_1 &= \nu x_1 + x_2 - \frac{2}{3} \tau x_1^2 + \tau y \\
\dot{x}_2 &= 2\nu x_2 - 2\tau x_1^2 + 2\tau y.
\end{align*}
\]

From these equations, we pursue the reduction to bring the system into canonical form of the Bogdanov-Takens bifurcation as for instance given by [27, 21], and obtain, with our notations,

\[
\begin{align*}
\dot{x}_3 &= z \\
\dot{z} &= 2\tau y + 2\nu z - 2\tau x_3^2 - \frac{4}{3}\tau x_3 z.
\end{align*}
\]

This is the canonical normal form of the subcritical Bogdanov-Takens bifurcation. In particular, we know that beyond the Hopf and saddle-node bifurcations already identified, the system presents a curve of saddle-homoclinic bifurcations. The local bifurcation diagram is plotted in Fig. 8, where we depict the local unfolding of the bifurcation in our parameters \((y, \tau)\), i.e. the curve of saddle-node bifurcations given by \(\{y = 2/3, J \in \mathbb{R}\}\), the Hopf bifurcation locally reduced to the curve \(\{y = 2/3 - 2(J\tau - 1)/\tau, J\tau > 1\}\) and the homoclinic bifurcation (saddle connection) \(\{y = 2/3 - 98/25(J\tau - 1)^2/\tau, J\tau > 1\}\). Additional curves will be depicted in this diagram, that are found numerically.

**Dynamics of the fast system**

The bifurcations allows characterizing locally the behavior of the system as a function of the parameters. From the presence of a Bogdanov-Takens bifurcation at \(y = \pm 2/3\) and \(\tau = 1/J\), we know that locally, there exists a curve of saddle-homoclinic bifurcations, unstable cycles. It is very complex to unfold globally the system. Aided by simulations (using XPPAut), we obtained a relatively simple bifurcation diagram (see Fig. 9). This is due to the fact that the system can actually be very well understood as a two-dimensional system. Saddle-node and Hopf bifurcation are known analytically and displayed in Fig. 9. As for the saddle-homoclinic bifurcation, we only know the local expansion around the BT point. The curve depicted in the figure is hand-drawn, between points that were computed through extensive numerical computations. Moreover, we find a pair of limit cycles arising for sufficiently large delays, collapsing at a fold of limit cycles. The small, unstable cycle arising from this bifurcation can have two different fates, depending on the value of \(\tau\). Below a certain value, the branch of stable cycles ends at saddle homoclinic bifurcations (at both saddle nodes for \(y = \pm 2/3\)), and above this value, it connects to the branch of unstable limit cycles arising from the Hopf bifurcation.

Let us first discuss the symmetric case corresponding to \(y = 0\) (the system is then symmetric with respect to the reflection \(x \leftrightarrow -x\)). We describe the
Figure 9: Bifurcation diagram of the fast system in the parameter space \((J, y)\). The diagram is symmetrical with respect to the reflection \(y \mapsto -y\). (a) Full diagram. The Bogdanov-Takens points (BT) at \(\tau = 1/J\) unfolds into a line of saddle-node bifurcations (SN, blue) at \(y = 2/3\), a curve of Hopf bifurcations (H, red), together with a saddle-homoclinic bifurcation (Sh1, green) enclosing one fixed point. The fold of limit cycles (FLC) is indicated in orange, and the second line of saddle homoclinics (Sh2) in dashed green, enclosing two fixed points. Saddle homoclinic lines intersect along the line \(y = 0\), giving rise to a double homoclinic loop (Dh). Seven specific locations are indicated with blue stars, and the phase portraits provided in Fig. 10.

behavior of the system as the delay is increased. For small delay, the system has two stable focuses, a saddle equilibrium and no cycle. Its phase space (in the coordinates \((u_{t-\tau}, u_{t})\)) corresponds to Fig. 10(A). A fold of limit cycles appears (orange curve in Fig. 9), together with a pair of stable and unstable cycles (Fig. 10(B)). In this regime, the stable manifold of the saddle fixed point winds around the unstable cycle, while the unstable manifolds converge towards the stable fixed points. The unstable cycle progressively shrinks, getting closer from the stable and unstable limit cycle, until the system reaches a double-homoclinic bifucation (Dh point in Figure 9), arising from the symmetry of the system and the continuation of the saddle-homoclinic bifurcation curve emerging from the BT bifurcations. At this point, stable and unstable manifolds of the saddle fixed point coincide (Fig. 10(Dh)). For slightly larger values of the delay, the system shows two unstable periodic orbits around the stable fixed points, towards which the stable manifold of the saddle fixed point converge, and the unstable manifold converges towards the large stable cycle (Fig. 10(C)). The unstable orbits cycling around the stable fixed points shrink, until
Figure 10: Phase space representation of the fast system in the axes \((x_{t-\tau}, x_t)\) for \(y = 0\) and different values of \(\tau\), corresponding to the stars in Fig. 9 (see text). The trajectories were obtained with XPP Aut. As before, \(J = 2\). (A-D): \(y = 0\) and distinct values of \(\tau\). (A) \(\tau = 0.4\), (B) \(\tau = 0.6\), (Dh) \(\tau = 0.664\), (C) \(\tau = 0.7\), (D) \(\tau = 1\). (E-G) \(y = 0.8\). (E): \(\tau = 1.5\), (F) \(\tau = 0.8\), (G): \(\tau = 0.5\). All curves are actual solutions of the fast system, except the dashed lines corresponding to the unstable trajectories, namely the unstable limit cycles and the stable manifold of the saddle fixed point. See text for a description of the phase portraits.

\(\tau\) reaches the value of the Hopf bifurcation (arising simultaneously here due to symmetry). At this point, the stable fixed point become unstable focuses and the unstable cycles disappear. The unstable manifold of the saddle fixed point keeps converging towards the stable relaxation cycle, and the stable manifold winds around the unstable focuses (Fig. 10 (D)).

Asymmetric situations arise when \(y \neq 0\). When \(|y| > 2/3\), the system has a unique fixed point, which can be unstable if \(|y| < \sqrt{1 + 2J}\). The system can also feature an additional pair of limit cycles (one stable and one unstable). A typical case for \(2/3 < y < \sqrt{1 + 2J}\) is depicted in Fig. 10 (E-G). Let us eventually discuss the situation where \(0 < y < 2/3\). In that case, homoclinic bifurcations do not arise simultaneously at a double homoclinic loop, but sequentially. An
Figure 11: Phase space representation of the fast system in the axes \((x_{t-\tau}, x_t)\) for \(y = 0.1\) and different values of \(\tau\). The trajectories were obtained with XPP Aut. As before, \(J = 2\). (a) \(\tau = 0.6449\), (b) \(\tau = 0.652\), (c) \(\tau = 0.656\), (d) \(\tau = 0.67\), (e) \(\tau = 0.6832\), (f) \(\tau = 0.7\). See text for a description.

Example of phase portraits is provided in Fig. 11 for \(y = 0.1\). In that case, the first homoclinic bifurcation \(\text{Sh1}\) already arises before the fold of limit cycles. A similar situation as in Fig. 10(A) occurs for small delays, and a homoclinic loop arises (Fig. 11(a)), turning into an unstable periodic orbit enclosing the smallest fixed point (Fig. 11(b)). After the fold of limit cycles, a pair of stable and
unstable cycles emerges (Fig. 11(b)). The unstable cycle progressively shrinks towards the saddle fixed point, until reaching the homoclinic bifurcation Sh2 (Fig. 11(c)). At this point, as delays are further increased, the unstable loop disappears, and the system is left with an unstable orbit enclosing the smallest fixed point, a stable fixed point, a saddle and a stable limit cycle (Fig. 11(d)). Further increasing the delays leads to crossing the saddle homoclinic line that emerges from the BT point at $y = -2/3$: a homoclinic loop encloses the largest fixed point (Fig. 11(2)). For larger delays, the loop becomes an unstable periodic orbit enclosing the largest fixed point (Fig. 11(f)), and we in a similar picture as in Fig. 10(C). As delays are further increased, the lowest fixed point will undergo a Hopf bifurcation and lose stability as the loop enclosing it collapses (not shown). The phase portrait is similar to Fig. 11(2), except that the loop around the smallest fixed point disappeared and that fixed point lost stability. The stable manifold of the saddle fixed point that converged towards this cycle now winds around this stable fixed point.

3.4 Canard explosion

In this section we discuss the details of canard explosion, namely the local explosion. First note that hypotheses (H1) and (H2) hold based on the analysis in Section 3.3. It follows that there exists a Hopf bifurcation followed by a local canard explosion (transition to small canard cycles). The analysis of these phenomena is the same as for a system in two dimensions [26], due to center manifold reduction, see Section 2.2. The existence of a global canard explosion relies on the hypotheses (H3)-(H4). In this section we will discuss the local canard explosion, verification of the hypotheses (H3) and (H4) and the transition from the canard explosion region to the bursting region through the Bogdanov-Takens point.

Local canard explosion

It follows from the calculations in Section 3.3 that the line line segment $a = 1, 0 \leq \tau < 1/J$ in the parameter space $(a, \tau)$ corresponds to the locus of canard points. To understand the details of local canard explosion we derive the reduction of (5) to a center manifold at the canard point, see Section 2.2 for a general description of such a reduction. To carry out the reduction we use the fact that a canard point is a special case of a a Bogdanov-Takens point. Further we observe that (5) has the same structure as (3.8) in [14], hence we obtain our normal form by following closely the approach of [14]. In addition we take advantage of the fact that our nonlinearity is independent of the delay. See Appendix B for a similar reduction, in the context of Hopf bifurcation in [5]. The reduction (we leave the details to the reader), followed by a reflection

\[ \text{The method of Stone, Campbell and Erneux showing canard explosion for small delays is outlined in appendix D.} \]
in the $\tilde{x}$ variable, yields the following system on the center manifold:

$$\begin{align*}
\frac{d\tilde{x}}{d\theta} &= -\tilde{y} + \tilde{x}^2 - \frac{\tilde{x}^3}{3} + a_1\varepsilon\tilde{x} + \text{hot} \\
\frac{d\tilde{y}}{d\theta} &= \bar{\varepsilon}(\tilde{x} - \bar{a} + \text{hot}),
\end{align*}$$

(24)

where $\bar{a} = 1 - a$,

$$a_1 = \frac{J\tau^2}{2(1 - J\tau)},$$

(25)

and hot denotes higher order terms which have no influence on qualitative and low order quantitative features of canard explosion. If the hot terms are omitted (24) differs from the classical van der Pol system by the term $a_1\varepsilon\tilde{x}$. Note that $a_1$ is positive and blows up as $\tau$ approaches $1/J$. The coefficient $a_1$ does not influence the coefficient $A$ defined in [25], which determines the criticality of canard explosion. The feature changed by $a_1$ is the position of the Hopf and canard curves in the $(a, \varepsilon)$ plane. For $\tau = 0$ the Hopf curve is given by $a = 1$ and the canard curve is in the half plane $a < 1$. As $\tau$ increases, the two curves turn to the right and eventually are both located in the $a > 1$ half plane. This feature allows for a very interesting version of a canard explosion: starting with $a < 1$ and $\tau = 0$ one can follow the evolution of the stable limit cycle as $\tau$ is increases while $a$ is kept fixed. Due to the movement of the canard line the parameter point $(a, \tau)$ approaches and eventually passes through the canard line, which gives a canard explosion. This is shown in Figure 12.

**Verification of hypotheses (H3) and (H4)**

Analytical demonstration of the persistence of fast connections from the saddle fixed point to one branch of the critical manifold (hypothesis (H3)) are complex to perform, and were not performed. We provide here ample simulations showing that such connections exist when $J = 2$ and $0 < \tau < 0.5$. For such delays, the system has a dynamics corresponding to the green region in Fig. 9 (corresponding to Fig. 10(A)). Fig. 13 shows the persistence of connections for varying values of $\tau$ or $y$.

To verify (H4) we have computed the contraction/expansion rates given by formula (16) together with the closed-form expression of the eigenvalues (in terms of the Lambert function $W_0$). These computations showed that (H4) holds for smaller values of $\tau$, but for $\tau > \tau_* \approx 0.354$ there exists a subinterval of $(-2/3, 2/3)$ where (H4) is violated, see Fig. 14.

It follows from Theorem 2 that for $\tau < \tau_*$ there exists a canard explosion analogous to the one occurring in the classical van de Pol system. However, for $\tau > \tau_*$ this version of canard explosion may no longer be present. There is still a weak version of canard explosion. Indeed, canards with ‘large head’ must exist and be stable. Moreover, canard explosion occurs and is locally supercritical (as shown above). All canard cycles are exponentially close in the parameter space to a segment of the canard solution and therefore the locus of the canard cycle is exponentially close to that of the maximal canard (see Remark 2). It is very difficult to determine numerically whether complex dynamics occurs during a
Figure 12: Delay-induced canards in the delayed van der Pol system. $\varepsilon = 0.05$, $a = 0.995$. Dotted lines correspond to the nullclines (red: $x$ nullcline, green: $y$ nullcline) and the curves represent trajectories in the phase plane $(x, y)$ for different values of the delay. 1. $\tau = 0.01$, 2. $\tau = 0.07$, 3. $\tau = 0.085$, 4. $\tau = 0.08951569008$, 5. $\tau = 0.08951569009$, 6. $\tau = 0.089516$, 7. $\tau = 0.0896$.

Figure 13: Persistence of connections for $J \tau < 1$. Here, $J = 2$ fixed. (a) $y = 0$ and different values of $\tau$ (from right to left, $\tau = 0.1$, 0.2, 0.3, 0.4). (b) $\tau = 0.3$ fixed, and $y$ ranging from 0 to 0.6 (steps 0.1). Initial condition was always set to $x_t = x_0$ for $t \in [-\tau, 0]$ and $x(0) = x_0 + 0.01$ (where $x_0$ is the saddle fixed point). We observe in (a) that the connection persists for increasing values of $\tau$, but the attractivity of the fixed point decays, and in (b) that the connections persist all along the branch of unstable fixed points.
canard transition. Partial verification is provided by period doubling cascades of small amplitude cycles, found for $\tau \approx 0.4$ (see Fig 4). Such period doubling cascades can no longer be observed as $\varepsilon \to 0$, but it is possible that they turn into period doubling cascades of canards when $\varepsilon$ is sufficiently small.

End of canard explosion at a Bogdanov-Takens point

The parameter point $(a, \tau) = (1, 1/J)$ is a Bogdanov-Takens (BT) bifurcation point. As a result for $\tau > 1/J$ there is a Hopf bifurcation of the fast system on $S_+$. As shown in Section 3.3 this bifurcation is subcritical. Moreover, along a curve in the parameter plane $(a, \tau)$, the Hopf bifurcation coincides with the equilibrium of the slow system. This curve, which we denote by $\gamma_{He}$, is obtained by substituting $a$ for $x_*$ in (22) and is parabolic, see Figure 8 (b). Clearly $\gamma_{He}$ perturbs to a subcritical Hopf bifurcation curve for $\varepsilon > 0$. As shown in Section 3.1 system (5) has a unique Hopf curve. Now we observe that for $\tau < 1/J$ this curve corresponds to supercritical Hopf bifurcations originating from canard explosion and for $\tau > 1/J$ to subcritical Hopf bifurcations arising by perturbation from $\gamma_{He}$. It follows that near the BT point there must be a Bautin point corresponding to a change of criticality of the Hopf bifurcation. This transition was indeed found in the normal form reduction of the full system in appendix B (see Figure 18). Moreover, the limit as $\varepsilon \to 0$ of the Hopf curve must be the union of the line $a = 1$ and $\gamma_{He}$, which is non-smooth at $(a, \tau) = (1, 1/J)$. This explains the sharp turn in the Hopf curve, see Figure 8.

The dynamics near a singularity given by a BT point in the fast subsystem is governed by an intricate sequence of bifurcations and includes chaos [29, 4]. Here we do not go into the details of this dynamics, but focus on a new type of solution which we call mixed-mode oscillations (MMOs). MMOs replace relaxation oscillations as the basic solutions approximately tracing the shape of $S$ and will be discussed in detail in Section 4.1.
4 Complex dynamics of the delayed van der Pol system

Now that we characterized the dynamics of the fast system and the canard explosion, we focus on the dynamics of the full system for $\tau > 1/J$. For such parameters, the system displays complex dynamics including Mixed-Mode Oscillations and Bursting and a chaotic transition.

4.1 Mixed Mode Oscillations

For $\tau$ slightly larger than $1/J$, we have observed that the fast system shows a subcritical Hopf bifurcation, associated with a family of limit cycles undergoing a saddle-homoclinic bifurcation, and no stable cycle is present in the fast system. For $\varepsilon > 0$, the variable $y$ will slowly vary and trajectories will travel around the bifurcation diagram. Because of the symmetry of the system, we consider that $a > 0$. For $a$ larger than the value corresponding to the value of $y$ at the Hopf bifurcation (‘stationary’ yellow region in Fig. 15), the system will converge towards a fixed point on the slow manifold. But if $a$ is smaller than this value (pink region in Fig. 15), the full system does not have any stable equilibrium.

![Figure 15: The MMO cycle: $J = 2$ and $\tau = 0.65$. The bifurcation diagram of the fast system is superimposed with a trajectory of the system in the plane $(x,y)$. Blue lines are fixed points (solid: stable, dashed: unstable). SN: saddle-node, H: Hopf, red dashed lines: unstable periodic orbits, and orange cycle is the saddle-homoclinic cycle. Dashed horizontal line corresponds to the value of $a$. (a) $\varepsilon = 0.05$, (b) $\varepsilon = 0.01$. Yellow and pink regions correspond to choices for $a$ leading respectively to stationary or periodic (relaxation oscillations or MMO) behaviors.](image)

For a given initial condition, the system will converge rapidly towards the slow manifold (blue line in Fig. 15). If starting from the lower branch of the slow manifold, the slow variable $y$ will be increasing along the branch, until reaching the Hopf bifurcation, where the system jumps towards the upper branch of the slow manifold. For the fast system, this fixed point is a stable focus, and
the system will display fast damped oscillations while converging towards this branch of equilibria. The branch being above the value of \(a\), the variable \(y\) will slowly decrease while oscillations are damped. For such values of \(a\), the trajectory will eventually cross the Hopf bifurcation of the upper branch where it looses stability and jumps towards the lower branch of fixed points where it will find a branch of stable focus, and will converge to it through damped oscillations. This creates a periodic solution showing small oscillations interspersed with large amplitude relaxation oscillations, corresponding to Mixed-Mode Oscillations. Note that for \(a \neq 0\), the oscillation is highly asymmetric: for \(a > 0\), the variation of \(y\) is much faster on the lower branch than on the upper branch. Therefore, on the upper branch, the trajectory gets closer of the slow manifold than on the lower branch, which is highly visible for larger values of \(\varepsilon\) (left panel of Fig. 15 for \(\varepsilon = 0.05\)) compared to cases where \(\varepsilon\) is smaller (right panel, \(\varepsilon = 0.01\)). Note that in this later case, we also see a delayed Hopf bifurcation phenomenon: the trajectories closely follow the unstable branch of the slow manifold. In the phase portraits of the fast system pictures, the system slowly switches back and forth between the situation described in Fig. 10(G) to the symmetric situation.

Remark 3. The MMOs described in this section are different than canard mediated MMOs arising near folded singularities \([2, 9]\), for which transitions between regions of different numbers of small oscillations occur by means of a passage through a canard solution (similar to canard explosion). MMOs of this type require the presence of two slow variables and a fold singularity. Interestingly, canard transitions seem to also play a role in the BT mediated MMOs discussed in this paper. Typical trajectories pointing towards the presence of this phenomenon can be computed in the delayed vdP system. Such a trajectory is depicted in Fig. 5 (left side).

The dynamics of the small oscillations in the vicinity of a BT point is slow (when \(\tau - 1/J\) is small). As \(\tau\) increases the small oscillations become fast and the solutions become similar to an elliptic burster \([31]\).

4.2 Bursting and periodic averaging

We now consider the system for larger values of \(\tau\), in the regime where the saddle-homoclinic bifurcation vanished. In that case, we have seen that the fast system shows two folds of limit cycles. The bifurcation diagram of the fast system (obtained with DDE Biftool \([12, 13]\)) shows that the folds of limit cycles connects the family of unstable limit cycles of the subcritical Hopf bifurcations with a branch of stable periodic orbits. These stable limit cycles appear relatively symmetrical, and the average value of typical stable limit cycles are relatively small. This quantity is relevant in order to understand the behavior of the system as a function of the value of the parameter \(a\).

Indeed, let us denote \(x_w(t)\) the fast system dynamics for a fixed value of the slow variable \(y = w\) (which is a parameter of the fast system), and let us virtually uncouple the slow and fast equation. Integrating the slow dynamics
for $x(t) = x_w(t)$ yields:

$$y(t) = y(0) + \varepsilon t (a - \frac{1}{t} \int_0^t x_w(s) \, ds)$$

which in the slow timescale $\theta = \varepsilon t$, reads:

$$y(\theta) = y(0) + \theta \left( a - \frac{\varepsilon}{\theta} \int_0^\theta x_w(s) \, ds \right) \quad (26)$$

We have shown that $x_w(t)$ is one of two types: either a fixed point or a periodic orbit. In the limit $\varepsilon \to 0$, the slow dynamics is therefore written, using equation $\quad \Box$, as an implicit system depending on the temporal average of the fast variable

$$m(w, \tau) = \lim_{t \to \infty} \frac{1}{t} \int_0^t x_w(s) \, ds.$$ 

In the case of a stationary behavior of the fast subsystem, this quantity is precisely equal to the value of the fixed point, and in the case of a periodic activity of period $T$, this quantity is equal to the average of the cycle $\frac{1}{T} \int_0^T x_w(s) \, ds$.

Therefore, at the slow timescale, introducing the delay has the effect of modifying the slow manifold by adding to the stationary solution manifold a branch related to stable periodic orbits of the fast system.

The dynamics of the slow variable $y(\theta)$ on the slow manifold given by equation $\quad \Box$, is now nonlinear, and the variation of $y$ is given by the sign of $a - m(y(\theta), \tau)$, hence increasing when $m(y(\theta), \tau) > a$ and decreasing otherwise. As depicted in this figure, the slow manifold is discontinuous: loss of stability of the manifold corresponding to fixed points occur in the delayed system through the delay-induced Hopf bifurcations of the fast dynamics identified in section 3.3. The manifold also present folds, either corresponding to the fold of limit cycles in the delayed system (see section 3.3) or through the saddle-node bifurcations on the manifold corresponding to equilibria. When the slow dynamics reaches one of these extremal value of the slow attractive manifold, a fast switch will occur, depicted in this figure by black vertical line with double arrows.

This analysis allows to characterize the behavior of the system as a function of $a$: if the value of $a$ is associated to a stable branch of the limit cycles manifold (light blue region of Fig. 16), the value of $y$ will stabilize at $a$ and the system will display fast oscillations. In the light yellow region, the value of $a$ corresponds to a stable fixed point of the fast system, and the full system will stabilize. In contrast, in the light pink region of intermediate values of $a$, neither the cycle nor the periodic orbits are stable, and the system switches between the branch of oscillations of the fast system and the branch of fixed point of the slow manifold.

Figure 16 shows that along the cycles, the slow variable increases with a lowerbounded speed. This is true until the slow variable $y$ reaches the value associate to the fold of limit cycles of the fast subsystem, at which time the system switches fast and reaches the slow manifold associated to the stable
Figure 16: vdP system, $\tau = 1$. (a): Bifurcation diagram of the fast system as a function of $y$. Fixed points are in blue, average amplitude of cycles in green (plain line: stable, dashed: unstable). Regions of fast oscillations, bursting and stationary solutions are depicted for $a < 0$ (completed by symmetry in the region $a > 0$). An example for $a = 1.1$ indicates the slow dynamics along these manifolds, and present a bursting orbit (black thick line) globally attractive. The three possible transient trajectories towards the bursting cycle are depicted with dashed gray line, and the system will follow one of these dynamics depending on the initial condition. (b): an actual bursting orbit in the non-zero $\varepsilon$ case. For legibility, we chose $\varepsilon$ relatively large $\varepsilon = 0.1$, and this explains why the system slightly departs from bursting cycle depicted in subfigure (a). Smaller $\varepsilon$ more closely match the bursting cycle, but the number of oscillations in the burst dramatically increase (as $1/\varepsilon$) affecting legibility.

fixed point of the fast dynamics. On that manifold, the slow variable $y$ decreases with a lowerbounded speed, and therefore will reach the Hopf bifurcation, at which time the fixed point of the fast subsystem looses stability and the fast variable $x$ starts oscillating. This scenario hence occurs periodically in time, defining a periodic orbit composed of fast oscillations followed by a silent period. A trajectory of the dynamical system superimposed to the slow manifold is plotted in Figure Fig. 16. In the phase portraits of the fast system pictures, the system slowly switches between the situation described in Fig. 11(a) to that of Fig. 10(E).

4.3 Chaotic transition

We now investigate the occurrence of chaotic orbits in the system. In the chaotic orbits observed (Fig. 7), we have observed the presence of relaxation cycles of relatively regular period, but during the increasing phase of the slow variable, the fast variable shows chaotic alternations between at least 3 behaviors: (i) damped oscillations convergence towards the stable manifold, corresponding to a cycle of Mixed Mode Oscillation (no fast oscillation cycle), (ii) large periodic
oscillations, corresponding to a cycle of bursting or (iii) a mixture between the two orbits. These phenomena are related to an expansion region of the phase plane for specific values of the parameters, which we now describe.

In order to understand the emergence of chaos, one needs therefore to understand the fate of the trajectories as they are leaving the upper branch of the stable manifold (in the case \( a > 0 \) that we have been considering thus far). The system will follow the manifold until approaching the Hopf bifurcation where it will lose stability and either switch to the other branch of fixed points (MMO case) or to the periodic orbit (busting case). This choice actually depends on the possible trajectories emerging from the neighborhood of the Hopf bifurcation. At this point, a delayed destabilization emerges. In the MMO case described in section 4.1, there is no periodic orbit at all and the system goes towards the stable fixed point, and in the bursting case described in section 4.2, in a wide neighborhood of the Hopf bifurcation, the only stable orbit is the fast periodic orbit. The chaotic phase occurs when both attractors are stable in the vicinity of the Hopf bifurcation. In Fig. 17(a) we show the phase plane representation of the fast system (in the coordinates \((x(t), x(t−τ))\)) for the value of \( τ \) corresponding to the chaotic region in Fig. 7. We observe indeed that in that case, the system can either switch towards the stable fixed point or to the stable periodic orbit: one branch the unstable manifold of the saddle fixed point converges on one side to the fixed point and the other branch to the periodic orbit, while both branches of the stable manifold wind around the unstable fixed point. In this two-dimensional representation, this stable manifold separates those trajectories converging towards the fixed point and those converging towards the periodic orbit. In the vicinity of the fixed point after the Hopf bifurcation, the expansion rate becomes very large because of this separation of trajectories. This is the phenomenon at the origin of the emergence of chaotic orbits.

In Fig. 17 we represent different branches of the trajectories on the phase plane \((x, y)\), together with the bifurcation diagram of the fast system. In this diagram, we represented the regions of \( y \) for which the fast system has a topology similar to that represented in Fig. 17(a), i.e. the presence of a stable periodic orbit and a stable fixed point, two unstable fixed point and no unstable periodic orbit. This region overlaps with the Hopf bifurcation point, and makes the system very sensitive to the specific shape of the trajectory. We represented on this diagram an actual trajectory of the system during one period of the relaxation oscillation, in all three cases (MMO, burst and mixed trajectory).

A Fenichel theory for delayed differential equations

For simplicity we assume that \( g(0, y, 0) > 0 \) for \( y_0 \leq y \leq y_1 \) (this is the only case we need in our application). By modifying the slow flow we can make sure that there exist equilibrium points \( y_{e0} < y_0 \) and \( y_{e1} > y_1 \) such that the hypothesis (H1) still holds, the slow flow is unchanged on \([y_0, y_1]\) and there is a solution
Figure 17: Chaotic orbits. $\tau = 0.7$, $\varepsilon = 0.05$. (a) Phase plane in a chaotic situation ($y = -0.4$) the stable manifold of the saddle fixed point winds around the unstable orbit. (b) the bifurcation diagram of the system as a function of $y$. The saddle homoclinic bifurcations (Sh2) and the Hopf bifurcations (H) are such that there exists a range of values of $y$ (gray region) in which the phase portrait is of type (a): no unstable cycle acts as a separatrix between the fast periodic orbit and the fixed point. (c,d,e) represent three trajectories for $\varepsilon = 0.05$ (large for legibility). (c) The first case is an MMO-type oscillation: the fast system closely converges towards the stable manifold branch, it leaves after a relatively long delay, away from the gray region, and falls on the branch of fixed points (no cycle exist). (d) is a bursting-type trajectory. In that case, the trajectory leaves the the branch of stable fixed points early, and converges towards the fast periodic orbit where it remains until the cycles disappear, (e) is a mixed trajectory: the system leaves the manifold of fixed points at an intermediate location, the convergence towards the branch of fixed point is loose, and the system reaches the fast periodic orbit instead of converging towards the fixed point.
The linear problem for (6) is as follows:

\[ \xi(t) \text{ of } \dot{y} = g(0, y, \varepsilon). \]  

(27)

The linear problem for (6) is as follows:

\[
x' = \int_0^h d\xi(y, \tau)x(t - \tau)d\tau, \]

\[
y' = \varepsilon g(0, y). \]

(28)

Using the solution \( \xi(t) \) we get the following equivalent formulation of (28)

\[
v' = \int_0^h d\xi(\tau, \tau)v(t - \tau)d\tau \]

(29)

and let \( T(t) \) be the solution operator. Let

\[
X^- = \{ v \in BC_\eta(\mathbb{R}; X) : \lim_{t \to \infty} \| T(t)v \| = 0 \}, \quad X^-(\tau) = \{ v(\tau) : v \in X^- \}
\]

\[
X^+ = \{ v \in BC_\eta(\mathbb{R}; X) : \lim_{t \to \infty} \| T(t)v \| = 0 \}, \quad X^+(\tau) = \{ v(\tau) : v \in X^+ \}. \]

(30)

Let \( P_-(\tau) \) and \( P_+(\tau) \) be projections onto the spaces \( X^-\) and \( X^+\) with kernel \( X^-(\tau) \) and \( X^-(\tau) \), respectively. We also define the operator \( r \) which assigns to \( a \) an element of \( X \) its value at 0. Further, we modify \( r \), replacing it with \( r_\text{mod} \), to ensure that there exists a small invariant neighborhood near \( S_0 = \{(0, y), \ y \in [y_0, y_1]\}, \) see [10] for details of a similar construction in the context of the center manifold theorem. Finally, we define the operator \( \mathcal{F} = (\mathcal{F}_f, \mathcal{F}_s) \) which maps the space \( BC_\eta(\mathbb{R}; X \times \mathbb{R}) \) into \( BC_\eta(\mathbb{R}; X \times \mathbb{R}) \) \((\eta \) has to be chosen within the spectral gap\), given by

\[
\mathcal{F}_f(\psi(t), \phi(t)) = T(t)\psi(0) + \int_{-\infty}^t T(t - \tau)P_-(\tau)r_\text{mod}(f(\psi(\tau), \phi(\tau), \varepsilon))d\tau
\]

\[
\quad + \int_{-\infty}^t T(t - \tau)P_+(\tau)r_\text{mod}(f(\psi(\tau), \phi(\tau), \varepsilon))d\tau\]

(31)

\[
\mathcal{F}_s(\psi(t), \phi(t)) = \phi(0) + \varepsilon \int_0^t r_\text{mod}(g(\psi(\tau), \phi(\tau), \varepsilon))d\tau.
\]

For each choice of \( r_\text{mod} \) there exists a unique fixed point of \( \mathcal{F} \) corresponding to a solution of (6) which defines a slow manifold.

To construct \( W^u(S_\varepsilon) \) we use a different version of the operator \( \mathcal{F} \), now defined on \( BC_\eta(\mathbb{R}_-; X \times \mathbb{R}) \) into \( BC_\eta(\mathbb{R}_-; X \times \mathbb{R}) \), where \( \mathbb{R}_- \) are non-positive real numbers. First of all, we note that the space \( X_+ \) is finite, see Chapter IV in [10]. Let \( k = \dim X_+ \). The fixed point equation

\[
(\psi, \phi) = \mathcal{F}(\psi(t), \phi(t)) \]

defines a \( k \) dimensional submanifold of \( BC_\eta(\mathbb{R}_-; X \times \mathbb{R}) \), for any \( \eta > 0 \) within the spectral gap. The elements of this manifold define solutions of (6) on \( \mathbb{R} \). The manifold \( W^u(S_\varepsilon) \) is obtained as the union of these solutions.
B Normal form at the Hopf bifurcation in the fast vdP equation

In this section compute the normal form reduction of full van der Pol system in the neighborhood of the Hopf bifurcation. To this end, we shall use classical reduction methods to normal forms for delayed differential equations, as exposed for instance in [22], and reviewed in [4]. The method used in our case is standard. We work in the Banach space $X$ of continuous functions from $[-\tau, 0]$ to $\mathbb{R}^2$ endowed with the uniform norm:

$$\|z\| = \sup_{\theta \in [-\tau, 0]} |z_\theta|.$$  

where $z^\tau = (x^\tau, y^\tau)$ and the norm on the righthand side is the Euclidian norm on $\mathbb{R}^2$. The delayed differential equation is expressed as a functional differential equation on this space. Denoting $z_t$ the portion of solution $(x(t), y(t), t \geq 0)$ restricted to the interval $[t - \tau, t]$, with the definition

$$z_t(\theta) = z(t + \theta), \quad -\tau \leq \theta \leq 0,$$

we rewrite the equation as:

$$\frac{d}{dt} x_t(\theta) = \begin{cases} \frac{d}{d\theta} x_t(\theta) \\ y_t(0) + x_t(0) + J(x_t(0) - x_t(-\tau)) - \frac{x_t(0)^3}{3} & -\tau \leq \theta < 0 \\ \theta = 0 \end{cases}$$

and similarly,

$$\frac{d}{dt} y_t(\theta) = \begin{cases} \frac{d}{d\theta} y_t(\theta) \\ \varepsilon(a - x_t(0)) & -\tau \leq \theta < 0 \\ \theta = 0 \end{cases}.$$

The terms within the brackets is the affine part of the flow, separated from the nonlinear cubic term in the equation on the first variable, which does not involve delays. Calculations are much simplified when changing variables so that the equilibrium is at the origin. This change of variable simply amounts changing the origin, i.e. consider the equation in terms of $\tilde{x} = x - x^*$. The equations now read:

$$\frac{d}{dt} \tilde{x}_t(\theta) = \begin{cases} \frac{d}{d\theta} \tilde{x}_t(\theta) \\ (1 - a^2)\tilde{x}_t(0) + J(\tilde{x}_t(0) - \tilde{x}_t(-\tau)) - a\tilde{x}_t(0)^2 - \frac{2\tilde{x}_t(0)^3}{3} & -\tau \leq \theta < 0 \\ \theta = 0 \end{cases},$$

$$\frac{d}{dt} \tilde{y}_t(\theta) = \begin{cases} \frac{d}{d\theta} \tilde{y}_t(\theta) \\ -\varepsilon\tilde{x}_t(0) & -\tau \leq \theta < 0 \\ \theta = 0 \end{cases}.$$

The linear operator decomposes into a part only depending on $\tilde{z}_t(0)$:

$$A_0 = \begin{pmatrix} 1 - a^2 + J & 1 \\ -\varepsilon & 0 \end{pmatrix}.$$
and an operator depending on $\bar{x}_t(-\tau)$:

$$A_1 = \begin{pmatrix} -J & 0 \\ 0 & 0 \end{pmatrix}.$$  

At Hopf bifurcation points, i.e. when the characteristic equation

$$\det(\lambda I_2 - A_0 - e^{-\lambda \tau} A_1) = 0$$

has a complex solution $\lambda = \pm i\omega$, a complex right eigenvector is:

$$v = \begin{pmatrix} 1 \\ \frac{i}{\omega} \end{pmatrix},$$

which corresponds to a two-dimensional center eigenspace $N$:

$$\begin{pmatrix} \cos(\omega \theta) & \sin(\omega \theta) \\ \sin(\omega \theta) \varepsilon & \cos(\omega \theta) \varepsilon \end{pmatrix}$$

and an infinite-dimensional stable eigenspace $S$. The corresponding center manifold is given by:

$$M = \{ \phi \in X, \phi = \Phi u + h(u) \}$$

where $u = (u_1, u_2)^t$ are the coordinates on the nullspace $N$ and $h(u) \in S$. Classically, we project the solutions to the delay differential equation on $M$ and obtain:

$$z_t(\sigma) = \begin{pmatrix} \cos(\omega \sigma) u_1(t) + \sin(\omega \sigma) u_2(t) \\ \sin(\omega \sigma) \varepsilon u_1(t) + \cos(\omega \sigma) \varepsilon u_2(t) \end{pmatrix} + \begin{pmatrix} h_{11}(\sigma) u_1(t)^2 + h_{12}(\sigma) u_1(t) u_2(t) + h_{22}(\sigma) u_2(t)^2 \\ h_{11}(\sigma) u_1(t)^2 + h_{12}(\sigma) u_1(t) u_2(t) + h_{22}(\sigma) u_2(t)^2 \end{pmatrix} + O(\|x\|^3)$$

where the $h_{jk}$ characterize the Taylor expansion of the solution on the center manifold. These coefficients satisfy a system of affine ODEs, in which the affine term depends on the orthogonal basis of the nullspace, denoted $\psi(\theta)$ (which is straightforward to calculate) arising from the projection of the original equation on $N$. These are now classical methods, introduced and used in \[1,22,16\]. In our case, the solution to the linear ordinary differential equation of $h_{ijk}$ yield relatively complex expressions, in terms of six constants $C_1 \cdots C_6$, that are then solved in order to match boundary values (and solve the original DDE on the center manifold). One finds:

$$h_{11}^1 = \frac{a}{3\omega} \left( \cos(\omega \theta) \Psi_{21}(0) - \sin(\omega \theta) \Psi_{11}(0) \right) + C_2 - C_5 \sin(\omega \theta) \cos(\omega \theta) + C_6 \left( \cos(\omega \theta)^2 - \frac{1}{2} \right)$$

and similar expressions for the other terms, and it is not hard to determine the constants. From these expressions, we obtain a system of ODEs describing the evolution in time of $u$:

$$\begin{cases} \dot{u}_1 = \omega u_2 + \Psi_{12}(0)(G_1(u_1, u_2)) + O(\|u\|^4) \\ \dot{u}_2 = \omega u_1 + \Psi_{22}(0)(G_1(u_1, u_2)) + O(\|u\|^4) \end{cases}$$
with $G$ a cubic polynomial, whose coefficients can be deduced from the evaluation of $h_{j_{ik}}$ and $C_i$, and the first Lyapunov exponent, characterizing the type of the Hopf bifurcation, is a simple function of these coefficients. Computer-aided thorough computation with Maple allows to compute all these coefficients. It happens that, as of quadratic terms, only the coefficients of $u_1^2$ in $G_1$ and $G_2$ are non zero, and cubic coefficients involved in the computation of the Lyapunov exponents enjoy a relatively simple form. These coefficients are given by:

$$
\begin{cases}
G_1(u_1, u_2) = -a\Psi_{11}(0)u_1^2 + u_1^3\Psi_{11}(0) \left( -\frac{1}{3} - 2h_{11}(0)a \right) - 2\Psi_{11}(0)ah_{22}(0) + \cdots \\
G_1(u_1, u_2) = -a\Psi_{21}(0)u_1^2 - 2ah_{12}(0)u_1^2u_2 + \cdots
\end{cases}
$$

where the dots correspond to terms that are not necessary to compute the first Lyapunov exponent. From this expression, classical formulae (see e.g. [27]) yield a very complex expression for the first Lyapunov exponent as a function of the parameters. Fixing $J = 2$ (as done in most figures), we obtain the coefficient depicted in Fig. 18 as a function of $\tau$. This shows in particular the fact that the type of Hopf bifurcation changes close from the kink (here, $\tau \simeq 0.503 \simeq 1/J$), as was predicted from the fact that (i) the canard explosion is supercritical and (ii) the Hopf bifurcation of the fast system is supercritical, and emerges from a supercritical Bogdanov-Takens bifurcation at $\tau = 1/J$ and $a = 1$. Moreover, we note that the first Lyapunov coefficient diverges with $\tau$: this divergence has the same origin, namely the fact that both the value of $\tau$ at the bifurcation and the first Lyapunov exponent involve a factor $1/\omega$ ($\omega$ is the imaginary part of the eigenvalue), and both diverge when $\omega \to 0$. For legibility, the Lyapunov exponent was thresholded at 1, and plotted as a function of $\tau$ at the Hopf bifurcation: this allows having better legibility, since $\tau$ varies very fast as a function of $a$.

Figure 18: First Lyapunov coefficient of the Hopf bifurcation as a function of $\tau$ (thresholded at 1 for legibility) for $J = 2$ and $\varepsilon = 0.05$. Positive (negative) coefficients correspond to supercritical (subcritical) Hopf bifurcations. We observe the sudden switch, for $\tau$ close to 1, from positive to negative, indicating a possible Bautin bifurcation.
C The Fitzhugh-Nagumo system

This appendix is devoted to the analysis of the delayed self-coupled Fitzhugh Nagumo system. This system is given by the equations:

\[
\begin{align*}
\dot{x} & = x - \frac{x^3}{3} + y + J(x(t) - x(t - \tau)) \\
\dot{y} & = \varepsilon(a - x + \gamma y)
\end{align*}
\] (33)

The fixed points of the system are given by \( y = (x - a)/\gamma \), where \( x \) is the solution of the cubic polynomial:

\[
\frac{\gamma + 1}{\gamma} x - \frac{x^3}{3} - \frac{a}{\gamma} = 0.
\]

This equation can be solved using Cardano method, exactly as we solve the cubic equation of the fast system. In detail, the Cardano discriminant of the equation writes:

\[
\Delta = \frac{27}{b^2} (9a^2 - b + 1) b.
\]

When \( \Delta > 0 \), the system has a unique solution:

\[
x_0 = \left(-\frac{3a}{2b} + \sqrt[3]{\Delta}\right) - \left(\frac{3a}{2b} + \sqrt[3]{\Delta}\right)
\]

and when \( \Delta < 0 \), we find 3 roots:

\[
x_k = 2 \left(\frac{b + 1}{b}\right)^{1/3} \cos \left(\frac{1}{3} \arccos \left(-\frac{3a\sqrt{b}}{2\sqrt{(b + 1)^3}}\right) + \frac{2k\pi}{3}\right).
\]

The stability of the fixed points can be analyzed in a similar fashion as done before. In that case, denoting by \( x_* \) one of the equilibria found, the dispersion relationship corresponds to the determinant of the matrix:

\[
\xi \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} A & 1 \\ -\varepsilon & \varepsilon\gamma \end{pmatrix}
\]

with \( A = 1 - (x_*)^2 + J(1 - e^{-\xi\tau}) \), and the dispersion relationship therefore reads:

\[
\xi^2 - (A + \gamma\varepsilon)\xi + \varepsilon = 0,
\]

or:

\[
e^{-\xi\tau} = \frac{\xi^2 - \xi(1 - (x_*)^2 + J + \gamma\varepsilon) + \varepsilon(1 - (x_*)^2 + J + 1)}{J\xi + \gamma\varepsilon J}.
\]

This appears much more complex to solve in closed form.

The fast dynamics is identical as that of the delayed vdP system, and therefore the analysis of section 3.3 applies. This allows to account for most of the behaviors of the FhN system. The only phenomenon requiring some more work
is the presence of canard explosion and their type. Similarly to the non-delayed case, the details of the slow dynamics matter for the demonstration of the presence of canard explosions. Here, we provide numerical evidences for canard explosion, and show that similar phenomena of emergence of Mixed-Mode oscillations and bursting appear, for the same values of parameters, in the parameter regime where FhN system has a unique fixed point. Let us for instance consider the FhN model for $b = -1$ and $\gamma = -0.3$. We set $a = 0.85$ (which leads the system to a similar situation as studied in the vdP system with $a = 1.01$). For $\tau > 1/J$, we have seen that geometric analysis of the fast system accounted for most phenomena arising in the vdP system. This property persists for the Fitzhugh-Nagumo system, and we therefore observe the very same behaviors and transitions at similar delays.

D Stone-Campbell small delay expansion

In the whole manuscript, we have worked with arbitrary delays. General analysis was provided for canard explosions, and an analysis of the delayed van der Pol equation was provided and showed a vast repertoire of behaviors above $\tau = 1/J$. In [32], Stone, Campbell and Erneux proposed a method in order to characterize canard explosions for delayed equations in the limit of small delays. In that regime, one may use the perturbation result of Chicone [7] showing, in our case of the delayed vdP system, that the system has a two-dimensional inertial manifold for $\tau$ sufficiently small. On this manifold, and in the limit of small delays, the term $x(t) - x(t-\tau)$ is well approximated by $\tau x'(t) - \tau^2/2x''(t) + O(\tau^3)$. The method consists in using the fact that equation (5) can be approximated at first order in $\tau$ by the solutions of the following ordinary differential equation:

\[
\begin{cases}
(1 - J\tau)x' = x - \frac{x^3}{3} + y \\
y' = \varepsilon(a - x)
\end{cases}
\]

which can be written, through the change of time $\theta = t/(1 - J\tau)$:

\[
\begin{cases}
\frac{dx}{d\theta} = x - \frac{x^3}{3} + y \\
\frac{dy}{d\theta} = \varepsilon(1 - J\tau)(a - x)
\end{cases}
\] (34)

which precisely corresponds to the non-delayed van der Pol equation with a modified slow timescale $\tilde{\varepsilon}(\tau) = \varepsilon(1 - J\tau)$. Classical methods from the ODE domain can thus be applied in order to show that a canard explosion occurs, and to provide an approximate formula for the canard point. This follows classical theory that we review here in the context of the vdP equation. Under a few geometrical conditions [26], a slow-fast dynamical system generically presents canard explosion. These results can be summarized as follows. We consider a two-dimensional slow-fast system of type:

\[
\begin{cases}
\varepsilon\dot{x} = f(x, y, \lambda, \varepsilon) \\
\dot{y} = g(x, y, \varepsilon)
\end{cases}
\]
Figure 19: Self-coupled delayed Fitzhugh-Nagumo system for $\tau > 1/J$ and $\varepsilon = 0.05$. (a) $\tau = 0.6$, (b) $\tau = 0.7$, (c) $\tau = 0.9$. The system transitions from MMOs to bursting through a period of chaos, arising for similar parameter values as in the delayed vdP system.
where $\lambda \in (-\lambda_0, \lambda_0)$ is a parameter, and make the following assumptions:

A1. The critical manifold $\Sigma = \{x, y : f(x, y, \lambda, 0) = 0\}$ is S-shaped for all $\lambda$, i.e. can be written as $y = \varphi_\lambda(x)$, and $\varphi_\lambda$ has exactly one non-degenerate minimum $x_l(\lambda)$ and maximum $x_r(\lambda)$.

A2. The submanifolds $S_l = \Sigma \cap \{x < x_l\}$ and $S_r = \Sigma \cap \{x > x_r\}$ are attracting ($\partial f/\partial x < 0$) and $S_m = \Sigma \cap \{x_l < x < x_r\}$ is repulsive ($\partial f/\partial x > 0$) for the layer problem.

A3. Both folds are generic for $\lambda \neq 0$, i.e. for $x^* = x_l$ or $x_r$,

$$\frac{\partial^2 f}{\partial x^2}(x^*, \varphi(x^*), \lambda, 0) \neq 0 \quad \frac{\partial f}{\partial x}(x^*, \varphi(x^*), \lambda, 0) \neq 0 \quad g(x^*, \varphi(x^*), \lambda, 0) \neq 0$$

and for $\lambda = 0$, one of the folds is a non-degenerate canard point, i.e. satisfies the two first differential conditions of the fold and

$$\frac{\partial g}{\partial x}(x^*, \varphi(x^*), 0, 0) \neq 0 \quad \frac{\partial g}{\partial \lambda}(x^*, \varphi(x^*), 0, 0) \neq 0$$

A4. When $\lambda = 0$, the slow flow on $\Sigma$, namely $g(x, \varphi_0(x), 0, 0)/\varphi_0'(x)$, is strictly positive on $S_l \cup S_m \cup \{x_r\}$ and strictly negative on $S_r$.

Then for $\varepsilon$ and $\lambda$ sufficiently small, the system has a unique equilibrium converging to the canard point as $(\varepsilon, \lambda) \to 0$, and this point looses stability as $\lambda$ is increased through a Hopf bifurcation. The small cycles arising from the Hopf bifurcation (canard cycles) joins relaxation oscillations within an exponentially small interval of $\lambda$ of order $O(e^{-K/\varepsilon})$.

It is very easy to see that these conditions readily apply to the case of the non-delayed van der Pol equation, implying the existence of a canard explosion as a function of the parameter $\alpha$. This is also true of the small delay ODE (34). Indeed, $\tilde{\varepsilon}(0) = \varepsilon$, and for $J > 0$ as assumed here, the function $\tau \mapsto \varepsilon(\tau)$ is non-increasing. This type of variation of the parameter is not usual in the analysis of the van der Pol equation, since our delay parameter acts precisely on the timescale of the slow variable. In order to follow blow-up method used in [25, 26], we shall define $\tilde{x} = -(x - 1)$ and $\tilde{y} = (y - 2/3)$. These variables satisfy the equations:

$$\begin{cases}
\frac{d\tilde{x}}{d\theta} = \tilde{x}^2 - \tilde{x}^3 - \tilde{y} \\
\frac{d\tilde{y}}{d\theta} = \tilde{\varepsilon}(\tilde{x} - \tilde{a})
\end{cases}$$

with $\tilde{a} = 1 - a$. It is then trivial to reduce it to canonical form [25, Section 3.1]:

$$\begin{cases}
\frac{d\bar{x}}{d\theta} = -\tilde{y}h_1 + \tilde{x}^2h_2(\tilde{x}) + \varepsilon h_3 \\
\frac{d\bar{y}}{d\theta} = \tilde{\varepsilon}(\bar{x}h_4 - \lambda h_5 + yh_6)
\end{cases}$$

with $h_1 = h_4 = -h_5 = 1, h_2 = 1 - x/3$ and $h_3 = 0$, compute the coefficients

$$a_1 = \frac{\partial h_1}{\partial x} = 0, \quad a_2 = \frac{\partial h_1}{\partial x} = 0, \quad a_3 = \frac{\partial h_2}{\partial x} = -\frac{1}{3}, \quad a_4 = \frac{\partial h_4}{\partial x} = 0, \quad a_5 = h_6 = 0$$
and we therefore conclude that the Hopf bifurcation (which we found in closed form in the delay system in section 3) arises at \( \tilde{a}_H(\varepsilon, \tau) = O(\varepsilon^2) \), this bifurcation is supercritical, and the maximal canard appears at

\[
\tilde{a}_c(\varepsilon, \tau) = \frac{\varepsilon}{8} + O(\varepsilon^2) \sim \frac{\varepsilon(1 - J\tau)}{8}.
\]

Let us now consider the system with fixed \( \tilde{a} > 0 \) small enough so that \( \tilde{a}_H(\varepsilon, 0) < \tilde{a} < \tilde{a}_c(\varepsilon, 0) \). In that case, the system with \( \tau = 0 \) present small canard oscillations. As \( \tau \) is increased, the value of the effective parameter \( \tilde{a}_c(\varepsilon, \tau) \) decreases. Increasing \( \tau \) with fixed \( \tilde{a} \) will hence induce a canard explosion. Simulations of the ODE system for small delay, provided in Fig. 20(a), indeed show that canard explosion as a function of the delays. This diagram is compared to simulations of the original delayed van der Pol system and the same qualitative scenario arises. However, for our choice of parameters, the delay corresponding to the bifurcation in the approximated ODE, close to 0.11, is not very small and therefore it happens to slightly different from the value corresponding to the delay differential equation (close to 0.09). Decreasing the value of \( \varepsilon \) and taking a closer to 1 reduces the value of the \( \tau \) corresponding to the bifurcation, making it closer from the simulations of the actual system.

The simulations of the original delayed van der Pol equations for this set of parameters shows a very clear delay-induced canard explosion, as shown in Fig. 20 for values of \( \alpha \) smaller than 1 and no delay, the system presents small oscillations corresponding to the presence of the Hopf bifurcation for \( \tau = 0 \) and \( \alpha = 1 \). When increasing the value of the delay, the amplitude of this small cycle suddenly becomes very large, corresponding to relaxation oscillations. However, these cycles depart from the actual system and non-perturbative analysis in the delay is necessary in order to uncover these phenomena, as provided in the main text. This is even more true for \( \alpha > 1 \), in which case canard explosion arise for relatively large delays.

References


\footnote{We chose the parameters for Fig. 20 because it allows more flexibility for illustrating the phenomena (the canard explosion arises on a broader interval of values for \( \tau \).}
Figure 20: Delay-induced canards in small delay approximated ODE (compare with Fig. 12). \( \varepsilon = 0.05, a = 0.995 \). Dotted lines correspond to the nullclines (red: \( x \) nullcline, green: \( y \) nullcline) and the curves represent trajectories in the phase plane (\( x, y \)) for different values of the delay. 1. \( \tau = 0.01 \), 2. \( \tau = 0.11 \), 3. \( \tau = 0.112 \), 4. \( \tau = 0.112167578 \), 5. \( \tau = 0.112167579 \), 6. \( \tau = 0.1122 \).


[27] Kuznetsov, Y.: Elements of applied bifurcation theory, 1995


49