The Differentiability of Real Functions
on Normed Linear Space Using
Generalized Subgradients*

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Submitted by F. Clarke
Received June 20, 1986

The modification of the Clarke generalized subdifferential due to Michel and
Penot is a useful tool in determining differentiability properties for certain classes of
real functions on a normed linear space. The Gâteaux differentiability of any real
function can be deduced from the Gâteaux differentiability of the norm if the
function has a directional derivative which attains a constant related to its
generalized directional derivative. For any distance function on a space with
uniformly Gâteaux differentiable norm, the Clarke and Michel–Penot generalized
subdifferentials at points off the set reduce to the same object and this generates a
continuity characterization for Gâteaux differentiability. However, on a Banach
space with rotund dual, the Fréchet differentiability of a distance function implies
that it is a convex function. A mean value theorem for the modified generalized sub-
differential has implications for Gâteaux differentiability. © 1987 Academic Press, Inc.

La modification de Michel et Penot du sous-differentiel généralisé de Clarke est
utilisée pour déterminer les propriétés dérивables de certaines classes de fonctions

* Ce rapport a été publié, en partie, grâce à une subvention du Fonds FCAR pour l’aide et
le soutien à la recherche.

‡ The first and third authors thank Professor Frank Clarke for providing the opportunity
for them to complete this research at CRM, Université de Montréal.
1. Introduction

In this paper we will consider conditions which relate to differentiability
of a real function \( \phi \) defined on a real normed linear space \( X \). In particular,
we will refer to two special types of real functions. A function \( \phi \) on \( X \) is said
to be \textit{locally Lipschitz} on \( X \) if for every \( x \in X \) there exists an \( M > 0 \) and
\( r > 0 \) such that

\[
|\phi(y) - \phi(z)| \leq M \|y - z\| \quad \text{for all } \|y - x\|, \|z - x\| < r.
\]

Given a non-empty closed subset \( K \) of \( X \) we will refer to the \textit{distance function} \( \phi \) generated by \( K \) as that defined by

\[
\phi(x) = \inf_{y \in K} \|x - y\|.
\]

Now a distance function is a Lipschitz one function.

For a function \( \phi \) on \( X \), given \( x, y \in X \), we say that \( \phi \) has a \textit{right-hand Gâteaux derivative} at \( x \) in the direction \( y \) if

\[
\lim_{t \to 0^+} \frac{\phi(x + ty) - \phi(x)}{t}
\]

exists. We denote such a limit by \( \phi'_+(x)(y) \). If

\[
\lim_{t \to 0} \frac{\phi(x + ty) - \phi(x)}{t}
\]

exists then we say that \( \phi \) has a \textit{Gâteaux directional derivative} at \( x \) in the direction \( y \). We denote such a limit by \( \phi'(x)(y) \). If this limit exists for all \( y \in X \) and is linear in \( y \) we say that \( \phi \) is \textit{Gâteaux differentiable} at \( x \) and we denote the limiting linear functional by \( \phi'(x) \) and call it the \textit{Gâteaux derivative} of \( \phi \) at \( x \). If \( \phi \) is Gâteaux differentiable at \( x \) and the limit \( \phi'(x)(y) \) is approached uniformly for all \( y \in X, \|y\| = 1 \), we say that \( \phi \) is \textit{Fréchet differentiable} at \( x \) and we refer to \( \phi'(x) \) as the \textit{Fréchet derivative} of \( \phi \) at \( x \).
Differentiability theory for convex functions on normed linear spaces is well developed. The most successful attempts to generalize the theory for non-convex functions have been for locally Lipschitz functions and a large measure of that success is due to ideas pioneered by Clarke [11]. Here we exploit a modification of Clarke's approach to prove several differentiability properties for real functions in general and for locally Lipschitz and distance functions in particular.

An important tool in non-smooth analysis, especially as it relates to optimization, is the Clarke generalized subdifferential. Given a locally Lipschitz function $\phi$ on a normed linear space $X$, the Clarke generalized directional derivative of $\phi$ at $x \in X$ in the direction $y \in X$ is defined by

$$
\phi^\circ(x)(y) = \lim_{t \to 0^+} \limsup_{z \to x} \frac{\phi(z + ty) - \phi(z)}{t}.
$$

The function $y \to \phi^\circ(x)(y)$ is continuous and sublinear. The Clarke generalized subdifferential of $\phi$ at $x \in X$ is defined by

$$
\partial \phi^\circ(x) = \{ f \in X^*: f(y) \leq \phi^\circ(x)(y) \text{ for all } y \in X \}
$$

and $\partial \phi^\circ(x)$ is non-empty, convex, and weak * compact. The elements $f \in \partial \phi^\circ(x)$ are called Clarke generalized subgradients for $\phi$ at $x$.

Now the Clarke generalized subdifferential has been used in the geometry of Banach spaces as a way of studying the differentiability of distance functions [4, p. 302]. However, for this purpose, the Clarke generalized subdifferential has a serious disadvantage: even where the function $\phi$ is Gâteaux differentiable at a point $x \in X$, the set $\partial \phi^\circ(x)$ need not necessarily be singleton.

In [8], Michel and Penot introduced a modified Clarke generalized subdifferential which has the advantage that it is smaller than the Clarke subdifferential and it reduces to the Gâteaux derivative when the Gâteaux derivative exists.

Given a locally Lipschitz function $\phi$ on a normed linear space $X$, the Michel–Penot generalized directional derivative of $\phi$ at $x \in X$ in the direction $y \in X$ is defined by

$$
\phi^{\circ}(x)(y) = \sup_{z \in X} \lim_{t \to 0^+} \frac{\phi(x + tz + ty) - \phi(x + tz)}{t}.
$$

Again, the function $y \to \phi^{\circ}(x)(y)$ is continuous and sublinear. The Michel–Penot generalized subdifferential of $\phi$ at $x \in X$ is defined by

$$
\partial \phi^{\circ}(x) = \{ f \in X^*: f(y) \leq \phi^{\circ}(x)(y) \text{ for all } y \in X \}.$$
and again $\partial \phi^\circ(x)$ is non-empty, convex, and weak * compact. The elements $f \in \partial \phi^\circ(x)$ are called Michel–Penot generalized subgradients for $\phi$ at $x$.

In both cases when $\phi$ is a convex function, then

$$\phi^\circ(x)(y) = \phi^\circ(x)(y) = \lim_{t \to 0^+} \frac{\phi(x + ty) - \phi(x)}{t} = \phi'_+(x)(y)$$

and $\partial \phi^\circ(x) = \partial \phi^\circ(x) = \partial \phi(x)$, the familiar subdifferential of a convex function. So both of these generalized subdifferentials are genuine generalizations from convex to locally Lipschitz functions.

However, they generalize different aspects of the convex situation: the Clarke generalized subdifferential mapping $x \mapsto \partial \phi^\circ(x)$ is weak * upper semi-continuous; the Michel–Penot generalized subdifferential mapping $x \mapsto \partial \phi^\circ(x)$ has the property that $\phi$ is Gâteaux differentiable at $x$ if and only if $\partial \phi^\circ(x)$ is singleton.

It is clear then that any study of differentiability of real functions on normed linear spaces using generalized subgradient methods will find the Michel–Penot formulation to be of considerable advantage.

It should be noted that for a locally Lipschitz function $\phi$, if for some $x \in X$, $\phi'_+(x)(y)$ exists for all $y \in X$ and is convex in $y$, then $\partial \phi^\circ(x) = \partial \phi'_+(x)$. So significantly the Michel–Penot subdifferential provides a way of generalizing the Pshenichnyi study of quasi-differentiable functions [10, p. 68].

Most of our conditions relating to differentiability of real functions are linked to particular properties of the norm. Now the norm as a convex function always possesses a subgradient at each point; that is, given $x \in X$ there exists a continuous linear functional $f$ on $X$ such that

$$f(y) \leq \phi'_+(x)(y) \text{ for all } y \in X.$$ 

It is convenient to denote a subgradient of the norm at $x$ by $f_x$.

2. Differentiability of Real Functions Through a Modified Lipschitz Constant

Fitzpatrick [5, p. 546] showed that a function $\phi$ on a normed linear space $X$ is Fréchet differentiable at $x \in X$ if there exists a $y_0 \in X$, $\|y_0\| = 1$ such that the norm of $X$ is Fréchet differentiable at $y_0$ and the Gâteaux directional derivative of $\phi$ exists at $x$ in the direction $y_0$ and attains a modified Lipschitz constant at $x$. Here we show that by using a modified Lipschitz constant associated with the Michel–Penot generalized directional derivative, a similar result can be given for Gâteaux differentiability.
For a function $\phi$ on a normed linear space $X$, given $x \in X$ we define the constant

$$P_\phi(x) \equiv \sup_{\|y\| = 1} \sup_{z \in X} \lim_{t \to 0^+} \sup \frac{\phi(x + tz + ty) - \phi(x + tz)}{t}.$$ 

Now

$$-P_\phi(x) = \inf_{\|y\| = 1} \inf_{z \in X} \lim_{t \to 0^+} \inf \frac{\phi(x + tz + ty) - \phi(x + tz)}{t}.$$ 

When $\phi$ is locally Lipschitz then

$$P_\phi(x) = \sup_{\|y\| = 1} \phi_\circ(x)(y)$$

and

$$-P_\phi(x) = \inf_{\|y\| = 1} (\phi_\circ(x)(-y)).$$

If $\phi$ is also Gâteaux differentiable at $x$ then

$$P_\phi(x) = \|\phi'(x)\|.$$ 

This constant does bear a closer relation to the derivative of the function than the local Lipschitz constant. This can be seen from the function $\phi$ defined on $\mathbb{R}$ by

$$\phi(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0, \end{cases}$$

where the local Lipschitz constant at 0 is 1 but the function has $\phi'(0) = 0$ and $P_\phi(0) = 0$.

Moreover, in the following theorem we demonstrate the significance of this constant in determining the differentiability of a function.

**Theorem 1.** Consider a function $\phi$ on a normed linear space $X$ where for some $x \in X$ there exists a $y_\circ \in X$, $\|y_\circ\| = 1$ such that the Gâteaux directional derivative at $x$ in the direction $y_\circ$ exists and $\phi'(x)(y_\circ) = P_\phi(x)$. If the norm of $X$ is Gâteaux differentiable at $y_\circ$ with derivative $f_{y_\circ}$, then $\phi$ is Gâteaux differentiable at $x$ and

$$\phi'(x)(y) = P_\phi(x)f_{y_\circ}(y) \quad \text{for all} \quad y \in X.$$
Proof. Given \( y \in X, \| y \| = 1, \) and \( \varepsilon > 0 \) there exists a \( \gamma(\varepsilon, y_0, y) > 0 \) such that

\[
\left| \frac{\| y_0 + \lambda y \| - \| y_0 \| - f_{y_0}(y)}{\lambda} \right| < \varepsilon \quad \text{for} \quad 0 < |\lambda| \leq \gamma,
\]

so

\[
\| y_0 + \gamma y \| - \| y_0 \| = \gamma f_{y_0}(y) < \varepsilon \gamma.
\]

Also there exists a \( \delta > 0, \delta(\varepsilon, x) \) such that

\[
\left| \frac{\phi(x + ty_0) - \phi(x)}{t} - P_\phi(x) \right| \leq \varepsilon \gamma \quad \text{for} \quad 0 < |t| < \delta.
\]

Clearly, from the definition of \( P_\phi(x) \), there exists a \( \delta' > 0, \delta'(\varepsilon, x, -y_0, y_0 + \gamma y) \) such that

\[
\phi(x + ty) - \phi(x - ty) = \phi(x + t(-y_0) + t(\gamma y)) - \phi(x + t(-y_0))
\]

\[
\leq P_\phi(x) \| y_0 + \gamma y \| t + \varepsilon \gamma t
\]

for \( 0 < t < \delta' \). Then

\[
\phi(x + ty) - \phi(x) = \phi(x + ty) - \phi(x + ty) + \phi(x + ty) - \phi(x)
\]

\[
< P_\phi(x) \| y_0 + \gamma y \| t + \varepsilon \gamma t - P_\phi(x) \| y_0 \| t + \varepsilon \gamma t
\]

for \( 0 < t < \min(\delta, \delta') \)

\[
= P_\phi(x) t(\| y_0 + \gamma y \| - \| y_0 \|) + 2\varepsilon \gamma t
\]

\[
< P_\phi(x) t(\gamma y + \varepsilon \gamma) + 2\varepsilon \gamma t.
\]

So

\[
\frac{\phi(x + ty) - \phi(x)}{ty} < P_\phi(x) f_{y_0}(y) + (P_\phi(x) + 2) \varepsilon
\]

for \( 0 < t < \min(\delta, \delta') \).

On the other hand, there exists a \( \delta'' > 0, \delta''(\varepsilon, x, y_0, -y_0 + \gamma y) \) such that

\[
\phi(x + ty) - \phi(x + ty_0) = \phi(x + ty_0 + t(-y_0 + \gamma y)) - \phi(x + ty_0)
\]

\[
> -P_\phi(x) \| y_0 - \gamma y \| t - \varepsilon \gamma t
\]
for $0 < t < \delta'$. Then

$$\phi(x + ty) - \phi(x) = \phi(x + ty) - \phi(x + ty_0) + \phi(x + ty_0) - \phi(x)$$

$$> -P(x)\|y_0 - \gamma y\| t - \varepsilon \gamma t + P(x)\|y_0\| t - \varepsilon \gamma t$$

for $0 < t < \min(\delta, \delta')$

$$= -P(x) t(\|y_0 - \gamma y\| \|y_0\|) 2\varepsilon \gamma t$$

$$> P(x) t(f_{y_0}(\gamma y) - \varepsilon \gamma) - 2\varepsilon \gamma t.$$

So

$$\frac{\phi(x + ty) - \phi(x)}{t} > P(x)f_{y_0}(y) - (P(x) + 2) \varepsilon$$

for $0 < t < \min(\delta, \delta'')$.

We conclude from (i) and (ii) that

$$\lim_{t \to 0^+} \frac{\phi(x + ty) - \phi(x)}{t}$$

exists and is equal to $P(x)f_{y_0}(y)$ for all $y \in X$. This implies that $\phi$ is Gâteaux differentiable at $x$ and

$$\phi'(x)(y) = P(x)f_{y_0}(y)$$

for all $y \in X$.

We now apply this theorem in examining the differentiability of distance functions.

**Corollary 2.** Consider a non-empty closed set $K$ in a normed linear space $X$ and a point $x \in X \setminus K$ which has a closest point $p(x)$ in $K$. If the distance function $\phi$ generated by $K$ is Gâteaux directionally differentiable at $x$ in a direction $x - p(x)$ and the norm of $X$ is Gâteaux differentiable at $x - p(x)$, then $\phi$ is Gâteaux differentiable at $x$ and

$$\phi'(x)(y) = f_{x - p(x)}(y)$$

for all $y \in X$.

**Proof.** From the closest point property we can deduce that $p(x)$ is a closest point for all $x_\lambda = x + \lambda(x - p(x))$, where $-1 < \lambda < 0$, and so
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\[ \phi'_p(x)(- (x - p(x))) \] exists and is equal to \(-\|x - p(x)\|\). Therefore, if \( \phi \) is Gâteaux directionally differentiable at \( x \) in the direction \( x - p(x) \) then

\[ \phi'(x)(x - p(x)) = \|x - p(x)\| \]

and since \( \phi \) is Lipschitz one, we deduce that \( P_\phi(x) = 1 \) and the conclusion follows from Theorem 1. 

The proof of this corollary indicates that the following general property holds.

**Corollary 3.** Consider a non-empty closed set \( K \) in a normed linear space \( X \) which has the property that each point of a dense subset of \( X \setminus K \) has a closest point in \( K \). If the norm of \( X \) is Gâteaux differentiable on \( X \setminus \{0\} \) then the distance function \( \phi \) generated by \( K \) is Gâteaux differentiable on a dense subset of \( X \setminus K \).

**Proof.** Given a point \( y \in X \setminus K \) and \( 0 < \varepsilon < \phi(y) \), consider \( x \in X \), \( \|x - y\| < \varepsilon \) such that \( x \) has a closest point \( p(x) \) in \( K \). Then \( p(x) \) is a closest point for all \( x_\lambda = x + \lambda(x - p(x)) \), where \( -1 < \lambda < 0 \). So \( \phi \) is Gâteaux directionally differentiable at \( x_\lambda \) for all \( -1 < \lambda < 0 \) in the direction \( x_\lambda - p(x) \) and \( \phi'(x_\lambda)(x_\lambda - p(x)) = \|x_\lambda - p(x)\| \). We deduce from Corollary 2 that \( \phi \) is Gâteaux differentiable at each \( x_\lambda \) for \( -1 < \lambda < 0 \).

A non-empty closed set \( K \) in a normed linear space \( X \) is said to be a *sun* if for each \( x \in X \setminus K \) there exists a closest point \( p(x) \) in \( K \) and each point \( x_\lambda = x + \lambda(x - p(x)) \) for \( \lambda > -1 \) has the same closest point \( p(x) \). So, in particular, we have the following property for suns.

**Corollary 4.** For a sun \( K \) in a normed linear space \( X \), if the norm of \( X \) is Gâteaux differentiable on \( X \setminus \{0\} \) then the distance function \( \phi \) generated by \( K \) is Gâteaux differentiable on \( X \setminus K \).

It is interesting to notice that even without the assumption of Gâteaux differentiability of norm in Corollary 2 the Michel–Penot subdifferential at a point \( x \in X \setminus K \) which has a closest point \( p(x) \) in \( K \) always contains a subgradient of the norm at \( x - p(x) \). To show this we need the following elementary property for the generalized subdifferential of a locally Lipschitz function.

Given a locally Lipschitz function \( \phi \) on a normed linear space \( X \), it is clear from the definition that the Michel–Penot generalized subdifferential has the following properties similar to those of the Clarke generalized subdifferential:

\[ \partial \phi^\circ(x) = \{ f \in X^* : - \phi^\circ(x)(-y) \leq f(y) \leq \phi^\circ(x)(y) \text{ for all } y \in X \} \]
and
\[ -\phi^\diamond(x)(-y) = \inf_{t \to 0^+} \liminf_{z \to X} \frac{\phi(x + tz + ty) - \phi(x + tz)}{t}. \]

But also given \( y_0 \in X \) for each \( \alpha \) where
\[ -\phi^\diamond(x)(-y_0) \leq \alpha \leq \phi^\diamond(x)(y_0), \]
there exists an \( f \in \partial\phi^\diamond(x) \) such that \( f(y_0) = \alpha. \)

**Theorem 5.** Consider a non-empty closed set \( K \) in a normed linear space \( X \) and distance function \( \phi \) generated by \( K \). For any \( x \in X \setminus K \) where there exists a closest point \( p(x) \) in \( K \), there exists an \( f \in \partial\phi^\diamond(x) \) where \( f \in \partial\|x - p(x)\| \).

**Proof.** Given \( y \in X \),
\[ \phi^\diamond(x)(y) = \sup_{z \in X} \limsup_{t \to 0^+} \frac{\phi(x + tz + ty) - \phi(x + tz)}{t} \geq \limsup_{t \to 0^+} \frac{\phi(x) - \phi(x - ty)}{t} \]
putting \( z = -y \)
\[ \geq \limsup_{t \to 0^+} \frac{\|x - p(x)\| - \|x - p(x) - ty\|}{t} \]
using the closest point property. So for \( y = x - p(x) \),
\[ \phi^\diamond(x)(x - p(x)) = \|x - p(x)\| \geq -\phi^\diamond(x)(-(x - p(x))). \]

But from the remarks made before this theorem, there exists an \( f \in \partial\phi^\diamond(x) \) such that
\[ f(x - p(x)) = \|x - p(x)\| \]
and
\[ \|f\| \leq 1 \quad \text{so} \quad f \in \partial\|x - p(x)\|. \]

This can be considered a gloss on a theorem due to Vlasov [11]. We may recapture his result using the special advantages of the Michel–Penot generalized subdifferential.
Corollary 6. Consider a non-empty closed set $K$ in a normed linear space $X$ and distance function $\phi$ generated by $K$. If $\phi$ is Gateaux differentiable at $x \in X \setminus K$ where there exists a closest point $p(x)$ in $K$, then

$$\phi'(x) \in \partial \|x - p(x)\|.$$  

3. A Characterization of the Generalized Subdifferentials for Distance Functions and A Continuity Characterization for Gateaux Differentiability

Given a locally Lipschitz function $\phi$ on a normed linear space $X$, it is evident that the Michel–Penot generalized subdifferential mapping $x \mapsto \partial \phi^\circ(x)$ has no general continuity property corresponding to the weak* upper semicontinuity of Clarke’s generalized subdifferential mapping $x \mapsto \partial \phi^\circ(x)$. However, for the distance function $\phi$ generated by a non-empty closed set $K$ in a normed linear space $X$ with uniformly Gateaux differentiable norm, we prove the surprising fact that the two generalized subdifferentials are the same object for points off the set and so the generalized subdifferential mapping in this case possesses the properties of both. This enables us to derive a continuity characterization for the Gateaux differentiability of $\phi$.

A normed linear space $X$ is said to have uniformly Gateaux differentiable norm if for each $y \in X$, given $\varepsilon > 0$, there exists a $\delta(\varepsilon, y) > 0$ such that for every $x \in X$, $\|x\| = 1$, there is a continuous linear functional $f_x$ on $X$ and

$$\left| \frac{\|x + ty\| - \|x\|}{t} - f_x(y) \right| < \varepsilon$$

for all $0 < |t| < \delta$.

We will use the following well-known characterizations of this property; see [13, p. 299].

Lemma 7. A normed linear space $X$ has uniformly Gateaux differentiable norm if and only if for each $y \in X$, either one of the following holds for any $r > 0$, for any $f_x \in \partial \|x\|$ and $f_{x+ty} \in \partial \|x+ty\|$

(i) $\lim_{t \to 0} \sup_{\|x\| > r} |(f_{x+ty} - f_x)(y)| = 0,$

(ii) the mapping $x \to f_x(y)$ is uniformly continuous on $\{x: \|x\| > r\}$.

The following theorem highlights the special differentiability properties inherent in a distance function on a normed linear space with uniformly Gateaux differentiable norm.
THEOREM 8. Consider the distance function $\phi$ generated by a non-empty closed set $K$ in a normed linear space $X$ with uniformly Gâteaux differentiable norm. For any $x \in X \setminus K$,

(i) the distance function $\phi$ always has a right-hand Gâteaux derivative and

$$-\phi'_+(x)(y) = (-\phi)^\circ(x)(y) = (-\phi)^\circ(y)$$

for all $y \in X$ and consequently $-\phi$ is always quasi-differentiable and regular,

(ii) $\partial\phi^\circ(x) = \partial\phi^\circ(x)$ and this set is the weak * closed convex hull of all weak * cluster points of

(a) $f_{z-u}$, where $\|z-u\| \to \phi(x)$, $u \in K$ as $z \to x$,

(b) $f_{x-v}$, where $\|x-v\| \to \phi(x)$, $v \in K$.

Proof. (i) We show that

$$\limsup_{t \to 0^+} \frac{\phi(x + ty) - \phi(x)}{t} \leq \liminf_{t \to 0^+} \frac{\phi(z + ty) - \phi(z)}{t}.$$ 

Now there exists $t_n \to 0^+$ and $z_n \to x$, such that

$$\liminf_{t \to 0^+} \frac{\phi(z + ty) - \phi(z)}{t} = \lim_{t \to 0^+} \frac{\phi(z_n) - \phi(z_{n} - t_n y)}{t_n}.$$ 

For each $n$, choose $u_n \in K$ such that

$$\|z_n - u_n\| \geq \phi(z_n) > \|z_n - u_n\| - t_n^2.$$ 

Then

$$\liminf_{t \to 0^+} \frac{\phi(z + ty) - \phi(z)}{t} \geq \liminf_{t \to 0^+} \frac{\|z_n - u_n\| - \|z_n - t_n y - u_n\|}{t_n}$$

$$\geq \liminf f_{z_n - t_n y - u_n}(y)$$

$$\geq \liminf f_{z_n - u_n}(y)$$

by Lemma 7(i),

$$\geq \liminf f_{x-u_n}(y)$$

by Lemma 7(ii),

$$\geq \liminf \{f_{x-v}(y) : \|x-v\| \to \phi(x), v \in K\}$$
since
\[ \|x - u_n\| \leq \|x - z_n\| + \|z_n - u_n\| \]
and
\[ \|z_n - u_n\| \to \phi(x) \quad \text{as } z_n \to x. \]

Now there exists \( v_n \in K \), where \( \|x - v_n\| \to \phi(x) \), such that
\[ \lim f_{x - v_n}(y) = \lim \inf \{ f_{x - v}(y) : \|x - v\| \to \phi(x), v \in K \}. \]

Since the norm is uniformly Gâteaux differentiable, given \( \epsilon > 0 \) there exists a \( \delta(\epsilon, y) > 0 \) such that
\[ \|b + v - v_n\| - \|b - u_n\| < \epsilon \]
for all \( 0 < |t| < \delta \) and all \( n \) sufficiently large. Then
\[ f_{x - v_n}(y) + \epsilon \geq \frac{\phi(x + ty) - \phi(x)}{t} + \frac{\phi(x) - \|x - v_n\|}{t}. \]

So
\[ \lim \inf \{ f_{x - v}(y) : \|x - v\| \to \phi(x), v \in K \} + \epsilon \]
\[ \geq \frac{\phi(x + ty) - \phi(x)}{t} \quad \text{for all } 0 < |t| < \delta. \]

Since
\[ \lim \inf \frac{\phi(x + ty) - \phi(x)}{t} \geq \lim \inf \frac{\phi(z + ty) - \phi(z)}{t}, \]
we conclude that the right-hand Gâteaux derivative of \( \phi \) exists and
\[ \phi_+^\circ(x)(y) = -\phi^\circ(x)(-y). \]

But
\[ \phi^\circ(x)(-y) - (-\phi)^\circ(x)(y) \]
and since always
\[ (-\phi)^\circ(x)(y) \geq (-\phi)^\circ(x)(y) \geq (-\phi)_+^\circ(x)(y) \]
for all \( y \in X \), we see that equality holds. This also implies that \(-\phi\) is quasi-differentiable \([10, \text{p. 68}]\) and regular \([1, \text{p. 39}]\).

(ii) Now

\[
\phi^\circ(x)(y) = (-\phi)^\circ(x)(-y)
\]

and

\[
\phi^{\circ\circ}(x)(y) = (-\phi)^\circ(x)(-y).
\]

From (i)

\[
(-\phi)^\circ(x)(y) = (-\phi)^\circ(x)(y)
\]

for all \( y \in X \)

so

\[
\phi^\circ(x)(y) = \phi^{\circ\circ}(x)(y)
\]

for all \( y \in X \),

which implies that \( \partial \phi^\circ(x) = \partial \phi^{\circ\circ}(x) \).

From the proof of (i), we see that

\[-\phi^\circ(x)(-y) = \liminf \{ f_{z-u}(y) : \|z-u\| \to \phi(x), u \in K \text{ as } z \to x \} \]

\[= \liminf \{ f_{x-v}(y) : \|x-v\| \to \phi(x), v \in K \}\]

and so

\[
\phi^{\circ\circ}(x)(y) = \limsup \{ f_{z-u}(y) : \|z-u\| \to \phi(x), u \in K \text{ as } z \to x \} - \limsup \{ f_{x-v}(y) : \|x-v\| \to \phi(x), v \in K \}.
\]

Therefore, for each \( y \in X \), \( \partial \phi^\circ(x)(y) \) is the closed convex hull of the cluster points of each of

\[f_{z-u}(y), \quad \text{where } \|z-u\| \to \phi(x), \ u \in K \text{ as } z \to x\]

and

\[f_{x-v}(y), \quad \text{where } \|x-v\| \to \phi(x), \ v \in K,\]

which establishes the theorem statements (ii)(a) and (b).

From the remarkable properties of the subdifferential of the distance function established in Theorem 8, we can deduce immediately the following continuity characterizations for Gâteaux differentiability.
COROLLARY 9. Consider a distance function $\phi$ generated by a non-empty closed set $K$ in a normed linear space $X$ with uniformly Gâteaux differentiable norm. For any $x \in X \setminus K$ and $y \in X$, $\phi$ is Gâteaux differentiable at $x$ in the direction $y$ if and only if any one of the following hold:

(i) $\partial \phi^0(x)(y)$ is singleton,
(ii) $\partial \phi^0(x)(y)$ is singleton,
(iii) $f_{x-}(y)$ is convergent as $\|z - u\| \to \phi(x)$, $u \in K$ and $z \to x$,
(iv) $f_{x-}(y)$ is convergent as $\|x - v\| \to \phi(x)$, $v \in K$.

Of course in (ii) and (iii) convergence is to the Gâteaux directional derivative $\phi'(x)(y)$. This implies that $\phi$ is Gâteaux differentiable at $x \in X \setminus K$ if and only if $\partial \phi^0(x)$ is singleton and this is the linear Gâteaux derivative at $x$.

So in a normed linear space with uniformly Gâteaux differentiable norm, the subdifferential mapping for a distance function combines the properties of both of the Clarke and Michel–Penot generalized subdifferential mappings. But we should note that Corollary 9 is a stronger characterization for Gâteaux differentiability than we might expect even from the Michel–Penot subdifferential, because it provides a characterization for directional Gâteaux differentiability.

It was Zahiček [13, p. 300] who first pointed out that in a normed linear space with uniformly Gâteaux differentiable norm a distance function $\phi$ always has right-hand Gâteaux derivatives at every point off the set and $\phi'_+(x)(y)$ is concave in $y$. Moreover, he also showed that this occurs for all sets $K$ if and only if the norm is uniformly Gâteaux differentiable. We should also note, with Zahiček, that Theorem 8 provides an extension of Mises theorem [13, p. 292] from $\mathbb{R}^n$ to a significant class of infinite-dimensional normed linear spaces.

Theorem 8 can be applied in a variety of significant spaces. Every Hilbert space and $L_p$-space ($1 < p < \infty$) has uniformly Gâteaux differentiable norm. Furthermore, any separable Banach space can be equivalently renormed to have uniformly Gâteaux differentiable norm [14, p. 199] and so too can any super-reflexive Banach space.

It is interesting to see that the satisfying conclusions of Theorem 8 do not extend to a point on the boundary of a set as is shown by the following simple example.

EXAMPLE 10. Consider the set $K \equiv \{(\lambda, \mu): \lambda^2 + \mu^2 \leq 1 \text{ or } \mu \leq -1\}$ in $\mathbb{R}^2$ which generates the distance function $\phi$, where

\[
\phi(\lambda, \mu) = (\sqrt{\lambda^2 + \mu^2} - 1) \land (\mu + 1) \quad \text{for } (\lambda, \mu) \in \mathbb{R}^2 \setminus K
\]

\[
= 0 \quad \text{for } (\lambda, \mu) \in K.
\]
Now
\[
\phi((0, -1) + t(\lambda, \mu)) - \phi(0, -1)
\]
\[
= 0 \quad \text{if } \mu \leq 0
\]
\[
= 0 \quad \text{if } \mu > 0 \text{ and } 0 < t \leq \frac{2\mu}{\lambda^2 + \mu^2}.
\]

Therefore, \(\phi\) is Gâteaux differentiable at \((0, -1)\) and
\[
\phi'(0, -1) = 0
\]
but also
\[
\phi^\circ(0, -1)(\lambda, \mu) = |\mu|.
\]

Now a closed set \(K\) is convex if and only if its distance function \(\phi\) is convex. Theorem 8 has implications for distance functions to closed convex sets.

**Corollary 11.** If \(K\) is a non-empty closed convex set in a normed linear space \(X\) with uniformly Gâteaux differentiable norm, then the distance function \(\phi\) generated by \(K\) is a convex function Gâteaux differentiable on \(X\setminus K\).

**Proof.** Since \(\phi\) is convex on \(X\), for all \(x \in X\), \(\phi'(x)(y)\) is a convex function of \(y\). But since the norm is uniformly Gâteaux differentiable, from Theorem 8, for all \(x \in X\setminus K\), \(\phi'(x)(y)\) is concave in \(y\). Therefore \(\phi\) is Gâteaux differentiable on \(X\setminus K\). \(\blacksquare\)

It is clear that this will not extend to points on the boundary; we need only consider \(K\) as the origin in any normed linear space with uniformly Gâteaux differentiable norm.

Recently, Fabian and Zhivkov [3, p. 71] have shown that a Banach space which contains as a dense subset the image of an Asplund space under a continuous linear mapping has the property that any continuous function where the right-hand Gâteaux derivative exists at each point of an open set is Gâteaux differentiable on a dense \(G_\delta\) subset. So in such a Banach space with uniformly Gâteaux differentiable norm, the distance function \(\phi\) generated by a non-empty closed set \(K\) is Gâteaux differentiable on a dense \(G_\delta\) subset of \(X\setminus K\).

However, it is even more fascinating to see that under mild spatial and differentiability conditions, a distance function is a convex function. To do this, we generalize the classic arguments of Vlasov [12], where he provides
conditions under which a Chebyshev set is convex. The steps in Vlasov’s work may be followed in Giles [6, pp. 239–245].

**Lemma 12.** In a Banach space $X$ consider a non-empty closed set $K$ which generates a distance function $\phi$ satisfying the property that

$$\limsup_{\|y\| \to 0} \frac{\phi(x + y) - \phi(x)}{\|y\|} = 1$$

for all $x \in X \setminus K$. Then given $x \in X \setminus K$, $r > 0$, and $\sigma > 1$, there exists an $x_0 \in X \setminus K$ such that

(i) $\phi(x) + 1/\sigma \|x - x_0\| \leq \phi(x_0)$,

(ii) $\phi(y) < \phi(x_0) + 1/\sigma \|y - x_0\|$ for all $y \neq x_0$ and $\|y - x\| \leq r$,

(iii) $\|x - x_0\| = r$.

**Proof.** Applying the primitive form of Ekeland’s theorem [2] to the complete metric space $B[x; r]$ and the continuous real function $-\phi$ on $B[x; r]$, we have that there exists an $x_0 \in B[x; r]$ such that

$$\phi(x) + 1/\sigma \|x - x_0\| \leq \phi(x_0)$$

and

$$\phi(y) < \phi(x_0) + 1/\sigma \|y - x_0\| \quad \text{for all} \quad y \neq x_0 \quad \text{and} \quad \|y - x\| \leq r.$$

From (i), we see that $x_0 \in X \setminus K$. Suppose that $\|x - x_0\| < r$. Now there exists a $y_0 \in X$, $0 \leq \|y_0\| < r - \|x - x_0\|$ such that

$$\frac{\phi(x_0 + y_0) - \phi(x_0)}{\|y_0\|} > 1/\sigma.$$

But then $x_0 + y_0 \in B[x; r]$, which contradicts (ii). We conclude that $\|x - x_0\| = r$.

A closed set $K$ in a normed linear space $X$ is said to be *almost convex* if for any closed ball $B$ which does not intersect $K$ there exists a closed ball $B'$ of arbitrarily large radius which contains $B$ and still does not intersect $K$.

**Lemma 13.** In a Banach space $X$, a non-empty closed set $K$ which generates a distance function $\phi$ satisfying the property that

$$\limsup_{\|y\| \to 0} \frac{\phi(x + y) - \phi(x)}{\|y\|} = 1$$

for all $x \in X \setminus K$ is almost convex.
Proof. Given $x \in X \setminus K$, choose $\alpha > \beta > 0$ such that $0 < \beta < \phi(x) < \alpha$. Then we can choose $\sigma > 1$ and $r > 0$ such that $\sigma(\alpha - \phi(x)) < r < \alpha - \beta$. By Lemma 12, there exists an $x_0 \in X \setminus K$ such that $r = \|x - x_0\| \leq \sigma(\phi(x_0) - \phi(x))$. This implies that $B[x_0; \alpha]$ contains $B[x; \beta]$ and does not intersect $K$. \[\square\]

A normed linear space $X$ is said to be rotund if for every $x, y \in X$, $x \neq y$, $\|x\| = \|y\| = 1$ we have $\|x + y\| < 2$. Vlasov showed that in a normed linear space $X$ every closed almost convex set is convex if and only if the dual space $X^*$ is rotund. So we can see that Vlasov's theorem could be stated in more generality as follows.

**Theorem 14.** In a Banach space $X$ with rotund dual $X^*$, a non-empty closed set $K$ which generates a distance function $\phi$ satisfying the property

$$\limsup_{\|y\| \to 0} \frac{\phi(x + y) - \phi(x)}{\|y\|} = 1$$

for all $x \in X \setminus K$ is a convex set.

Clearly, a distance function $\phi$ which is Gâteaux differentiable on $X \setminus K$ and where $\|\phi'(x)\| = 1$ for all $x \in X \setminus K$ satisfies the differentiability condition of this theorem. But even in Hilbert space, the Gâteaux differentiability of a distance function $\phi$ at a point $x \in X \setminus K$ does not necessarily imply that $\|\phi'(x)\| = 1$ [4, p. 308]. However, in any normed linear space, the Fréchet differentiability of a distance function $\phi$ is sufficient to satisfy the differentiability condition of the theorem.

**Lemma 15.** Given a non-empty closed set $K$ in a normed space $X$, if the distance function $\phi$ generated by $K$ is Fréchet differentiable at $x \in X \setminus K$ then $\|\phi'(x)\| = 1$.

**Proof.** Given $x \in X \setminus K$ and $t > 0$, choose $u(x, t) \in K$ such that $\|x - u(x, t)\| > \phi(x) > \|x - u(x, t)\| - t^2$.

For $x_t \equiv x - t(x - u(x, t))$,

$$\frac{\phi(x_t) - \phi(x)}{\|x_t - x\|} \leq \frac{\|x_t - u(x, t)\| - \|x - u(x, t)\| + t^2}{\|x_t - x\|}$$

$$\leq - \frac{\|x_t - x\| + t^2}{\|x_t - x\|}$$

$$\leq -1 + \frac{t}{\phi(x)}.$$
But since $\phi$ is Fréchet differentiable at $x$ we deduce that, for $y_t = t(x-u(x,t))$,

$$\limsup_{\|y_t\| \to 0} \frac{\phi(x+y_t) - \phi(x)}{\|y_t\|} = 1$$

and $\|\phi'(x)\| = 1$.

We can use the proof of Lemma 15 to determine a necessary condition for $K$ to be convex.

**Corollary 16.** If $K$ is a closed convex set in a normed linear space $X$ and the distance function $\phi$ generated by $K$ is Gâteaux differentiable at a point $x \in X \setminus K$ then $\|\phi'(x)\| = 1$.

**Proof.** Since $\phi$ is convex on $X$ and Gâteaux differentiable at $x$,

$$\phi'(x) \left( \frac{x_t - x}{\|x_t - x\|} \right) \leq \frac{\phi(x_t) - \phi(x)}{\|x_t - x\|} \leq -1 + \frac{t}{\phi(x)}$$

and so $\|\phi'(x)\| = 1$.

We are now in a position to see that in a Banach space with spatial conditions less stringent than those of Corollary 9, a distance function satisfying rather mild differentiability conditions is in fact a convex function.

From Corollary 2 and Theorem 14 we have the following result.

**Theorem 17.** In a Banach space $X$ with rotund dual $X^*$ consider a non-empty closed set $K$ with the property that to each $x \in X \setminus K$ there exists a closest point $p(x)$ in $K$. If the distance function $\phi$ generated by $K$ is Gâteaux directionally differentiable at each $x \in X \setminus K$ in the direction $x - p(x)$, then $\phi$ is a convex function, Gâteaux differentiable on $X \setminus K$.

More generally we have the following associated result.

**Theorem 18.** In a Banach space $X$ with rotund dual $X^*$ if the distance function $\phi$ generated by a non-empty closed set $K$ is Gâteaux differentiable on $X \setminus K$ and $\|\phi'(x)\| = 1$ for all $x \in X \setminus K$ then $\phi$ is a convex function.

Recently, Preiss has shown [9, p.129] that a real locally Lipschitz function $\phi$ on an open subset $G$ of an Asplund space is Fréchet differentiable on a dense subset of $G$. So, Lemma 15 and his result imply that in an Asplund space $X$ the distance function $\phi$ generated by the non-empty closed set $K$ satisfies $\|\phi'(x)\| = 1$ for points $x$ belonging to a dense subset of $X \setminus K$. 

Lebourg [7] developed a useful Mean Value Theorem for Clarke generalized subgradients. We now show that a similar result holds for Michel–Penot generalized subgradients. Since the Michel–Penot generalized subdifferential is smaller than that of Clarke, this Mean Value Theorem is correspondingly more powerful.

**Theorem 19 (The Mean Value Theorem).** Given a locally Lipschitz function $\phi$ on a normed linear space $X$, for every $x, y \in X$, $x \neq y$ there exists an $a \in (x, y)$ and an $f \in \partial \phi^\circ(a)$ such that

$$f(y-x) = \phi(y) - \phi(x).$$

**Proof.** Consider the real function $g$ defined on $\mathbb{R}$ by

$$g(t) = \phi(ty + (1-t)x) - t\phi(y) - (1-t)\phi(x).$$

Now $g(0) = g(1) = 0$ and since $g$ is continuous on $[0, 1]$, $g$ has a minimum or a maximum at some $t_0 \in (0, 1)$. Then for $a \equiv t_0y + (1-t_0)x$,

$$\phi(a + t(y - x)) = g(t + t_0) + (t + t_0)\phi(y) + (1-t-t_0)\phi(x)$$

so

$$\phi(a + t(y - x)) - \phi(a) = g(t + t_0) - g(t_0) + t(\phi(y) - \phi(x)).$$

If $g$ has a minimum at $t_0$ then

$$\phi(a + t(y - x)) - \phi(a) \geq t(\phi(y) - \phi(x))$$

for $-t_0 < t < 1 - t_0$, and so

$$-\phi^\circ(a)(- (y-x)) = \inf_{z \in X} \liminf_{t \to 0+} \frac{\phi(a + t \varepsilon(t(y-x))) - \phi(a + t\varepsilon)}{t}$$

$$\leq \liminf_{t \to 0+} \frac{\phi(a) - \phi(a - t(y-x))}{t}$$

$$\leq \phi(y) - \phi(x)$$

$$\leq \limsup_{t \to 0+} \frac{\phi(a + t(y-x)) - \phi(a)}{t}$$

$$\leq \phi^\circ(a)(y - x).$$
If \( g \) has a maximum at \( t_0 \), then
\[
\phi(a + t(y - x)) - \phi(a) \leq t(\phi(y) - \phi(x))
\]
and a similar argument gives the same result that
\[
-\phi^\circ(a) \leq \phi(y) - \phi(x) \leq \phi^\circ(a)(y - x).
\]
By the generalized subdifferential properties mentioned before Theorem 5, we have that there exists an \( f \in \partial \phi^\circ(a) \) such that
\[
f(y - x) = \phi(y) - \phi(x).
\]

It is clear that a necessary condition for a locally Lipschitz function \( \phi \) on a normed linear space \( X \) to have a maximum or a minimum at a point \( a \in X \) is that \( 0 \in \partial \phi^\circ(a) \).

We should point out that this Mean Value Theorem implies that there is a very neat relationship between the Clarke and Michel–Penot subdifferentials for any locally Lipschitz function \( \phi: \delta^\circ\phi(x) \) is the set of weak* cluster points of \( \partial \phi^\circ(z) \) as \( z \to x \).

In Corollary 9, we developed a continuity characterization for Gâteaux differentiability of a distance function on a normed linear space with uniformly Gâteaux differentiable norm and we subsequently commented on the spatial conditions for this characterization to hold. The continuity condition implying Gâteaux differentiability is in fact quite strong and we can use the Mean Value Theorem to show how a similar result can be given more generally to imply the existence of Gâteaux directional derivatives. For a locally Lipschitz function \( \phi \) on a normed linear space \( X \), it is convenient to denote a generalized subgradient of \( \phi \) at \( x \in X \) by \( f^\phi_x \).

**Theorem 20.** Given a locally Lipschitz function \( \phi \) on a normed linear space \( X \), if for any given \( y \in X \) and all the generalized subgradients \( f^\phi_{x + ty} \in \partial \phi^\circ(x + ty) \), \( f^\phi_{x + ty}(y) \) is convergent as \( t \to 0 \), then \( \phi \) is Gâteaux differentiable at \( x \) in the direction \( y \).

**Proof.** Given \( t > 0 \), by the Mean Value Theorem there exists an \( 0 < \alpha < 1 \) and an \( f^\phi_{x + \alpha ty} \in \partial \phi^\circ(x + t\alpha y) \) such that
\[
\phi(x + ty) - \phi(x) = f^\phi_{x + \alpha ty}(ty).
\]
But the continuity hypothesis tells us that \( f^\phi_{x + ty}(y) \) is convergent as \( t \to 0^+ \). So
\[
\lim_{t \to 0^+} \frac{\phi(x + ty) - \phi(x)}{t}.
\]
exists and has the same value. But also there exists an $0 < \alpha' < 1$ and an $f_{x - t\alpha' y}^\phi \in \partial \phi^\odot (x - t\alpha' y)$ such that
\[ \phi(x) - \phi(x - ty) = f_{x - t\alpha' y}^\phi (ty) \]
and again
\[ \lim_{t \to 0} \frac{\phi(x + ty) - \phi(x)}{t} \]
exists and has the same value. So we conclude that $\phi$ is Gâteaux differentiable at $x$ in the direction $y$. \[\]

For a distance function in a normed linear space $X$, we have an improved criterion for Gâteaux differentiability.

**Corollary 21.** Consider a non-empty closed set $K$ in a normed linear space $X$ where to each point $x \in X \setminus K$ there exists a closest point $p(x)$ in $K$. If given $y \in X$, every $f_{x + ty - p(x + ty)}(y)$ is convergent as $t \to 0$, then $\phi$ is Gâteaux differentiable at $x$ in the direction $y$ and $\phi'(x)(y) = f(y)$ for some $f \in \partial \|x - p(x)\|$.

**Proof.** From the proof of Theorem 5 we see that if $s$ is such that $q_{x + sy}(y)$ exists, then it is of the form
\[ f_{x + sy - p(x + sy)}(y). \]
But the real function $t \to \phi(x + ty)$ is differentiable almost everywhere by Rademacher's theorem, so $\phi'(x + sy)(y)$ exists almost everywhere and
\[\frac{1}{t} (\phi(x + ty) - \phi(x)) = \frac{1}{t} \int_0^r \phi'(x + sy)(y) \, ds \]
\[ - \frac{1}{t} \int_0^r f_{x + sy - p(x + sy)}(y) \, ds. \]
But we are given that $f_{x + sy - p(x + sy)}(y)$ converges as $s \to 0$, so $\phi$ is Gâteaux differentiable at $x$ in the direction $y$ and $\phi'(x)(y)$ is of the form $f_{x - p(x)}(y)$. \[\]

As with the Clarke generalized subdifferential [1, p. 37], the Mean Value Theorem enables us to establish that the monotonicity of the Michel–Penot generalized subdifferential mapping $x \to \partial \phi^\odot(x)$ guarantees the convexity of the function $\phi$. 

THEOREM 22. A locally Lipschitz function $\phi$ on a normed linear space $X$ is convex if and only if

$$(f^\phi_y - f^\phi_x)(y - x) \geq 0$$

for all $x, y \in X$ and

$$f^\phi_x \in \partial \phi^\circ(x), \quad f^\phi_y \in \partial \phi^\circ(y).$$

Proof. For any given $x, y \in X$ and $0 < \lambda < 1$, we have from the Mean Value Theorem that

$$\phi(x) - \phi(\lambda x + (1 - \lambda) y) = f^\phi_x + \xi(1 - \lambda)(y - x)(1 - \lambda)(x - y)$$

for some $0 < \xi < 1$ and $f^\phi_x + \xi(1 - \lambda)(y - x) \in \partial \phi^\circ(x + \xi(1 - \lambda)(y - x))$. But also

$$\phi(y) - \phi(\lambda x + (1 - \lambda) y) = f^\phi_y + \xi'(1 - \lambda)(x - y)$$

for some $0 < \xi' < 1$ and $f^\phi_y + \xi'(x - y) \in \partial \phi^\circ(y + \xi'(x - y))$. If $x \to \partial \phi^\circ(x)$ is monotone,

$$(f^\phi_y + \xi'(x - y) - f^\phi_x + \xi(1 - \lambda)(y - x))(1 - \xi' \lambda - \xi(1 - \lambda))(y - x) \geq 0.$$ 

But $0 < \xi' \lambda + \xi(1 - \lambda) < 1$ and so

$$\lambda \phi(x) - \lambda \phi(\lambda x + (1 - \lambda) y) + (1 - \lambda) \phi(y) - (1 - \lambda) \phi(\lambda x + (1 - \lambda) y) \geq 0;$$

that is,

$$\phi(\lambda x + (1 - \lambda) y) \leq \lambda \phi(x) + (1 - \lambda) \phi(y).$$

However, we should notice that for certain distance functions, convexity follows if a restricted monotonicity property holds.

THEOREM 23. Consider a non-empty closed set $K$ in a rotund normed linear space $X$ where to each point $x \in X \setminus K$ there exists a closest point $p(x)$ in $K$. If the norm is Gateaux differentiable on $X \setminus \{0\}$ and for each $x \in X \setminus K$ and $x_\lambda \equiv x + \lambda(x - p(x))$, where $\lambda > 0$,

$$(f^\phi_x - p(x))(x - p(x)) \geq 0,$$

then the distance function $\phi$ generated by $K$ is convex.

Proof. If the restricted monotonicity property holds then

$$f^\phi_x - p(x)(x - p(x)) = \|x - p(x)\|.$$
Since the norm is Gâteaux differentiable at $x - p(x)$ then

$$f_{x - p(x)} = f_x - p(x)$$

and since $X$ is rotund $p(x_l) = p(x)$; that is, $K$ is a sun. It is not difficult to show that in a normed linear space $X$ with norm Gâteaux differentiable on $X\setminus\{0\}$, a sun is convex. It follows that $\phi$ is convex.

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