Matching 2-lattice polyhedra: finding a maximum vector

Shiow-yun Chang\textsuperscript{a}, Donna C. Llewellyn\textsuperscript{b}, John H. Vande Vate\textsuperscript{b,}\textsuperscript{*}

\textsuperscript{a}Department of Industrial Management Science, National Cheng Kung University, Tainan 70101, Taiwan, Republic of China

\textsuperscript{b}ISyE, School of Industrial and Systems Engineering, Georgia Institute of Technology, Groseclose Bldg., Room 433, Atlanta, GA 30332-0205, USA

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Abstract

Matching 2-lattice polyhedra are a special class of lattice polyhedra that include network flow polyhedra, fractional matching polyhedra, matroid intersection polyhedra, etc. In this paper we develop a polynomial-time extreme point algorithm for finding a maximum cardinality vector in a matching 2-lattice polyhedron. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Although there are a number of polynomial algorithms for finding a maximum cardinality matching in a representable matroid \cite{14,15,4}, there can be no efficient algorithm for the problem in general matroids that relies on an oracle to determine ranks \cite{14,9}. The distinction in the tractability of these two problems does not carry over to their natural relaxations via 2-lattice polyhedra. This paper presents an efficient extreme point algorithm for finding a vector with maximum sum of components in a 2-lattice relaxation of the general matroid matching problem.

A companion paper \cite{2} explored duality relationships for the problem of finding a vector in a 2-lattice polyhedron with maximum sum of components. This problem generalizes the problems of finding a maximum cardinality matching in a bipartite graph, finding a maximum cardinality intersection in two polymatroids, and other related problems. In particular, \cite{2} characterized a minimum capacity cover, called the dominant cover, of a matching 2-lattice polyhedron in terms of the collection of all maximum cardinality vectors. This characterization is at the heart of our algorithm for finding

\textsuperscript{*}Corresponding author.

E-mail address: john.vandevate@isye.gatech.edu (J.H. Vande Vate).
a maximum cardinality vector in a matching 2-lattice polyhedron. Chang et al. [2] also showed that the problem of determining whether a given half-integral vector is an extreme point of a matching 2-lattice polyhedron is equivalent to finding a maximum word in a greedoid on a possibly infinite alphabet.

This paper develops these results into an efficient extreme point algorithm for finding a maximum cardinality vector in a matching 2-lattice polyhedron. This algorithm generalizes augmenting path algorithms for finding a maximum cardinality intersection in two matroids, although the possibility of half-integral components makes it more complicated. It also provides an extreme point method for finding a maximum cardinality vector in a fractional matching polytope.

Section 2 gives notation and preliminaries. Sections 3–5 describe our algorithm. The algorithm relies on the structure of extreme points and, in the appendix, we show how to ensure that we maintain an extreme point at each iteration. Section 6 describes the computational effort required.

2. Preliminaries

Let \( L \) be a finite set of elements (called lines) and let \( \Gamma \) be a finite lattice with partial order \((\Gamma, \leq)\), which induces meet operation \( \wedge \) and join operation \( \vee \). Let \( \beta : \Gamma \to \mathbb{Z} \) be submodular and, for each element \( \ell \in L \), let \( \alpha_\ell : \Gamma \to \mathbb{Z} \) be supermodular. Given \( S \in \Gamma \) and \( x \in \mathbb{R}^{|L|}_+ \), let \( \alpha(S)x = \sum(\alpha_\ell(S)x(\ell) : \ell \in L) \). Then

\[
\{ x \in \mathbb{R}^{|L|}_+ : \alpha(S)x \leq \beta(S) \text{ for each } S \in \Gamma \}
\]

is a lattice polyhedron. Lattice polyhedra were introduced by Hoffman and Schwartz [8] and independently by Johnson [10], and further studied by Hoffman [6], and Gröfll and Hoffman [5] (we use the term ‘lattice polyhedron’ somewhat differently than its coiners, who further restrict \( x \) to be \( 0, \pm 1 \) valued).

Here we consider those lattice polyhedra in which we allow \( \Gamma \) to be infinite, but require a finite bound on the length of chains in \( \Gamma \). This ensures that \( \Gamma \) is a complete lattice and includes, for example, the lattice of linear subspaces of a finite-dimensional vector space. We further require that \( \beta : \Gamma \to \mathbb{Z}_+ \) and for each \( \ell \in L \), \( \alpha_\ell \) is not only supermodular, but also non-decreasing and maps \( \Gamma \) onto \( \{0, 1, 2\} \). The set

\[
P(\alpha, \beta) = \{ x \in \mathbb{R}^{|L|}_+ : \alpha(S)x \leq \beta(S) \text{ for each } S \in \Gamma \}
\]

is called a 2-lattice polyhedron and each vector \( x \in P(\alpha, \beta) \) is called a 2-lattice vector. Examples of 2-lattice polyhedra include bipartite matching polyhedra [7,12], the intersection of two integral polymatroids [3], and the perfectly matchable subgraph polytope of a bipartite graph [1].

Classic examples of 2-lattice polyhedra relate \( \alpha \) and \( \beta \) in some way. We capture these relationships with the following general conditions. First, let \( \delta \) be a (possibly
We also extend the range of \( f_{ESC} \), \( f_{FF} \) while \( f_{VT} \) and let \( f_{VT} \) be a finite subset of \( 2^\mathcal{E} \) (generally chosen to be a collection of pairs from \( \mathcal{E} \)). We also require that \( \Gamma \) include \( \mathcal{E} \), the empty set \( \emptyset \) and be closed under intersections. In fact, we require that the partial order \( (\Gamma, \leq) \) simply be \( (\Gamma, \subseteq) \) so that for each pair \( S \) and \( T \) of members of \( \Gamma \), \( S \cap T = S \cap T \) and the smallest element of \( \Gamma \) is the empty set. Note that this does not imply that for each pair \( S \) and \( T \) of members in \( \Gamma \), \( S \cup T = S \cup T \). The lattice of subspaces of a finite-dimensional linear space is, for example, ordered by inclusion and has meet defined by intersection, but is not generally even closed under unions.

We associate with each set \( S \subseteq \mathcal{E} \) the smallest member, \( \sigma(S) \), of \( \Gamma \) containing \( S \), which we refer to as the span of \( S \). We further require that \( \beta : \Gamma \to \mathbb{Z}_+ \) be normalized, i.e., \( \beta(\emptyset) = 0 \), increasing and satisfy \( \beta(\{e\}) = 1 \) for each \( e \in \mathcal{E} \), and \( \beta(\sigma(\ell)) = 2 \) for each \( \ell \in L \). Thus, \( \Gamma \) and \( \beta \) define a (possibly infinite) matroid on \( \mathcal{E} \). We employ many of the concepts and terminology of matroids in this setting. In particular, we refer to a finite subset \( S \subseteq \mathcal{E} \) such that \( \beta(\sigma(S)) = |S| \) as an independent set and as a base of \( \sigma(S) \). We refer to a minimal dependent subset \( S \subseteq \mathcal{E} \) as a circuit. A subset \( S \subseteq \mathcal{E} \) with \( \beta(\sigma(S)) = |S| - 1 \) contains a unique circuit.

We model the relationship between \( \alpha \) and \( \beta \) via the condition \( \alpha_{\ell}(S) = \beta(\sigma(\ell) \cap S) \) for each \( \ell \in L \) and \( S \in \Gamma \). We often refer to \( \alpha_{\ell}(S) \) as the number of points of \( \ell \) in \( S \). It is easy to see that \( \alpha_{\ell} \) is normalized and non-decreasing. It is also straightforward to prove (see [16]) that \( \alpha_{\ell} \) is supermodular. We call such 2-lattice polyhedra, denoted by \( P(\mathcal{E}, \Gamma, \beta, L) \), matching 2-lattice polyhedra.

To avoid awkward notation, we extend the meet and join operations of \( \Gamma \) to all subsets of \( \mathcal{E} \) so that for \( S \) and \( T \subseteq \mathcal{E} \), \( S \cap T = \sigma(S) \cap \sigma(T) \) and \( S \cup T = \sigma(S) \cup \sigma(T) \). We also extend the range of \( \alpha_{\ell} \) and \( \beta \) to \( 2^\mathcal{E} \) as follows. For \( S \subseteq \mathcal{E} \), let \( \beta(S) = \beta(\sigma(S)) \) and let \( \alpha_{\ell}(S) = \beta(S \cap \ell) \). We must exercise some care in employing this extension: while \( \alpha_{\ell} : \Gamma \to \{0, 1, 2\} \) is supermodular, its extension to \( 2^\mathcal{E} \) may not be. Likewise, while \( \beta : \Gamma \to \mathbb{Z}_+ \) is increasing, its extension to \( 2^\mathcal{E} \) may only be non-decreasing.

Given \( S \) and \( T \) in \( \Gamma \), \( \beta(S/T) \) is defined by

\[
\beta(S/T) = \beta(S \vee T) - \beta(T).
\]

Let \( x \) be a vector in \( \mathbb{R}^{|L|}_+ \). We denote the support of \( x \) by \( \text{supp}(x) = \{ \ell \in L : x(\ell) > 0 \} \). For each scalar \( \lambda \in \mathbb{R} \), we let \( L_{\lambda}(x) = \{ \ell \in L : x(\ell) = \lambda \} \) and for each \( S \subseteq L \), we define \( x_S \in \mathbb{R}^{|L|}_+ \) by

\[
x_S(\ell) = \begin{cases} x(\ell) & \text{if } \ell \in S, \\ 0 & \text{otherwise}. \end{cases}
\]

Let \( 1 \in \mathbb{R}^{|L|}_+ \) be the vector of all ones. We denote the unit vector that is the characteristic vector of \( \ell \in L \) by \( 1_{\ell} \), rather than the more cumbersome \( 1_{(\ell)} \).

We refer to the members of \( \Gamma \) as flats and we denote by \( \Gamma(x) = \{ S \in \Gamma : \alpha(S)x = \beta(S) \} \) the collection of flats tight with respect to the 2-lattice vector \( x \). The following lemma shows that \( \Gamma(x) \) is a sublattice of \( \Gamma \).
Lemma 2.1 (Chang et al. [2]). Let $x$ be a 2-lattice vector and suppose $S$ and $T$ are in $\Gamma(x)$. Then

1. $S \lor T \in \Gamma(x)$,
2. $S \land T \in \Gamma(x)$, and
3. $\alpha(S) + \alpha(T) = \alpha(S \lor T) + \alpha(S \land T)$ for each $\ell \in \text{supp}(x)$.

Since $\Gamma(x)$ is a sublattice of a complete lattice, it has a largest member, which we denote by $\text{cl}(x)$.

The following lemma is an immediate consequence of Lemma 2.1 and will prove useful in arguing that certain vectors $x \in \mathbb{R}_{\geq 0}^{|L|}$ are 2-lattice vectors.

Lemma 2.2 (Chang et al. [2]). Let $x \in \mathbb{R}_{\geq 0}^{|L|}$ and suppose $S$ and $T$ are flats such that

$\alpha(S)x > \beta(S)$

and

$\alpha(T)x = \beta(T)$.

Then either $\alpha(S \land T)x > \beta(S \land T)$ or $\alpha(S \lor T)x > \beta(S \lor T)$.

Theorem 2.3 is a very slight generalization of Theorem 4.2 in [16] and provides a mechanism for describing extreme 2-lattice vectors in terms of perfect fractional matchings of graphs. Given a graph $G = (V, E)$ and an integral vector $b \in \mathbb{R}_{\geq 0}^{|V|}$, the perfect fractional $b$-matching polytope of $G$, denoted $\text{FP}(G, b)$, is

$$
\left\{ x \in \mathbb{R}_{\geq 0}^{|E|} : \sum (d_e(v)x(e) : e \in E) = b(v) \text{ for each } v \in V \right\}.
$$

Here, $d_e(v)$ is the degree of edge $e$ at node $v$. As the graph $G$ may have loops, $d_e(v) \in \{0, 1, 2\}$ and as the graph $G$ may have spurs (i.e., edges with only one end), $\sum (d_e(v) : v \in V) \in \{1, 2\}$. Letting $D$ be the $|V| \times |E|$ matrix with elements $d_e(v)$, $\text{FP}(G, b)$ may be written as:

$$
\text{FP}(G, b) = \{ x \in \mathbb{R}_{\geq 0}^{|E|} : Dx = b \}.
$$

Each vector $x \in \text{FP}(G, b)$ is a perfect fractional $b$-matching (or, more briefly, a fractional matching) of $G$.

A subset $T$ of edges in a graph $G$ is a bloom if the subgraph induced by the edges in $T$ is connected, contains exactly one cycle and that cycle has an odd number of edges. A subset $T$ of columns is a base of the node-edge incidence matrix of a graph $G$ if and only if the corresponding set of edges is a maximal set with the property that each component of the subgraph $(V, T)$ is either a tree or a bloom. (If $G$ has spurs, we add a distinguished node, called the root, incident to each spur edge. In this case, the component containing the root must be a tree.) When a set of columns is a base of the node-edge incidence matrix of a graph $G$, we also refer to the corresponding
set of edges as a base of $G$. When $E$ is a base of the graph $G = (V,E)$, we refer to $G$ as a basis graph.

Gröflin and Hoffman [5] showed that each extreme 2-lattice vector $x^*$ is defined by a subset $N$ of $L$ and a family $\mathcal{F} = \{S_i : i \in [1, \ldots, t]\}$ of flats with $S_1 \prec S_2 \prec \cdots \prec S_t$. The pair $(\mathcal{F}, N)$ induces a graph, denoted $G(\mathcal{F}, L \setminus N)$, defined as follows. For each $S_i \in \mathcal{F}$, there is a node $S_i$ in $G(\mathcal{F}, L \setminus N)$ and for each line $\ell \in L \setminus N$ there is an edge $\ell$ in $G(\mathcal{F}, L \setminus N)$. The edge $\ell$ is incident to node $S_i$ if $x(\ell) > x(S_{i-1}) = 1$ and is a loop at node $S_i$ if $x(\ell) = x(S_i) - x(S_{i-1}) = 2$. In the case of matching 2-lattice polyhedra, we let $S_0 = \emptyset$.

**Theorem 2.3** (Vande Vate [16]). A 2-lattice vector $x^*$ is extreme if and only if there is a subset $N$ of $L$ and a family $\mathcal{S} = \{S_i : i \in [1, \ldots, t]\}$ of flats with $S_1 \prec S_2 \prec \cdots \prec S_t$ such that

1. $x^*(\ell) = 0$ for each $\ell \in N$,
2. $G(\mathcal{F}, L \setminus N)$ is a basis graph,
3. The projection of $x^*$ onto the components indexed by lines in $L \setminus N$ is the unique, perfect fractional $b$-matching in $G(\mathcal{F}, L \setminus N)$, where $b(S_i) = \beta(S_i) - \beta(S_{i-1})$ for each $i \in [1, \ldots, t]$.

**Corollary 2.4** (Chang et al. [2]). Each extreme 2-lattice vector is half-integral.

A pair $(S, T)$ of possibly identical flats such that

$$z_\ell(S) + z_\ell(T) \geq 2 \quad \text{for each } \ell \in L$$

is a cover of $P(\mathcal{F}, \Gamma, \beta, L)$ and the capacity of a cover $(S, T)$, denoted $\beta(S, T)$, is

$$[\beta(S) + \beta(T)]/2.$$ 

Although matching 2-lattice vectors need not be integral, we refer to $\sum_{\ell \in L} x(\ell)$ as the cardinality of the matching 2-lattice vector $x$. In [2] we proved that the maximum cardinality of a matching 2-lattice vector is equal to the minimum capacity of a cover. Corollary 2.5 is an immediate consequence of this min–max relation.

**Corollary 2.5** (Chang et al. [2]). For each minimum cover $(S, T)$ and maximum 2-lattice vector $x$,

- $z(S)x = \beta(S)$
- $z(T)x = \beta(T)$ and,
- if $z_\ell(S) + z_\ell(T) > 2$, then $x(\ell) = 0$.

In [2], we also showed that the set of all minimum capacity covers forms an upper semi-lattice and so has a largest member $(S^*, T^*)$ such that $S^* \preceq S$ and $T \preceq T^*$ for every minimum cover $(S, T)$. We refer to this cover $(S^*, T^*)$ as the dominant cover. Let $\Omega$ denote the collection of all maximum cardinality matching 2-lattice vectors and
let $\Omega_{\text{ext}}$ denote the collection of all maximum cardinality extreme matching 2-lattice vectors. Theorem 2.6 characterizes the dominant cover in terms of $\Omega$.

Given a flat $T$, we denote by $L(T)$ the collection $\{\ell \in L : \alpha_\ell(T) = 1\}$.

**Theorem 2.6 (Chang et al. [2]).** Let $T^* = \cap (\text{cl}(x) : x \in \Omega) = \cap (\text{cl}(x) : x \in \Omega_{\text{ext}})$ and $S^* = \sigma(\{\ell \land T^* : \ell \in L(T^*)\})$. Then $(S^*, T^*)$ is the dominant cover of the matching 2-lattice polyhedron $P(\emptyset, \Gamma, \beta, L)$.

Characterizing the dominant cover in terms of extreme matching 2-lattice vectors is useful in that we can determine the closure of these matching 2-lattice vectors.

**Corollary 2.7 (Chang et al. [2]).** If $x \in \Omega_{\text{ext}}$ and $(S^*, T^*)$ is the dominant cover of the matching 2-lattice polyhedron $P(\emptyset, \Gamma, \beta, L)$, then $\text{cl}(x) = \sigma(\text{supp}(x)) = \sigma(L(1)(x) \lor T^*)$.

The following lemma is useful in proving that certain vectors are feasible.

**Lemma 2.8 (Chang et al. [2]).** Let $x$ and $\tilde{x}$ be matching 2-lattice vectors and let $(S, T)$ be the dominant cover. If $x$ satisfies

1. $\alpha(T)x_{L \setminus L(T)} = \beta(T/S)$ and
2. for $T' \subseteq T$, $\alpha(T')x_{L \setminus L(T)} \leq \beta(T'/S)$,
\[ \tilde{x} \text{ satisfies} \]
3. $\alpha(T)\tilde{x}_{L(T)} = \beta(S)$,
4. for $T' \subseteq T$, $\alpha(T')\tilde{x}_{L(T)} \leq \beta(T' \land S)$, and
\[ \text{and} \]
(a) $\beta(T/\text{cl}(\tilde{x}_{L(T)})) = \beta(T/S)$,
(b) $\alpha(\text{cl}(\tilde{x}_{L(T)}))x_{L \setminus L(T)} = 0$ and
(c) $\text{supp}(\tilde{x}_{L(T)}) \subseteq \text{cl}(\tilde{x}_{L(T)})$,
then $x' = \tilde{x}_{L(T)} + x_{L \setminus L(T)}$ is a matching 2-lattice vector.

In [2], we showed that the problem of recognizing whether a given vector is an extreme matching 2-lattice vector is equivalent to finding a maximum length word in a greedoid.

A language $A$ over finite ground set of letters, called the alphabet, is a collection of finite sequences of letters, called words. We distinguish the sequence $w = [e_1, e_2, \ldots, e_t]$ from the set $\hat{w} = \{e_1, e_2, \ldots, e_t\}$ and denote the concatenation of two sequences by $[e_1, e_2, \ldots, e_t][e_{i+1}, e_{i+2}, \ldots, e_t] = [e_1, e_2, \ldots, e_t]$. The length of a word $w$, denoted $|w|$, is the number of letters in the sequence. A sequence $w$ is simple if no letter is repeated, i.e., if $|w| = |\hat{w}|$, and a language is simple if all its words are simple sequences. A language $A$ such that

1. the empty set is in $A$, and
2. if $w = a || b$ is in $A$, then $a$ is in $A$ is said to be hereditary. A simple hereditary language $A$ such that
3. if \( w \) and \( w' \) are in \( \mathcal{FT} \) and \( w \) is longer than \( w' \), then there is a letter \( e \) in \( w \) such that \( w' || e \) is in \( \mathcal{FT} \).

is a greedoid. Korte and Lovasz introduced and explored greedoids in a series of papers including [11]. We relax the restriction that the alphabet be finite and instead require that there be a finite bound on the length of words in the language.

We say that the family \( \mathcal{S} = \{ S_i : i \in [1, \ldots, t] \} \) of flats is canonical with respect to \( x \in \mathbb{R}^{[L]} \) if there is a sequence \( I = [e_1, e_2, \ldots, e_t] \) of elements in \( \mathcal{F} \) such that

1. \( S_1 = L_1(x) \vee \{ e_1 \} \) is in \( \Gamma(x) \),
2. \( S_i = S_{i-1} \vee \{ e_i \} \) is in \( \Gamma(x) \) for \( i = 2, 3, \ldots, t \), and
3. \( G(\mathcal{S}, L_{1/2}(x)) \) is a collection of node-disjoint odd cycles.

Chang et al. [2] showed that it is enough to consider canonical families when characterizing extreme points of matching 2-lattice polyhedra.

**Theorem 2.9** (Chang et al. [2]). A vector \( x^* \in \{ 0, \frac{1}{2}, 1 \}^{[L]} \) is an extreme point of the matching 2-lattice polyhedron \( P(\mathcal{F}, \Gamma, \beta, L) \) if and only if \( \sigma(L_1(x^*)) \in \Gamma(x^*) \) and there is a canonical family \( \mathcal{F} \) with respect to \( x^* \).

This characterization leads to the recognition that each extreme point of \( P(\mathcal{F}, \Gamma, \beta, L) \) induces a greedoid ensuring that we may identify the flats of a canonical family one at a time.

For each extreme point \( x \), we construct a hereditary language \( \mathcal{L}(x) \) on \( \mathcal{F} \) representing the sequences of elements that can be extended to define a nested family \( \mathcal{F} \) canonical with respect to \( x \). With each simple sequence \( w = [e_1, e_2, \ldots] \) of elements, we associate the collection \( \mathcal{F}(w) = \{ S_1, S_2, \ldots \} \) of flats, where \( S_1 = L_1(x) \vee \{ e_1 \} \), and, for \( i = 2, 3, \ldots, S_i = S_{i-1} \vee \{ e_i \} \). The sequence \( w \) is a word in \( \mathcal{L}(x) \) if

1. each flat \( S_j \in \mathcal{F}(w) \) is in \( \Gamma(x) \), and
2. the vectors \( \{ \sigma(S_j) : S_j \in \mathcal{F}(w) \} \cup \{ 1_\ell : \ell \in L_0(x) \} \) are linearly independent.

**Lemma 2.10** (Chang et al. [2]). If \( x \) is an extreme point of \( P(\mathcal{F}, \Gamma, \beta, L) \), then \( (\mathcal{F}, \mathcal{L}(x)) \) is a greedoid.

The ideas behind this algorithm are analogous to those used in Edmonds’ non-bipartite matching algorithm. In the case of graphic matchings, a minimum cover of a subset of edges reduces the problem of finding a larger matching to a bipartite matching problem. In each iteration, the solution of the bipartite matching problem leads to either a larger matching or a minimum cover for a larger set of edges. In our algorithm for finding a maximum cardinality 2-lattice matching, the dominant cover of a subset of lines reduces the problem of finding a larger 2-lattice matching to solving matroid intersection problems. In each iteration, the solution of the matroid intersection problems leads to either a larger 2-lattice matching or the dominant cover for a larger set of lines.
3. Maximum 2-lattice matching algorithm

Consider a subset $L' \subset L$ of the lines. A vector $x \in P(\delta, \Gamma, \beta, L)$ is called a 2-lattice matching of $L'$ if $\text{supp}(x) \subseteq L'$. We denote the maximum cardinality of a 2-lattice matching of $L'$ by $\Omega(L')$, the collection of all maximum 2-lattice matchings of $L'$ by $\Omega(L')$ and the collection of all extreme points in $\Omega(L')$ by $\Omega_{\text{ext}}(L')$. We refer to a pair $(S, T)$ of (possibly identical) members of $\Gamma$ such that

$$\alpha_{\ell}(S) + \alpha_{\ell}(T) \geq 2 \quad \text{for each } \ell \in L'$$

as a cover of $L'$. The results of [2] characterize the dominant cover $(S^*, T^*)$ of $L'$.

Given a solution $x \in \Omega_{\text{ext}}(L')$ and the dominant cover $(S^*, T^*)$ for a subset $L' \subset L$ of lines, we consider a new line $\ell^* \in L \setminus L'$ and show how to find either a larger solution $x^*$ for $L' \cup \{\ell^*\}$ or the dominant cover of $L' \cup \{\ell^*\}$ proving that no larger solution exists. We proceed in this way, adding one line at a time, until we have a maximum 2-lattice matching and the dominant cover for $L$.

The most direct method for constructing a maximum 2-lattice matching in $L' \cup \{\ell^*\}$ is to find a solution $x' \in \Omega(L')$ such that $x' + 1_{\ell^*} \in \Omega(L' \cup \{\ell^*\})$. In fact, if $v(L' \cup \{\ell^*\}) = v(L') + 1$, this is the only method. If $v(L' \cup \{\ell^*\}) = v(L') + \frac{1}{2}$, however, there are other possibilities to consider. For example, there may be no $x' \in \Omega(L')$ such that $x' + \frac{1}{2}1_{\ell^*}$ is a maximum 2-lattice matching in $L' \cup \{\ell^*\}$. In this case, each maximum 2-lattice matching in $L' \cup \{\ell^*\}$ must be of the form $x' + 1_{\ell^*}$, where $x'$ is a 2-lattice matching of $L'$ with $\sum_{\ell \in L'} x'(\ell) = v(L') - \frac{1}{2}$.

We begin by considering the cases in which we can obtain a maximum 2-lattice matching $x' + \varepsilon 1_{\ell^*}$ in $L' \cup \{\ell^*\}$ from a maximum 2-lattice matching $x'$ in $L'$. Lemma 3.1 shows that this can occur only if there is $x' \in \Omega(L')$ such that $\ell^* \land \text{cl}(x') = \emptyset$.

**Lemma 3.1.** Given a line $\ell^*$ and a 2-lattice matching $x'$, there is $\varepsilon > 0$ such that $x' + \varepsilon 1_{\ell^*}$ is a 2-lattice matching if and only if $\ell^* \land \text{cl}(x') = \emptyset$.

Lemma 3.1 shows that if we can find $x' \in \Omega(L')$ such that $\ell^* \land \text{cl}(x') = \emptyset$, then we can construct a larger solution $x^* = x' + \varepsilon 1_{\ell^*}$ in $L' \cup \{\ell^*\}$. It does not guarantee, however, that we can construct a maximum 2-lattice matching of $L' \cup \{\ell^*\}$ in this way. For the moment, we ignore this difficulty and focus on the simpler problem of finding a larger 2-lattice matching in $L' \cup \{\ell^*\}$.

Another difficulty involved in applying Lemma 3.1 is that we do not know how to compute the closure of an arbitrary 2-lattice matching efficiently. We do, however, know that the closure of an extreme 2-lattice matching is simply the span of its support.

Now, the dominant cover $(S^*, T^*)$ of $L'$ is characterized by

$$T^* = \bigcap (\text{cl}(x) : x \in \Omega_{\text{ext}}(L')).$$

So, there can be $x \in \Omega_{\text{ext}}(L')$ such that $\ell^* \land \text{cl}(x) = \emptyset$ only if $\ell^* \land T^* = \emptyset$. The converse, however, need not hold. Namely, there may be no extreme solution $x$ in $L'$ such that $\ell^* \land \text{cl}(x) = \emptyset$ even if $\ell^* \land T^* = \emptyset$ (see Example 3.1).
Example 3.1. Let $\Gamma$ be the lattice of flats and let $\beta$ be the rank function of the cycle matroid of the graph in Fig. 1. Let $L' = \{\ell_1, \ell_2\}$ and $\ell^* = \ell_3$, where $\ell_1 = (e_1, \bar{e}_1)$, $\ell_2 = (e_2, \bar{e}_2)$, and $\ell_3 = (e_1, e_2)$. The extreme maximum 2-lattice matchings of $L'$ are $x^1 = (1, 0, 0)$ and $x^2 = (0, 1, 0)$. The dominant cover of $L'$ is $S^* = T^* = \{e_1, e_2\}$. Although $T^* \land \ell^* = \emptyset$, neither $\ell^* \land \cl(x^1)$ nor $\ell^* \land \cl(x^2)$ is empty.

Although there may be no extreme point $x \in \Omega_{\text{ext}}(L')$ such that $\ell^* \land \cl(x) = \emptyset$, there is an extreme point $x^1 \in \Omega_{\text{ext}}(L')$ such that $\ell^* \not\subseteq \cl(x^1)$. Similarly, there is an extreme point $x^2 \in \Omega_{\text{ext}}(L')$ such that $\ell^* \land \cl(x^2) \not= \emptyset$ and $\ell^* \land \cl(x^2) = \emptyset$ if and only if $\ell^* \land T^* = \emptyset$. This is, in fact, a stronger characterization of the dominant cover. Thus, we have proved the following.

Theorem 3.2. Let $(S^*, T^*)$ be the dominant cover of $L' \subset L$ and let $\ell^* \in L \setminus L'$. There is $x' \in \Omega(L')$ such that $\ell^* \land \cl(x') = \emptyset$ if and only if $\ell^* \land T^* = \emptyset$.

Since each extreme point of a 2-lattice polyhedron is half-integral, one immediate consequence of Theorem 3.2 is:

Corollary 3.3. Let $(S^*, T^*)$ be the dominant cover of $L' \subset L$ and suppose $\ell^* \in L \setminus L'$ satisfies $\ell^* \land T^* = \emptyset$. Then $v(L' \cup \{\ell^*\}) \leq v(L') + \frac{1}{2}$.

When $T^* \land \ell^* = \emptyset$ then, we know that $v(L' \cup \{\ell^*\}) \geq v(L') + \frac{1}{2}$, and so there are only two possibilities to consider: either $v(L' \cup \{\ell^*\}) = v(L') + \frac{1}{2}$ or $v(L' \cup \{\ell^*\}) = v(L') + 1$. To pin down the true value of $v(L' \cup \{\ell^*\})$ in this case, we consider separately the two possible values of $\beta(T^* \lor \ell^*)$.

If $\beta(T^* \lor \ell^*) = \beta(T^*) + 2$, we solve an induced matroid intersection problem to look for an extreme point $x' \in \Omega_{\text{ext}}(L')$ such that $\ell^* \land \cl(x') = \emptyset$. If the intersection problem identifies an extreme point $x' \in \Omega_{\text{ext}}(L')$ with this property, then, as we show in Lemma 4.3, $x' = x' + e_{\ell^*}$ is a 2-lattice matching of $L' \cup \{\ell^*\}$ proving that $v(L' \cup \{\ell^*\}) = v(L') + 1$. If we do not find an extreme point $x' \in \Omega_{\text{ext}}(L')$ with this property, then, as we show in Lemma 4.2, the minimum cover for the intersection problem leads to a minimum cover of $L' \cup \{\ell^*\}$ proving that $v(L' \cup \{\ell^*\}) = v(L') + \frac{1}{2}$. To find a maximum 2-lattice...
matching of $L' \cup \{\ell^*\}$ in this case, we look for $x' \in \Omega(L')$ such that $\ell^* \cap \text{cl}(x') = \emptyset$ by solving two induced matroid intersection problems. The first intersection problem finds an extreme point $x^1 \in \mathcal{E}_\text{ext}(L')$ such that $\ell^* \notin \text{cl}(x^1)$. The second intersection problem finds an extreme point $x^2 \in \mathcal{E}_\text{ext}(L')$ such that $\ell^* \cap \text{cl}(x^1) \notin \text{cl}(x^2)$. Now, $x' = x^1 + \frac{1}{2}x^2$ is a maximum 2-lattice matching of $L' \cup \{\ell^*\}$ such that $\ell^* \cap \text{cl}(x') = \emptyset$ and, as we show in Lemma 4.5, $x' + \frac{1}{2}1_{\ell^*}$ is a maximum 2-lattice matching of $L' \cup \{\ell^*\}$.

If $\beta(T^* \cup \ell^*) = \beta(T^*) + 1$, $(S^*, T^* \cup \ell^*)$ is a minimum cover of $L' \cup \{\ell^*\}$ proving that $v(L' \cup \{\ell^*\}) = v(L') + \frac{1}{2}$. In this case, we again find a maximum 2-lattice matching of $L' \cup \{\ell^*\}$ by solving two induced matroid intersection problems.

When $\ell^* \cap T^* \neq \emptyset$, on the other hand, we know that $\beta(T^* \cup \ell^*) = \beta(T^*) + 1$ and so $(S^*, T^* \cup \ell^*)$ is a cover of $L' \cup \{\ell^*\}$ proving that $v(L' \cup \{\ell^*\}) = v(L') + \frac{1}{2}$. So, in this case the two possibilities are: either $v(L' \cup \{\ell^*\}) = v(L') + \frac{1}{2}$ or $v(L' \cup \{\ell^*\}) = v(L')$. To pin down the true value of $v(L' \cup \{\ell^*\})$, we need only consider the case in which $\ell^*$ is not already covered by $(S^*, T^*)$, i.e., $\beta(\ell^* \cap T^*) = 1$ and $\beta(\ell^* \cap S^*) = 0$. In this case, since $T^* \cap \ell^* = \emptyset$, there is no $x' \in \Omega(L')$ such that $\ell^* \cap \text{cl}(x') = \emptyset$. So, to construct a larger 2-lattice matching of $L' \cup \{\ell^*\}$ we must find a 2-lattice matching $x'$ of $L'$ with cardinality equal to $v(L') - \frac{1}{2}$ such that $x' + 1_{\ell^*}$ is a 2-lattice matching. We begin by solving an induced matroid intersection problem to find $x \in \Omega(\text{ext}(L'))$ such that $\ell^* \notin \text{cl}(x)$. We then look for an augmentation in a basis graph of $x$. If there is an augmentation, it leads to a larger matching proving that $v(L' \cup \{\ell^*\}) = v(L') + \frac{1}{2}$. If there is no augmentation, we show how to construct the dominant cover of $L' \cup \{\ell^*\}$ proving that $v(L' \cup \{\ell^*\}) = v(L')$.

The following provides an overview of the algorithm. The major work takes place in Steps 3 and 4, where we solve the induced matroid intersection problems. We discuss the details of Step 3 in Section 4. The appendix discusses the problem of maintaining certain technical conditions and Section 5 discusses the details of Step 4.

**Algorithm:** MAXIMUM

**Input:** An extreme 2-lattice matching $x$ in $P(\ell, \Gamma, \beta, L)$.

**Output:** An extreme point $x \in \Omega(\text{ext}(L))$ and the dominant cover $(S^*, T^*)$ of $L$.

**Step 1:** Construct the dominant cover $(S^*, T^*)$ of $\text{supp}(x)$. In particular, $S^* = \emptyset$ and $T^* = \sigma(\text{supp}(x))$. Let $L' = \text{supp}(x)$.

**Step 2:** If $(S^*, T^*)$ covers each line in $L \setminus L'$; Stop — $x$ is in $\mathcal{E}_\text{ext}(L)$ and $(S^*, T^*)$ is the dominant cover of $L$. Otherwise, choose a line $\ell^*$ in $L \setminus L'$ that is not covered by $(S^*, T^*)$. If $\ell^* \cap T^* = \emptyset$, there is a larger 2-lattice matching in $L' \cup \{\ell^*\}$. Go to Step 3. Otherwise, go to Step 4.

**Step 3:** $\ell^* \cap T^* = \emptyset$:

- If $\beta(T^* \cup \ell^*) = \beta(T^*) + 2$, go to Step 3a. Otherwise, go to Step 3b.

**Step 3a:** Look for $x' \in \Omega(\text{ext}(L'))$ such that $\ell^* \cap \text{cl}(x') = \emptyset$. Solve the matroid intersection problem induced by $\ell^*$ and $(S^*, T^*)$ to determine whether there is a 2-lattice matching $x' \in \mathcal{E}_\text{ext}(L')$ such that $x' + 1_{\ell^*}$ is an extreme 2-lattice matching of $L' \cup \{\ell^*\}$ proving that $v(L' \cup \{\ell^*\}) = v(L') + 1$. If there is one, replace $x$ with $x' + 1_{\ell^*}$ and go to Step 1. Otherwise, $v(L' \cup \{\ell^*\}) = v(L') + \frac{1}{2}$. Go to Step 3b.
Step 3b: Look for \( x' \in \Omega(L') \) such that \( \ell^* \cap \text{cl}(x') = \emptyset \). Solve two matroid intersection problems induced by \((S^*, T^*)\) to find \( x' \in \Omega(L') \) such that \( \text{cl}(x') \cap \ell^* = \emptyset \). Then \( x' + \frac{1}{2} \mathbf{1}_{\ell^*} \) is a maximum 2-lattice matching of \( L' \cup \{ \ell^* \} \). Replace \( x \) with \( x' + \frac{1}{2} \mathbf{1}_{\ell^*} \) and go to Step 1.

Step 4: \( \ell^* \cap T^* \neq \emptyset \). Choose \( e \in \ell^* \) so that \( e \notin T^* \). Solve the matroid intersection problem induced by \( \{ e \} \) and \((S^*, T^*)\) to construct \( x' \in \Omega_{\text{ext}}(L') \) such that \( \ell^* \not\subseteq \text{cl}(x') \). Call Procedure \text{AUGMENT} to either find a larger solution \( x'' \) or construct the dominant cover of \( L' \cup \{ \ell^* \} \) proving there is no larger solution. If \text{AUGMENT} finds a larger solution, replace \( x \) with \( x'' \) and go to Step 1. Otherwise, go to Step 2.

4. When \( \ell^* \cap T^* = \emptyset \)

When \( \ell^* \cap T^* = \emptyset \), we showed that \( v(L' \cup \{ \ell^* \}) > v(L') \). In this section, we show how to find a larger 2-lattice matching in this case.

First, if \( \beta(T^* \cup \ell^*) = \beta(T^*) + 2 \), we look for \( x' \in \Omega_{\text{ext}}(L') \) such that \( \ell^* \cap \text{cl}(x') = \emptyset \). This reduces to the following matroid intersection problem.

Let \((S^*, T^*)\) be the dominant cover of \( L' \) and for each \( \ell \in L'(T^*) \), let \( t(\ell) \) be an element in \( \ell \cap T^* \). Define the matroid \( \mathcal{M}_1(L') \) with rank function \( r_1 \) on \( L'(T^*) \) as in [2], i.e., a set \( X \) of lines in \( L'(T^*) \) is independent in \( \mathcal{M}_1(L') \) if \( \beta(\{ t(\ell) : \ell \in X \}) = |X| \).

For each \( Z \subseteq S \), define the matroid \( \mathcal{M}_2(L', Z) \) with rank function \( r_2 \) on \( L'(T^*) \) as in [2], i.e., a set \( X \) of lines in \( L'(T^*) \) is independent in \( \mathcal{M}_2(L', Z) \) if \( \beta(X/T^* \cup Z) = |X| \).

Note that we use the less cumbersome, but potentially confusing notation \( \beta(X/T^* \cup Z) \) in place of the more correct \( \beta(\bigcup_{\ell \in X} \ell/T^* \cup Z) \).

Chang et al. [2] showed that, we can construct extreme 2-lattice vectors from \( \beta(S^*) \) intersections in \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \).

**Corollary 4.1** (Chang et al. [2]). Let \( x \in \Omega_{\text{ext}}(L') \), \((S^*, T^*)\) be the dominant cover of \( L' \) and \( e \in T^* \). If \( X \) is a \( \beta(S^*) \) intersection in \( \mathcal{M}_1(L') \) and \( \mathcal{M}_2(L', e) \), then \( x' \) defined by

\[
x'(\ell) = \begin{cases} 
1 & \text{if } \ell \in X, \\
0 & \text{if } \ell \in L(T^*) \setminus X, \\
x(\ell) & \text{otherwise}
\end{cases}
\]

is an extreme maximum matching 2-lattice vector with \( e \not\subseteq \text{cl}(x') \).

The following lemmas show that by solving a single induced matroid intersection problem we can determine whether \( v(L' \cup \{ \ell^* \}) \) is \( v(L') + 1 \) or \( v(L') + \frac{1}{2} \). In particular, Lemma 4.3 shows that each \( \beta(S^*) \)-intersection \( I \) in \( \mathcal{M}_1(L') \) and \( \mathcal{M}_2(L', \ell^*) \) corresponds to a 2-lattice matching of \( L' \cup \{ \ell^* \} \) with cardinality \( v(L') + 1 \). Lemma 4.2 shows that if there is no \( \beta(S^*) \)-intersection in these two matroids, then a minimum cover of the intersection problem leads to a minimum cover of \( L' \cup \{ \ell^* \} \) proving that \( v(L' \cup \{ \ell^* \}) = v(L') + \frac{1}{2} \).
Lemma 4.2. Let \((S^*, T^*)\) be the dominant cover of \(L'\) and suppose \(\ell^* \in L \setminus L'\) satisfies \(\beta(T^* \vee \ell^*) = \beta(T^*) + 2\). If the maximum cardinality of an intersection in \(\mathcal{M}_1(L')\) and \(\mathcal{M}_2(L', \ell^*)\) is less than \(\beta(S^*)\), then the maximum cardinality of a 2-lattice matching of \(L' \cup \{\ell^*\}\) is \(\nu(L') + \frac{1}{2}\).

Proof. We first show that if the maximum cardinality of an intersection in \(\mathcal{M}_1(L')\) and \(\mathcal{M}_2(L', \ell^*)\) is less than \(\beta(S^*)\), it must be \(\beta(S^*) - 1\). Let \(I\) be a maximum cardinality intersection and let \((X_1, X_2)\) be a minimum rank cover for the matroid intersection problem. Then

\[
r_1(X_1) + r_2(X_2) \leq |I|,
\]

that is,

\[
\beta(\{t(\ell) : \ell \in X_1\}) + \beta(X_2/(T^* \vee \ell^*)) \leq |I|,
\]

and so,

\[
\beta(\{t(\ell) : \ell \in X_1\}) + \beta(X_2 \vee T^* \vee \ell^*) \leq |I| + \beta(T^* \vee \ell^*).
\]

Since \(\beta(T^* \vee \ell^*) = \beta(T^*) + 2\),

\[
\beta(\{t(\ell) : \ell \in X_1\}) + \beta(X_2 \vee T^* \vee \ell^*) \leq |I| + \beta(T^*) + 2.
\]

Let \(S' = \sigma(\{t(\ell) : \ell \in X_1\})\) and \(T' = X_2 \vee T^* \vee \ell^*\). Clearly, \((S', T')\) is a cover of \(L' \cup \{\ell^*\}\). If \(|I| \leq \beta(S^*) - 2\), then \((S', T')\) is a minimum cover of \(L'\) with \(T^* \subseteq T'\); contradicting the assertion that \((S^*, T^*)\) is the dominant cover of \(L'\). It follows that the maximum cardinality of an intersection must be at least \(\beta(S^*) - 1\).

If \(|I| = \beta(S^*) - 1\), then \(\beta(S', T') = \beta(S^*, T^*) + \frac{1}{2}\) proving that the maximum cardinality of a 2-lattice matching of \(L' \cup \{\ell^*\}\) is at most \(\nu(L') + \frac{1}{2}\). By Corollary 4.3, the maximum cardinality of a 2-lattice matching in \(L' \cup \{\ell^*\}\) is exactly \(\nu(L') + \frac{1}{2}\). □

Lemma 4.3 shows that when \(\beta(T^* \vee \ell^*) = \beta(T^*) + 2\), each \(\beta(S^*)\)-intersection in \(\mathcal{M}_1(L')\) and \(\mathcal{M}_2(L', \ell^*)\) induces a 2-lattice matching \(x' \in \Omega_{\text{ext}}(L')\) such that \(\ell^* \wedge \text{cl}(x') = \emptyset\) and, in fact, \(x' + 1_{\ell^*}\) is an extreme 2-lattice matching of \(L' \cup \{\ell^*\}\).

Lemma 4.3. Given \(x \in \Omega_{\text{ext}}(L')\) and the dominant cover \((S^*, T^*)\) of \(L'\), suppose \(\ell^* \in L\) satisfies \(\beta(T^* \vee \ell^*) = \beta(T^*) + 2\). Then, for each \(\beta(S^*)\)-intersection \(I\) in \(\mathcal{M}_1(L')\) and \(\mathcal{M}_2(L', \ell^*)\), \(x'\) defined by

\[
x'(\ell) = \begin{cases} 1 & \text{if } \ell \in I, \\ 0 & \text{if } \ell \in L'(T^*) \setminus I, \\ x(\ell) & \text{otherwise} \end{cases}
\]

is in \(\Omega_{\text{ext}}(L')\) and \(x^* = x' + 1_{\ell^*}\) is in \(\Omega_{\text{ext}}(L' \cup \{\ell^*\})\).

Proof. Consider a \(\beta(S^*)\)-intersection \(I\) in \(\mathcal{M}_1(L')\) and \(\mathcal{M}_2(L', \ell^*)\). By Corollary 4.1, \(x' \in \Omega_{\text{ext}}(L')\). We show that \(x^*\) is a 2-lattice matching as follows. Since \(I\) is independent
Thus, by Lemma 2.8, \( \beta(I \cup T^* \cup \ell^*) = |I| = \beta(S^*) \). It follows that

\[
\beta(I \cup T^* \cup \ell^*) = \beta(S^*) + \beta(T^* \cup \ell^*).
\]

Let \( \{t(\ell) : \ell \in I\} \cup D \) be a base of \( T^* \), then \( \beta(D) = \beta(T^*) - \beta(S^*) \) and

\[
\beta((I \cup \ell^*)/D) = \beta(I \cup \ell^*, D) - \beta(D) = \beta(S^*) + \beta(T^* \cup \ell^*) - \beta(D) = 2\beta(S^*) + 2 = 2|I \cup \{\ell^*\}|.
\]

Hence, by the submodularity of \( \beta \), \( \beta(I \cup \ell^*) = 2|I \cup \{\ell^*\}| \). Therefore, \( z = 1_{I \cup \{\ell^*\}} \) is a 2-lattice matching. We further show that \( z \) satisfies the following properties:

1. \( \alpha(T^*)z = \beta(S^*) \),
2. for \( T' \subseteq T^* \), \( \alpha(T')z \leq \beta(T' \land S^*) \),
3. \( \text{cl}(z) = \sigma(\text{supp}(z)) \), and
4. \( \beta(T^*/\text{cl}(z)) = \beta(T^*/S^*) \).

To see (1), observe that \( \alpha(T^*)z = \sum_{\ell \in I \cup \{\ell^*\}} x_\ell(T^*) = |I| = \beta(S^*) \).

To see (2), observe that for \( T' \subseteq T^* \),

\[
\alpha(T')z = \alpha(T' \land \{t(\ell) : \ell \in I\})z = \alpha(T' \land S^*)z \leq \beta(T' \land S^*).
\]

To see (3), observe that \( z \) is an extreme 2-lattice matching.

To see (4), observe that

\[
\beta(T^*/\text{cl}(z)) = \beta(T^*(I \lor \ell^*)) = |D| = \beta(T^*/S^*).
\]

Finally, we see that \( \alpha(\text{cl}(z))x_{L \setminus L(T^*)} = 0 \) as follows. Since \( \text{supp}(x_{L \setminus L(T^*)}) \subseteq T^* \),

\[
\alpha(\text{cl}(z))x_{L \setminus L(T^*)} = \alpha((I \lor \ell^*) \land T^*)x_{L \setminus L(T^*)}.
\]

But \( (I \lor \ell^*) \land T^* = S^* \), so

\[
\alpha(\text{cl}(z))x_{L \setminus L(T^*)} = \alpha(S^*)x_{L \setminus L(T^*)} = 0.
\]

Thus, by Lemma 2.8, \( x^* = z + x_{L \setminus L(T^*)} \) is a 2-lattice matching.

To see that \( x^* \) is a maximum 2-lattice of \( L' \cup \{\ell^*\} \), observe that

\[
v(L' \cup \{\ell^*\}) \geq \sum_{\ell \in L' \cup \{\ell^*\}} x^*(\ell) = \sum_{\ell \in I \cup \{\ell^*\}} x^*(\ell) + \sum_{\ell \in L' \setminus L(T^*)} x(\ell) = \beta(S^*) + 1 + \sum_{\ell \in L'} x(\ell) - \sum_{\ell \in L(T^*)} x(\ell)
\]

\[ = \beta(S^*) + 1 + \beta(S^*, T^*) - \sum_{\ell \in L(T^*)} x(\ell) \]
\[ \geq \beta(S^*) + 1 + \beta(S^*, T^*) - \beta(S^*) \]
\[ = \beta(S^*, T^*) + 1 \]
\[ \geq v(L' \cup \{ \ell^* \}). \]

Since \(x^*\) is extreme, it follows that \(x^*\) is extreme as well. \(\Box\)

When \(\ell^* \cap T^* = \emptyset\), but \(v(L' \cup \{ \ell^* \}) = v(L') + \frac{1}{2}\), there may not be an extreme maximum 2-lattice matching \(x\) of \(L'\) such that \(\ell^* \cap \text{cl}(x) = \emptyset\). Thus, we extend our search to the set of all maximum 2-lattice matchings of \(L'\). Fortunately, as we indicated in Section 3, it is enough to consider only those maximum 2-lattice matchings that can be expressed as a convex combination of two extreme maximum 2-lattice matchings. Thus, we can construct \(x^*\) by solving at most two matroid intersection problems.

Although the fact that \(\ell^* \cap \text{cl}(x^*) = \emptyset\) does guarantee we can construct a larger 2-lattice matching \(x^* + \varepsilon 1_{\ell^*}\) of \(L' \cup \{ \ell^* \}\), it does not guarantee we can construct a maximum 2-lattice matching of \(L' \cup \{ \ell^* \}\) in this way. Lemma 4.4 establishes conditions under which \(x^* + \frac{1}{2} 1_{\ell^*}\) is a 2-lattice matching.

**Lemma 4.4.** Let \(\ell^*\) be a line in \(L\) and let \(x^1\) and \(x^2\) be half-integral 2-lattice matchings such that
1. \(\ell^* \not\subset \text{cl}(x^i) = \sigma(\text{supp}(x^i))\) for \(i = 1, 2\),
2. \(\ell^* \cap \text{cl}(x^1) \cap \text{cl}(x^2) = \emptyset\), and
3. \(L_{1/2}(x^1) = L_{1/2}(x^2)\).

Then, \(\frac{1}{2}(x^1 + x^2 + 1_{\ell^*})\) is a 2-lattice matching.

**Proof.** Let \(x^* = \frac{1}{2}(x^1 + x^2)\), then \(\ell^* \cap \text{cl}(x^*) = \emptyset\) and, since \(L_{1/2}(x^1) = L_{1/2}(x^2)\), \(x^*\) is half-integral. Let \(x^* = x^* + \varepsilon 1_{\ell^*}\), where \(\varepsilon = \min(\{\beta(S) - \alpha(S)x^*: \alpha_{\ell^*}(S) > 0\})\). By Lemma 3.1, \(\beta(S) - \alpha(S)x^*\) is half-integral, if \(\alpha_{\ell^*}(S) \in \{0, 1\}\) whenever \(\beta(S) - \alpha(S)x^* > \frac{1}{2}\), then \(\varepsilon \geq \frac{1}{2}\). We show that if \(\beta(S) - \alpha(S)x^* = \frac{1}{2}\), then \(\alpha_{\ell^*}(S) \in \{0, 1\}\) as follows. Consider \(S \in I^*\) such that \(\beta(S) - \alpha(S)x^* = \frac{1}{2}\). Without loss of generality, assume that \(\beta(S) - \alpha(S)x^* \leq \beta(S) - \alpha(S)x^1\). Clearly, then \(\beta(S) - \alpha(S)x^1 \in \{0, \frac{1}{2}\}\).

If \(\beta(S) - \alpha(S)x^1 = 0\) then \(S \subseteq \text{cl}(x^1)\) and so \(\alpha_{\ell^*}(S) \leq 1\). We show that if \(\beta(S) - \alpha(S)x^1 = \frac{1}{2}\), then \(S \subseteq \text{cl}(x^1)\) as well and so \(\alpha_{\ell^*}(S) \leq 1\). Since

\[ \beta(S \cup \text{cl}(x^1)) + \beta(S \cap \text{cl}(x^1)) \geq \alpha(S \cup \text{cl}(x^1)x^1 + \alpha(S \cap \text{cl}(x^1)x^1) \]
\[ \geq \alpha(S)x^1 + \alpha(\text{cl}(x^1))x^1 \]
\[ = \beta(S) - \frac{1}{2} + \beta(\text{cl}(x^1)) \]
\[ \geq \beta(S \cup \text{cl}(x^1)) + \beta(S \cap \text{cl}(x^1)) - \frac{1}{2}, \]

the difference between \(\beta(S \cup \text{cl}(x^1)) + \beta(S \cap \text{cl}(x^1))\) and \(\alpha(S \cup \text{cl}(x^1)x^1 + \alpha(S \cap \text{cl}(x^1))x^1\) is at most \(\frac{1}{2}\).
Since cl($x^1$) = $\sigma$(supp($x^1$)), $\alpha(S \vee \text{cl}(x^1))x^1 = \alpha(\text{cl}(x^1))x^1 = \beta(\text{cl}(x^1))$. And so, $\alpha(S \vee \text{cl}(x^1))x^1$ is integral. Since $x^1$ is feasible, 

$\alpha(S \vee \text{cl}(x^1))x^1 \leq \beta(S \vee \text{cl}(x^1))$

and 

$\alpha(S \wedge \text{cl}(x^1))x^1 \leq \beta(S \wedge \text{cl}(x^1))$.

Further since the difference between $\beta(S \vee \text{cl}(x^1)) + \beta(S \wedge \text{cl}(x^1))$ and $\alpha(S \vee \text{cl}(x^1))x^1 + \alpha(S \wedge \text{cl}(x^1))x^1$ is at most $\frac{1}{2}$ and $\beta$ is integral, it follows that $\alpha(S \vee \text{cl}(x^1))x^1 = \beta(\text{cl}(x^1)).$ Therefore, $S \subseteq \text{cl}(x^1)$, and so $\alpha(S) \leq 1$. □

By Corollary 4.1 we may construct extreme 2-lattice matchings $x^1$ and $x^2$ satisfying the conditions of Lemma 4.4 by solving at most two matroid intersection problems. Thus, we have proved the following.

**Lemma 4.5.** Let $(S^*, T^*)$ be the dominant cover of $L'$ and let $\ell^* \in L$ be such that $\ell^* \wedge T^* = \emptyset$. If $v(L' \cup \{\ell^*\}) = v(L') + \frac{1}{2}$, we may construct a maximum 2-lattice matching $x^*$ of $L'$, such that $x^* = x^* + \frac{1}{2}1_L$. is a maximum 2-lattice matching of $L' \cup \{\ell^*\}$ by solving at most two matroid intersection problems.

To summarize, we use the results of this section to obtain a maximum 2-lattice matching for $L' \cup \{\ell^*\}$ as follows. If $\beta(T^* \vee \ell^*) = \beta(T^*) + 2$, we first find a maximum intersection $I$ in $\mathcal{M}_1(L')$ and $\mathcal{M}_2(L', \ell^*)$. If $|I| = \beta(S^*)$, we construct a maximum solution $x^*$ as in Lemma 4.3. If $|I| = \beta(S^*) - 1$ or $\beta(T^* \vee \ell^*) = \beta(T^*) + 1$, we find a larger solution by solving two intersection problems as in Lemma 4.5. Note this latter construction does not apparently lead to an extreme point, the appendix shows that, if appropriately carried out, it does. In the next section, we show that Step 4 of Algorithm MAXIMUM constructs an extreme 2-lattice matching as well.

5. When $\ell^* \wedge T^* \neq \emptyset$

When $\ell^* \wedge T^* \neq \emptyset$ and $\ell^*$ is not covered, the cover $(S^*, T^* \vee \ell^*)$ proves that $v(L' \cup \{\ell^*\})$ is at most $v(L') + \frac{1}{2}$. Further, we know that there is no $x' \in \Omega(L')$ such that $x' + \frac{1}{2}1_L$ is a 2-lattice matching. To find a larger 2-lattice matching, in this case, we look for a 2-lattice matching $x'$ of $L'$ with cardinality equal to $v(L') - \frac{1}{2}$ such that $x' + 1_L$ is a 2-lattice matching. The procedure AUGMENT searches for such a 2-lattice matching by considering an extreme 2-lattice matching $x \in \Omega_{\text{ext}}(L')$ such that $\ell^* \not\in \text{cl}(x)$ and looking for an augmentation in an auxiliary graph.

Since $\ell^*$ is not covered by $(S^*, T^*)$, $\ell^* \not\in T^*$ and there is a 2-lattice matching $x \in \Omega_{\text{ext}}(L')$ such that $\ell^* \not\in \text{cl}(x)$. Consider a maximal word $w \in \mathcal{A}(x)$ and index the lines so that $L_i(x) = \{\ell_1, \ldots, \ell_k\}$. Let $M_0 = \emptyset$ and let $M_i = M_{i-1} \vee \ell_i$ for $i \in [1, \ldots, k]$. Then $G(\mathcal{P}, \text{supp}(x))$, where $\mathcal{P} = \{M_i : i \in [1, \ldots, k]\} \cup \mathcal{P}(w)$, is a collection of vertex-disjoint odd cycles. Since $\ell^* \not\in \text{cl}(x)$, there is a fractional graphic.
matching in the auxiliary graph $G(\mathcal{Y}, \text{supp}(x) \cup \{\ell^*\})$ of cardinality $\nu(L') + \frac{1}{2}$. We derive conditions under which this fractional graphic matching is also a larger 2-lattice matching.

Note, however, that there is a fractional graphic matching of cardinality $\nu(L') + \frac{1}{2}$ even when $\nu(L' \cup \{\ell^*\}) = \nu(L')$ (see Example 5.1). On the other hand, when $\nu(L' \cup \{\ell^*\}) = \nu(L') + \frac{1}{2}$, a maximum graphic fractional matching in the auxiliary graph need not be a 2-lattice matching (see Example 5.2).

**Example 5.1.** Let $\mathcal{Y}$ be the lattice of flats and let $\beta$ be the rank function of the cycle matroid of the graph in Fig. 2. Let $L' = \{\ell_1\}$ and $\ell^* = \ell_2$, where $\ell_1 = (e_1, \bar{e}_1)$ and $\ell_2 = (e_2, \bar{e}_2)$. The only maximum 2-lattice matching of $L'$ is $x = (1, 0)$ and the dominant cover of $L'$ is $\emptyset, \{e_1, \bar{e}_1\}$. The basis graph of $x$ is a loop, i.e., both ends of the edge $\ell_1$ are incident to the flat $\{e_1, \bar{e}_1, e_2\}$. In the auxiliary graph, the edge $\ell_2$ has one end incident to the flat $\{e_1, \bar{e}_1, e_2\}$ (see Fig. 3). The unique maximum fractional graphic matching in the auxiliary graph is $x^* = (\frac{1}{2}, 1)$. In the 2-lattice matching problem, however, $x$ is a maximum 2-lattice matching of $L' \cup \{\ell^*\}$.

**Example 5.2.** Let $\mathcal{Y}$ be the lattice of flats and let $\beta$ be the rank function of the cycle matroid of the graph in Fig. 4. Let $L' = \{\ell_1, \ell_2\}$ and $\ell^* = \ell_3$, where $\ell_1 = (e_1, \bar{e}_1)$, $\ell_2 = (e_2, \bar{e}_2)$, and $\ell_3 = (e_3, \bar{e}_3)$. One extreme maximum 2-lattice matching of $L'$ is $x = (1, \frac{1}{2}, 0)$ and the dominant cover of $L'$ is $\emptyset, \{e_1, \bar{e}_1, e_2, \bar{e}_2, e_3\}$. The basis graph of $x$ consists of two loops, i.e., both ends of the edge $\ell_1$ are incident to the flat $\{e_1, \bar{e}_1, e_3\}$ and both ends of the edge $\ell_2$ are incident to the flat $\{e_1, \bar{e}_1, e_2, \bar{e}_2, e_3\}$. In the auxiliary graph the edge $\ell_3$ has one end incident to the flat $\{e_1, \bar{e}_1, e_3\}$ (see Fig. 5). The unique
maximum graphic fractional matching in the auxiliary graph is \( x^* = (1, 1, 1) \), but \( x^* \) is not a 2-lattice matching. The unique maximum 2-lattice matching of \( L' \cup \{\ell^*\} \) is \((0, 1, 1, 1)\).

We construct a basis graph for \( x \) with the property that either the unique maximum fractional graphic matching in the corresponding auxiliary graph is a larger 2-lattice matching or we can construct the dominant cover of \( L' \cup \{\ell^*\} \) proving that no larger 2-lattice matching exists.

For each line \( \ell \in L_1(x) \), define \( x' \) so that

\[
x'(\ell') = \begin{cases} 
1 & \text{if } \ell' = \ell^*, \\
0 & \text{if } \ell' = \ell, \\
x(\ell') & \text{otherwise}.
\end{cases}
\]

If \( \ell^* \cap L_1(x) \neq \emptyset \), we define a subset \( L^c \) of \( L_1(x) \) as follows. Since \( \ell^* \cap L_1(x) \neq \emptyset \), there is an element \( p \) in \( \ell^* \cap L_1(x) \). Let \( C \) be the unique circuit in \( L_1(x) \cup \{p\} \), and let \( L^c \) be the collection of lines \( \ell \in L_1(x) \) such that \( \ell \cap C \neq \emptyset \). The following lemma establishes conditions under which there is no larger 2-lattice matching.

**Lemma 5.1.** Let \( (S^*, T^*) \) be the dominant cover of a subset \( L' \subseteq L \) of lines and let \( \ell^* \in L \setminus L' \) be such that \( \ell^* \cap T^* \neq \emptyset \), but \( \ell^* \) is not covered by \( (S^*, T^*) \). Consider an extreme 2-lattice matching \( x \in \Omega_{\text{ext}}(L') \) such that \( \ell^* \notin \text{cl}(x) \) and let \( w = [e_1, e_2, \ldots, e_t] \) be a canonical family for \( x \). If

1. \( \ell^* \cap L_1(x) \neq \emptyset \),
2. for each \( \ell \in L^c \), \( \ell \cap L_1(x') \neq \emptyset \), and
3. for each \( \ell \in L^c \), \( L_1(x') \cap \{e_1, e_2, \ldots, e_t\} \in \Gamma(x') \),

then, \( (S, T) \) is a minimum cover of \( L' \cup \{\ell^*\} \) proving that \( v(L' \cup \{\ell^*\}) = v(L') \), where \( T = T^* \setminus \left( \text{cl}(x') : \ell \in L^c \right) \) and \( S = \sigma(\{\ell \cap T : \ell \in L'(T) \cap L_1(x)\}) \).

**Proof.** To show that \( v(L' \cup \{\ell^*\}) = v(L') \), we show that \( (S, T) \) is the dominant cover of \( L' \cup \{\ell^*\} \) and \( \beta(S, T) = \beta(S^*, T^*) \). The construction of \( (S, T) \) ensures that if it is a minimum cover of \( L' \cup \{\ell^*\} \) then it is the dominant cover of \( L' \cup \{\ell^*\} \). We show that \( (S, T) \) is a cover of \( L' \cup \{\ell^*\} \) as follows.

We first show that \( \text{cl}(x') = L_1(x') \cup S = \sigma(\text{supp}(x')) \) for each \( \ell \in L^c \). Since \( L_1(x') \cup \{e_1, e_2, \ldots, e_t\} \in \Gamma(x') \), \( L_1(x') \cup \{e_1, e_2, \ldots, e_t\} \subseteq \text{cl}(x') \). Now, \( L_1(x) \cup \{e_1, e_2, \ldots, e_t\} \) is a base of \( \text{cl}(x) \), so \( \beta(L_1(x) \cup \{e_1, e_2, \ldots, e_t\}) = 2 \sum_{\ell' \in L} x(\ell') \). Since \( \ell^* \not\in \text{cl}(x) \), there is a unique circuit in \( L_1(x) \cup \{e_1, e_2, \ldots, e_t\} \cup \{\ell^*\} \) and, since \( \ell \in L^c_1, L_1(x') \cup \{e_1, e_2, \ldots, e_t\} \) is independent. So,

\[
\beta(L_1(x') \cup \{e_1, e_2, \ldots, e_t\}) = \beta(L_1(x) \cup \{e_1, e_2, \ldots, e_t\}) \\
= 2 \sum_{\ell' \in L} x(\ell') = 2 \sum_{\ell' \in L} x'(\ell').
\]
Since $L_1(x') \lor \{e_1, e_2, \ldots, e_i\} \in I(x')$, we must have $\Delta_t(L_1(x') \lor \{e_1, e_2, \ldots, e_i\}) = 2$ for each $t' \in \text{supp}(x')$. It follows that $\text{cl}(x') \subseteq \sigma(\text{supp}(x')) \subseteq L_1(x') \lor \{e_1, e_2, \ldots, e_i\}$, and hence $\text{cl}(x') = L_1(x') \lor \{e_1, e_2, \ldots, e_i\} = \sigma(\text{supp}(x'))$.

Consider a line $t' \in \text{supp}(x) \setminus L^c$. To see that $(S, T)$ covers $t'$, observe that $t' \subseteq \text{supp}(x')$ for each $t' \in L^c$, and so $t' \subseteq \text{cl}(x')$ for each $t' \in L^c$. Thus, $t' \cap T = t' \cap T^* \neq \emptyset$.

Consider a line $t' \in L^c$. To see that $(S, T)$ covers $t'$, observe that by the previous argument $t' \cap T = t' \cap \text{cl}(x') \cap T^*$. We see that $t' \cap \text{cl}(x') \cap T^* \neq \emptyset$ as follows. We first argue that $\beta(t' \cap \text{cl}(x')) = 1$. Since $t' \cap L_1(x') \neq \emptyset$, by assumption, and $L_1(x') \subseteq \text{supp}(x') \subseteq \text{cl}(x')$, $t' \cap \text{cl}(x') \neq \emptyset$ and so, $\beta(t' \cap \text{cl}(x')) > 0$. In addition, $\beta(t' \cap \text{cl}(x')) < 2$. To see this, observe that $t' \not\subseteq \text{cl}(x)$ and $\beta(\text{cl}(x)) = \beta(t' \cap \text{cl}(x'))$. It follows that $\beta(\text{cl}(x) \lor t^*) > \beta(t' \cap \text{cl}(x'))$. If $t' \subseteq \text{cl}(x')$, then $\text{cl}(x) \lor t^* \subseteq \text{cl}(x')$; a contradiction. Therefore, $t' \not\subseteq \text{cl}(x')$ and $\beta(t' \cap \text{cl}(x')) = 1$.

Now, let $f(t') \in \text{cl}(x')$ be an element in $t' \cap \text{cl}(x')$, we see that $f(t') \in T^*$ as follows. If $f(t') \not\in T^*$, then, by Lemma 4.3, there is $x' \in \Omega_{\text{ext}}(L^c)$ such that $f(t') \not\in \text{cl}(x')$ and $L_1(x') = L_1(x')$. By Lemma 4.4, $\frac{1}{2}(x' + x' + 1_{t'})$ is a 2-lattice matching. But

$$\frac{1}{2}(x' + x' + 1_{t'}) = \frac{1}{2}(x + 1_{t'}) = \frac{1}{2}(x + 1_{t'}) + x'$$

and since $t^* \cap T^* \neq \emptyset$, this cannot be a 2-lattice matching. So, $f(t') \in T^*$ and $t' \cap T = t' \cap \text{cl}(x') \cap T^* = t' \cap T^* \neq \emptyset$.

To see that $(S, T)$ covers $L^c \setminus \text{supp}(x)$, observe that if $t' \in L^c$ and $x(t') = 0$ then $t' \in L'(T^*)$ and since $S^* \subseteq S$, $t' \cap S \neq \emptyset$. Thus, $(S, T)$ is a cover of $L^c$.

Since $\text{cl}(x') = \sigma(\text{supp}(x'))$ for each $t' \in L^c$, $t^* \subseteq \text{cl}(x')$ for each $t' \in L^c$ and so $t^* \cap T = t^* \cap T^* \neq \emptyset$. We show that $t^* \cap T^* \subseteq S$ as follows. Since $t^* \cap T^* \subseteq t^* \cap \text{cl}(x)$ and $\beta(t^* \cap \text{cl}(x)) = \beta(t^* \cap T^*) = 1$, $t^* \cap T^* \subseteq t^* \cap \text{cl}(x)$. Similarly, since $t^* \cap L^c \subseteq t^* \cap \text{cl}(x)$ and $\beta(t^* \cap \text{cl}(x)) = \beta(t^* \cap L^c) = 1$, we have that $t^* \cap T^* = t^* \cap \text{cl}(x) = t^* \cap L^c$. Since $t^* \cap T^* \subseteq \text{cl}(x')$ for each $t' \in L^c$, we must have that $t^* \cap T^* = t^* \cap L^c \cap \text{cl}(x') : t' \in L^c)$. Recall that for each line $t' \in L^c$, $f(t') = t' \cap \text{cl}(x')$ and so, if we let $\tilde{f}(t')$ be an element such that $\tilde{f}(t') \lor f(t') = t'$, then $\tilde{f}(t') \not\subseteq \text{cl}(x')$. It follows that

$$t^* \cap T^* = t^* \cap L^c \cap (\text{cl}(x') : t' \in L^c)$$

$$\subseteq \sigma\{f(t') : t' \in L^c\}$$

$$\subseteq S^* \cup \{f(t') : t' \in L^c\} = S.$$

Therefore, $t^* \cap S \neq \emptyset$. If follows that $(S, T)$ covers $t^*$ as well.

To see that $(S, T)$ is minimum, we show that $\beta(S) \leq \beta(S^*) + |L^c \setminus L(T^*)|$ and $\beta(T) \leq \beta(T^*) - |L^c \setminus L(T^*)|$ as follows. By construction, $\{t(t') : t' \in L(T^*) \cap L_1(x)\} \cup \{f(t') : t' \in L^c \setminus L(T^*)\}$ spans $S$. Since $\{t(t') : t' \in L(T^*) \cap L_1(x)\}$ is a base of $S^*$, $\beta(S) \leq \beta(S^*) + |\{t(t') : t' \in L^c \setminus L(T^*)\}| = \beta(S^*) + |L^c \setminus L(T^*)|$. To see that $\beta(T) \leq \beta(T^*) - |L^c \setminus L(T^*)|$, index the lines in $L^c \setminus L(T^*)$ as $\{t_1, \ldots, t_k\}$ and let $x' = x + 1_{t_i} = 1_{t_i}$ for $i = 1, \ldots, k$. Let $T^* = T^* \cap \text{cl}(x) \cap \text{cl}(x^2) \cdots \cap \text{cl}(x')$. We
show that $\beta(T^i) \leq \beta(T^*) - i$ for $i = 1, \ldots, k$ as follows. Since $\ell_1 \notin \text{cl}(x^i)$ and $\ell_1 \subseteq T^*$, $\beta(T^i) = \beta(T^* \land \text{cl}(x^i)) \leq \beta(T^*) - 1$. Assume $\beta(T^i) \leq \beta(T^*) - i$ for some $i < k$. Since $\ell_{i+1} \subseteq T^*$ and $\ell_{i+1} \subseteq \text{cl}(x^i)$ for $j \leq i$, $\ell_{i+1} \subseteq T^i$. Since $\ell_{i+1} \notin \text{cl}(x^i)$, $\beta(T^{i+1}) = \beta(T^i \land \text{cl}(x^i+1)) \leq \beta(T^i) - 1 \leq \beta(T^*) - (i + 1)$.

Since $T \subseteq T^k$, $\beta(T) \leq \beta(T^x) \leq \beta(T^*) - k = \beta(T^*) - |L^x \setminus L(T^*)|$. Thus, we have that $(S, T)$ is a cover of $L' \cup \{\ell^*\}$ with

$$\beta(S, T) = \frac{1}{2}[\beta(S) + \beta(T)] \geq \frac{1}{2}[\beta(T^*) + \beta(S^*)] = \beta(S^*, T^*).$$

Hence, $(S, T)$ is a minimum cover of $L' \cup \{\ell^*\}$. \qed

In the remainder of this section we show how to find larger 2-lattice matchings when the conditions of Lemma 5.1 do not hold. The three different conditions lead to the following three types of augmentations.

Type I augmentations: If $\ell^* \land L_1(x) = \emptyset$, then the edge $\ell^*$ is incident to a component of $G(\mathcal{F}, \text{supp}(x))$ formed by edges in $L_1(x)$ and Lemma 5.3 shows that the maximum fractional graphic matching in the auxiliary graph $G(\mathcal{F}, \text{supp}(x) \cup \{\ell^*\})$ is an extreme 2-lattice matching.

Type II augmentations: If $\ell^* \land L_1(x) \neq \emptyset$, but $\ell \land L_1(x') = \emptyset$ for some line $\ell \in L^x$, we obtain a larger 2-lattice matching via a Type II augmentation. In this case, we reorder the lines in $L_1(x)$ so that the maximum fractional graphic matching in the auxiliary graph $G(\mathcal{F}, \text{supp}(x) \cup \{\ell^*\})$ is a 2-lattice matching (see Example 5.3).

Example 5.3. Let $\Gamma$ be the lattice of flats and let $\beta$ be the rank function of the cycle matroid of the graph in Fig. 6. Let $L' = \{\ell_1, \ell_2\}$ and $\ell^* = \ell_3$, where $\ell_1 = (e_1, \bar{e}_1)$, $\ell_2 = (e_2, \bar{e}_2)$, and $\ell_3 = (e_3, \bar{e}_3)$. The unique maximum 2-lattice matching of $L'$ is $(1, 1, 0)$ and the dominant cover of $L'$ is $(\emptyset, \{e_1, \bar{e}_1, e_2, \bar{e}_2, e_3\})$. The line $\ell_2 \in L^x$ and $\ell_2 \land (\ell_1 \lor \ell_3) = \emptyset$, so $x' = (1, \frac{1}{2}, 1)$ is a 2-lattice matching of $L' \cup \{\ell^*\}$. In fact, this is the unique maximum 2-lattice matching of $L' \cup \{\ell^*\}$. Note that, if we let $M_1 = \sigma(\ell_2)$ and $M_2 = M_1 \lor \ell_1$, $x' = (\frac{1}{2}, 1, 1)$ is the unique maximum fractional graphic matching in the auxiliary graph $G(M_1, M_2, \{\ell_1, \ell_2, \ell_3\})$, but it is not a 2-lattice matching.

Type III augmentations: If $\ell^* \land L_1(x) \neq \emptyset$ and $\ell \land L_1(x') \neq \emptyset$ for each line $\ell \in L^x$, but $L_1(x') \lor \{e_1, e_2, \ldots, e_i\} \notin \Gamma(x')$ for some line $\ell \in L^x$, where $w = [e_1, e_2, \ldots, e_i]$ is a maximal word in $A(x)$, we obtain a larger 2-lattice matching via a Type III augmentation. In this case, there is a larger 2-lattice matching, but it may not correspond to the unique maximum fractional graphic matching in the auxiliary graph (see Example 5.4). We revise the basis graph of $x$ so that the unique maximum fractional graphic matching in the auxiliary graph is a 2-lattice matching (see Example 5.4) (see Fig. 7).

Example 5.4. Let $\Gamma$ be the lattice of flats and let $\beta$ be the rank function of the cycle matroid of the graph in Fig. 8. Let $L' = \{\ell_1, \ell_2, \ell_3\}$ and $\ell^* = \ell_4$, where $\ell_1 = (e_1, \bar{e}_1)$, $\ell_2 = (e_2, \bar{e}_2)$, $\ell_3 = (e_3, \bar{e}_3)$ and $\ell_4 = (e_4, \bar{e}_4)$. The unique maximum 2-lattice matching of
Fig. 6. Example 5.3.

Fig. 7. The auxiliary graph of Example 5.3.

Fig. 8. Example 5.4.

Fig. 9. The maximum graphic fractional matching in this auxiliary graph is not a 2-lattice matching.

$L'$ is $(1, 1, \frac{1}{2}, 0)$ and the dominant cover of $L'$ is $(\emptyset, \{e_1, \bar{e}_1, e_2, \bar{e}_2, e_3, \bar{e}_3, e_4\})$. The lines $\ell_1$ and $\ell_2$ are in $L^c$, but $\ell_1 \land L_1(x^{e_1}) = \{e_1\}$ and $\ell_2 \land L_1(x^{e_2}) = \{e_2\}$. Now, $w = [e_3]$ is a maximal word in $A(x)$ and $L_1(x^{e_1}) \lor \{e_3\} \notin \Gamma(x^{e_1})$. So, there is a larger 2-lattice matching in $L' \cup \{\ell^*\}$. In fact, the unique maximum 2-lattice matching in $L' \cup \{\ell^*\}$ is $(0, 1, 1, 1)$. The unique maximum fractional graphic matching in the auxiliary graph of Fig. 9 is $(1, \frac{1}{2}, \frac{1}{2}, 1)$, which is not a 2-lattice matching. The unique maximum fractional graphic matching in the auxiliary graph of Fig. 10 is the maximum 2-lattice matching in $L' \cup \{\ell^*\}$.

Note that if none of these three cases arises, then the conditions of Lemma 5.1 are satisfied and we can construct the dominant cover of $L' \cup \{\ell^*\}$ proving that $v(L' \cup \{\ell^*\}) = v(L')$. 
We first give an overview of the procedure AUMENT and prove its correctness later.

**Procedure AUMENT:**

**Input:**
- The dominant cover \((S^*, T^*)\) of a subset \(L' \subset L\) and a line \(\ell^* \in L \setminus L'\) such that \(\ell^* \land T^* \neq \emptyset\), but \(\alpha_{\ell^*}(S^*) + \alpha_{\ell^*}(T^*) = 1\).
- A 2-lattice matching \(x \in \Omega_{\text{ext}}(L')\) such that \(\ell^* \not\in \text{cl}(x)\), and a maximal word \(w = [e_1, e_2, \ldots, e_t] \in A(x)\).

**Output:** A larger solution \(x^* \in \Omega_{\text{ext}}(L' \cup \{\ell^*\})\) or the dominant cover \((S, T)\) of \(L' \cup \{\ell^*\}\).

**Step 1:** **Type I augmentation:** Index the lines so that \(L_1(x) = \{\ell_1, \ell_2, \ldots, \ell_k\}\). Let \(M_1 = \sigma(\ell_1)\) and for \(i = 2, 3, \ldots, k\), \(M_i = M_{i-1} \lor \ell_i\). If the edge \(\ell^*\) is incident to a component of \(G(\sigma, \text{supp}(x))\) formed by edges in \(L_1/2(x)\), the unique maximum fractional graphic matching \(x^*\) in the auxiliary graph, is a larger 2-lattice matching. Return \(x^*\).

**Step 2:** If the edge \(\ell^*\) is incident to a loop of \(G(\sigma, \text{supp}(x))\) formed by an edge in \(L_1(x)\), and there is a line \(\ell \in L_\ell\) such that \(\ell \land L_1(x') = \emptyset\), then there is a Type II augmentation; Go to Step 3. Otherwise, if \(L_1(x') \lor \{e_1, e_2, \ldots, e_t\} \not\subseteq \Gamma(x')\) for some line \(\ell \in L_\ell\), then there is a Type III augmentation; Go to Step 4. Otherwise, go to Step 5.

**Step 3:** **Type II augmentation:** Let

\[
x^*(\ell') = \begin{cases} 
1 & \text{if } \ell' = \ell^*, \\
\frac{1}{2} & \text{if } \ell' = \ell, \\
x(\ell') & \text{otherwise.}
\end{cases}
\]

Return \(x^*\).

**Step 4:** **Type III augmentation:** Call procedure Type 3 to obtain a larger 2-lattice matching \(x^*\) and return \(x^*\).

**Step 5:** **Construct the dominant cover:** Let

\[
T = T^* \land (\text{cl}(x') : \ell \in L_\ell) \quad \text{and} \quad S = \sigma(\{\ell \land T : \ell \in L_1(T) \cap L_1(x)\}).
\]

Return \((S, T)\).
Lemma 5.2. Let $x \in \Omega_{\text{ext}}(L')$ and let $\ell^* \in L \setminus L'$. Index the lines of $L'$ so that $L_1(x) = \{\ell_1, \ell_2, \ldots, \ell_k\}$ and let $\mathcal{M} = \{M_1, M_2, \ldots, M_k\}$, where $M_1 = \sigma(\ell_1)$ and $M_i = M_{i-1} \cup \{\ell_i\}$. Suppose $\mathcal{S} = \{S_1, \ldots, S_t\}$ is a canonical family for $x$. The unique maximum fractional graphic matching $x^*$ in the auxiliary graph $G(\mathcal{M} \cup \mathcal{S}, \text{supp}(x) \cup \{\ell^*\})$ is an extreme 2-lattice matching if $S_i \cup \ell^* \in \Gamma(x^*)$ and $\alpha(S)x^* \leq \beta(S)$ for each $S \in \Gamma$ such that $S \subseteq \sigma(L_1(x))$.

Proof. We first show that $x^*$ is feasible. Since $x^*$ is non-negative, it follows that, if $x^*$ is infeasible, there is a violated member $S'$ in $\Gamma$, i.e., a member $S' \in \Gamma$ such that $\alpha(S')x^* > \beta(S')$. Since no member contained in $L_1(x)$ is violated, we have by Corollary 2.2, that $S' \cup L_1(x)$ must be a violated member. We show that there is no violated member containing $L_1(x)$.

Since $x^*$ is a fractional graphic matching in the auxiliary graph,
\[ \alpha(M_i)x^* = \beta(M_i) \quad \text{for } i = 1, 2, \ldots, k \]
and
\[ \alpha(S_i)x^* = \beta(S_i) \quad \text{for each } S_i \in \mathcal{S}. \]

Let $S_{t+1} = S_t \cup \ell^*$. Since $\ell^* \notin \text{cl}(x^*)$, $\ell^* \notin S_t$. Also, since $\ell^* \cap T^* = \emptyset$ and $T^* \subseteq S_t$, it follows that $\beta(S_{t+1}) = \beta(S_t) + 1$.

Let $S_0 = \emptyset$. Then, \{S_0, S_1, \ldots, S_{t+1}\} is a nested family of $\Gamma$ such that for each $i \in [1, \ldots, t+1]$, $S_i \in \Gamma(x^*)$ and $\beta(S_i) - \beta(S_{i-1}) = 1$. Since $G(\mathcal{M} \cup \mathcal{S}, \text{supp}(x))$ is a collection of cycles with no spurs and $\text{supp}(x) \subseteq \text{supp}(x) \cup \{\ell^*\}$, $\text{supp}(x^*) \subseteq S_{t+1}$. Thus, by arguments analogous to those used in the proof of Theorem 7.2 in [2], there is no violated member of \Gamma containing $M_k$. This contradicts the existence of $S'$. So, $x^*$ is 2-lattice matching.

To see that $x^*$ is extreme, observe that $\mathcal{S}' = \mathcal{M} \cup \mathcal{S} \cup \{S_{t+1}\} \subseteq \Gamma(x^*)$ and $G(\mathcal{S}', \text{supp}(x^*))$ is a basis graph. \qed

The following two lemmas show that Types I and II Augmentations satisfy the conditions of Lemma 5.2 and so in these cases $x^*$ is a larger extreme 2-lattice matching.

Lemma 5.3. A vector $x^*$ obtained from a Type I Augmentation in Procedure AUGMENT is an extreme 2-lattice matching.

Proof. Since $\beta(L_1(x)) = 2|L_1(x)|$ and $\alpha(L_1(x)) = 0$ for each $\ell \in L_1(x)$, $\alpha(S)x^* \leq \beta(S)$ for each $S \subseteq \sigma(L_1(x))$. Also, we see that $S_i \cup \ell^* \in \Gamma(x^*)$ as follows.
Observe that $G(\mathcal{S}, \text{supp}(x))$ is a collection of cycles with no spurs and $\ell^* \subseteq S \cup \ell^*$. Therefore, $\text{supp}(x^*) \subseteq S \cup \ell^*$. Thus,

$$
\alpha(S \cup \ell^*)x^* = \sum_{\ell \in \text{supp}(x^*)} \alpha(\ell) \xi(\ell)
= 2 \sum_{\ell \in \text{supp}(x^*)} x^*(\ell)
= 2 \left[ \sum_{\ell \in \text{supp}(x)} x(\ell) + \frac{1}{2} \right]
= \beta(S) + 1
= \beta(S \cup \ell^*).
$$

So, by Lemma 5.2, $x^*$ is an extreme 2-lattice matching. □

**Lemma 5.4.** A vector $x^*$ obtained from a Type II Augmentation in Procedure AUGMENT is an extreme 2-lattice matching.

**Proof.** Let $\mathcal{S} = \{S_1, \ldots, S_t\}$ be a canonical family for $x$. Index the lines so that $L_1(x) = \{\ell_1, \ldots, \ell_k\}$, where $\ell_k = \ell$ is a line in $L^c$ such that $\ell \land L_1(x') = \emptyset$, and let $M_1 = \sigma(\ell_1)$ and $M_i = M_{i-1} \cup \ell_i$ for $i \in [2, \ldots, t]$.

The edge $\ell^*$ is incident to the node $M_k$ in $G(\mathcal{S}, \text{supp}(x))$, where $\mathcal{S} = \{M_i : i \in [1, \ldots, k]\} \cup \mathcal{S}^*$. So, $x^*$ is the unique maximum fractional graphic matching in the auxiliary graph $G(\mathcal{S}, \text{supp}(x) \cup \{\ell^*\})$.

We see that $S \cup \ell^* \in I(x^*)$ as follows. Observe that $G(\mathcal{S}, \text{supp}(x))$ is a collection of cycles with no spurs and $\ell^* \subseteq S \cup \ell^*$. Therefore, $\text{supp}(x^*) \subseteq S \cup \ell^*$. Thus,

$$
\alpha(S \cup \ell^*)x^* = \sum_{\ell \in \text{supp}(x^*)} \alpha(\ell) \xi(\ell)
= 2 \sum_{\ell \in \text{supp}(x^*)} x^*(\ell)
= 2 \left[ \sum_{\ell \in \text{supp}(x)} x(\ell) + \frac{1}{2} \right]
= \beta(S) + 1
= \beta(S \cup \ell^*).
$$

We show that for each $S \subseteq M_k$, $\alpha(S)x^* \leq \beta(S)$ as follows. Suppose $\alpha(S')x^* > \beta(S')$ for some $S' \subseteq M_k$. Since $\beta(M_{k-1}) = 2k - 2$ and for each $\ell \in L_{1/2}(x)$, $\xi(M_{k-1}) = 0$, no
violated member of $\Gamma$ is contained in $M_{k-1}$. By Corollary 2.2, there must be a violated member of $\Gamma$ containing $M_{k-1}$. Therefore, we may choose a violated member $S'$ of $\Gamma$ such that $M_{k-1} \subseteq S' \subseteq M_k$.

Since $M_{k-1} \subseteq M_k$ and $\beta(M_k) - \beta(M_{k-1}) = 2$, $\beta(S') = \beta(M_{k-1}) + 1$. Note that $x_{\ell'}(M_k) = 0$ for $\ell' \in L_{1/2}(x') \setminus \ell$. So, $x_{\ell'}(S') = 0$ for $\ell' \in L_{1/2}(x') \setminus \ell$. By assumption,

$$x(S')x^* = \sum_{\ell' \in L_1(x) \setminus \ell'} x_{\ell'}(S')x^*(\ell') + x_{\ell'}(S')x^*(\ell*) + x_{\ell'}(S')x^*(\ell)$$

$$= 2(k - 1) + x_{\ell'}(S') + \frac{1}{2}x_{\ell'}(S')$$

$$= 2k - 1.$$ 

Therefore, $x_{\ell'}(S') = x_{\ell'}(S') = 1$.

Since $\ell^* \wedge M_{k-1} = \emptyset$ and $x_{\ell^*}(S') = 1$, $S' = M_{k-1} \lor (\ell^* \wedge S') \subseteq M_{k-1} \lor \ell^* = \sigma(L_1(x'))$. But $x_{\ell}(L_1(x')) = 0$, and so, $x_{\ell^*}(S') = 0$. So, there is no violated member of $\Gamma$ contained in $M_k$. By Lemma 5.2, $x^*$ is an extreme 2-lattice matching. \(\square\)

Finally, we consider Type III augmentations. Recall that in Type III augmentations there is no line $\ell \in L$ such that $\ell \cap L_1(x') = \emptyset$, but $L_1(x') \lor \{e_1, e_2, \ldots, e_t\} \not\subseteq \Gamma(x')$ for some line $\ell \in L_1$. We revise the basis graph as in procedure Type 3.

**Procedure** Type 3:

**Input:**

- The dominant cover $(S^*, T^*)$ for $L' \subseteq L$ and a line $\ell^* \in L \setminus L'$ such that $\ell^* \wedge T^* \neq \emptyset$, but $x_{\ell^*}(S^*) + x_{\ell^*}(T^*) = 1$.
- A 2-lattice matching $x \in \Omega_{\text{ext}}(L')$ and a maximal word $w = [e_1, e_2, \ldots, e_t]$ in $A(x)$ such that
  - $\ell^* \not\subseteq \text{cl}(x)$,
  - $\ell^* \cap L_1(x) \neq \emptyset$,
  - $\ell \cap L_1(x') \neq \emptyset$ for each $\ell \in L_1$,
  - $L_1(x') \lor \{e_1, e_2, \ldots, e_t\} \not\subseteq \Gamma(x')$ for some line $\ell' \in L_1$.

**Output:** A larger solution $x^* \in \Omega_{\text{ext}}(L' \cup \{\ell^*\})$.

**Step 1:** Index the lines so that $L_1(x) = \{\ell_1, \ldots, \ell_k\}$, where $\ell' = \ell_k$. Let $M_1 = \sigma(\ell_1)$ and $M_i = M_{i-1} \lor \ell_i$ for $i \in [2, \ldots, k]$. Let $S'_0 = L_1(x) \land L_1(x')$ and for $i = 1, 2, \ldots, t + 1$, let

$$S'_i = \begin{cases} 
S'_{i-1} \lor \{e_i\} & \text{if } i < t, \\
S'_i \lor \ell^* & \text{if } i = t + 1.
\end{cases}$$

Note that $S'_0 \in \Gamma(x)$ and, since $S'_{t+1} = L_1(x') \lor \{e_1, e_2, \ldots, e_t\} \not\subseteq \Gamma(x')$, there is a line $\ell \in L_{1/2}(x)$ such that $x_{\ell}(S'_{t+1}) < 2$. Therefore, $S'_{t+1} \not\subseteq \Gamma(x)$. Let $S'_i$ be the smallest member of $S'_1, \ldots, S'_{t+1}$ such that $S'_i \not\subseteq \Gamma(x)$. For $i = 0, 1, \ldots, t + 1$, define $S''_i$
as follows:

$$S_i'' = \begin{cases} S_i' & \text{if } i < s, \\ L_1(x) \cup S_i & \text{if } i \geq s. \end{cases}$$

**Step 2:** Let \( \mathcal{F}'' = \{ M_i : i \in [1, \ldots, k-1] \} \cup \{ S_i'': i \in [0, \ldots, t] \} \). Let \( x^* \) be the unique maximum fractional graphic matching in the auxiliary graph \( G(\mathcal{F}'', \text{supp}(x) \cup \{ \ell^* \}) \). Return \( x^* \).

Note that:
- \( S_i'' \in \Gamma(x) \) for each \( i = 0, 1, \ldots, t \),
- \( \alpha_i(S_i'') = \alpha_i(S_i) \) for each \( i = 0, 1, \ldots, t \) and each \( \ell \in \text{supp}(x) \setminus \ell' \),
- \( \alpha_{\ell'}(M_{k-1}) = 0 \),
- \( \alpha_{\ell'}(S_i'') = 1 \) for \( i = 0, \ldots, s - 1 \), and
- \( \alpha_{\ell'}(S_i'') = 2 \) for \( i \geq s \).

It follows that \( G(\mathcal{F}'', \text{supp}(x)) \) is the same as \( G(\mathcal{F}, \text{supp}(x)) \) except that one end of the edge \( \ell'' \) is incident to the node \( S_i'' \) and the other end is incident to the node \( S_i'' \). Also, in the auxiliary graph \( G(\mathcal{F}'', \text{supp}(x) \cup \{ \ell^* \}) \), the edge \( \ell^* \) is incident to the node \( S_i'' \).

An argument similar to the proof of Lemma 5.2 shows that the unique maximum fractional graphic matching in the auxiliary graph \( G(\mathcal{F}'', \text{supp}(x) \cup \{ \ell^* \}) \) is a 2-lattice matching as well.

**Lemma 5.5.** Let \( x^* \) be the unique maximum fractional graphic matching in the auxiliary graph \( G(\mathcal{F}'', \text{supp}(x) \cup \{ \ell^* \}) \). Then, \( x^* \) is an extreme 2-lattice matching.

**Proof.** Since \( x^* \) is a fractional graphic matching in the auxiliary graph \( G(\mathcal{F}'', \text{supp}(x) \cup \{ \ell^* \}) \),

$$\alpha(M_i)x^* = \beta(M_i) \quad \text{for each } i = 1, 2, \ldots, k - 1$$

and

$$\alpha(S_i'')x^* = \beta(S_i'') \quad \text{for each } i = 0, 1, \ldots, t.$$ 

Also, we see that \( S_i' \in \Gamma(x^*) \) as follows. Let \( \ell_p \) and \( \ell_q \) be two (possibly identical) lines in \( L_{1/2}(x) \) such that \( \ell_p \) and \( \ell_q \) are incident to the vertex \( S_i \). Then,

$$\alpha_{\ell_p}(S_i) > \alpha_{\ell_q}(S_i),$$

$$\alpha_{\ell_q}(S_i) > \alpha_{\ell_p}(S_i).$$

We may choose the maximal word \( w \) in Procedure Type 3 so that if the (possibly identical) edges \( \ell \) and \( \ell'' \) in \( L_{1/2}(x) \) are incident to the vertex \( S_i \), then \( e_i \in \ell \cap S_i \) or \( \ell'' \cap S_i \). Thus, there is an element \( p \in \ell_p \cap S_i \) or \( \ell_q \cap S_i \) such that \( p \notin \ell_p \cap (S_{i-1}) \) or \( p \notin \ell_q \cap (S_{i-1}) \). Since \( S_i' = S_i' \cup S_i \), \( S_i'_{i-1} = S_i' \cup S_i \) and \( S_i' \subseteq L_1(x) \),

$$\alpha_{\ell_p}(S_i') + \alpha_{\ell_q}(S_i') > \alpha_{\ell_p}(S_{i-1}) + \alpha_{\ell_q}(S_{i-1}).$$
Since \( S'_1 \not\in \Gamma(x) \), it must be the case that
\[
x_{\ell} (S'_1) + x_{\ell'} (S'_1) = x_{\ell} (S'_{s-1}) + x_{\ell'} (S'_{s-1}) + 1.
\]

Otherwise, if
\[
x_{\ell} (S'_1) + x_{\ell'} (S'_1) = x_{\ell} (S'_{s-1}) + x_{\ell'} (S'_{s-1}) + 2,
\]
then \( \alpha(S'_1)x = \alpha(S'_{s-1})x + 1 = \beta(S'_{s-1}) + 1 = \beta(S'_1) \); contradicting the assertion that \( S'_1 \not\in \Gamma(x) \).

Note that for each \( \ell \in L_{1/2}(x) \),
\[
x_\ell (S'_{s-1}) - x_\ell (S'_s) = \begin{cases} 1 & \text{if } \ell = \ell_p \text{ or } \ell_q, \\ 0 & \text{otherwise}. \end{cases}
\]

Since \( S'_{s-1} = S'_s \subseteq S'_t \subseteq S''_t \), it follows that for each \( \ell \in L_{1/2}(x) \setminus \{\ell_p, \ell_q\} \),
\[
x_\ell (S'') = x_\ell (S'_{s-1}).
\]

So, \( \alpha(S''_t)x^* = \beta(S'_{s-1}) + x_\ell (\ell_p) \) or \( \alpha(S'_{s-1})x^* + x_\ell (\ell_q) \). Since \( \alpha(S'_{s-1})x^* = \beta(S'_{s-1}) \) and \( x^*(\ell_p) = x^*(\ell_q) = 1 \), it follows that \( \alpha(S''_t)x^* = \beta(S'_{s-1}) + 1 = \beta(S'_t) \), so \( S'_t \in \Gamma(x^*) \).

We show that \( x^* \) is a 2-lattice matching as follows. Since \( x^* \) is non-negative, it follows that, if \( x^* \) is infeasible, there is a violated member \( S' \) in \( \Gamma \), i.e., a member \( S' \in \Gamma \) such that \( \alpha(S')x^* > \beta(S') \). By Corollary 2.2, either \( S' \cap M_{k-1} \) or \( S' \cup M_{k-1} \) must be a violated member. We first show that there is no violated member in \( \Gamma \) containing \( M_{k-1} \).

First, observe that \( \text{supp}(x^*) \subseteq \text{supp}(x) \cup \{\ell^*\} \) and \( \text{supp}(x) \subseteq S''_{t+1} \). Therefore, \( \text{supp}(x^*) \subseteq S''_{t+1} \). Also, \( \beta(S''_{t+1}) = \beta(S''_t) + 1 \). Thus,
\[
\alpha(S''_{t+1})x^* = \sum_{\ell \in \text{supp}(x^*)} x_\ell (S''_{t+1})x^*(\ell) = 2 \sum_{\ell \in \text{supp}(x^*)} x^*(\ell)
\]
\[
= 2 \left[ \sum_{\ell \in L'} \alpha(\ell) + \frac{1}{2} \right]
\]
\[
= \beta(S''_t) + 1 = \beta(S''_{t+1}).
\]

Let \( T_0 = M_{k-1} \) and for \( i = 1, 2, \ldots, t+2 \), let
\[
T_i = \begin{cases} S''_t & \text{if } i < s, \\ S'_t & \text{if } i = s, \\ S''_{t+1} & \text{if } i > s. \end{cases}
\]

It follows that \( \{T_0, \ldots, T_{t+2}\} \) is a nested family in \( \Gamma(x') \) such that \( \beta(T_i) - \beta(T_{i-1}) = 1 \) for \( i = 1, \ldots, t+2 \) and \( \text{supp}(x) \subseteq T_{t+2} \). By arguments analogous to those used in the proof of Theorem 7.2 in [2], there is no violated member of \( \Gamma \) containing \( T_0 = M_{k-1} \).
Since \( z(\mathcal{A}_{k-1}) = 0 \) for each \( \ell \in L_{1/2}(x) \), no violated member of \( \Gamma \) is contained in \( \mathcal{A}_{k-1} \). This contradicts the existence of \( S' \). Therefore, \( x^* \) is 2-lattice matching.

To see that \( x^* \) is extreme, observe that \( G(S'' \cup \{ S''_t \}, \text{supp}(x) \cup \{ \ell^* \}) \) is a collection of vertex-disjoint odd cycles with one bloom. \( \square \)

6. Complexity

The algorithm begins with a (possibly empty) subset \( L' \) of lines and the dominant cover for \( L' \). Since \( L \) is 8GGnite and the algorithm 8Gnds a larger 2-lattice matching or constructs the dominant cover for a larger set of lines in each iteration, it repeats Step 2 at most \( \beta(L) |L| \) times. Since \( \beta(L) \leq 2 |L| \), the algorithm terminates in \( O(|L|^2) \) iterations.

In each iteration, the major work takes place in Steps 3 and 4. Step 3 solves at most three matroid intersection problems and Step 4 solves one matroid intersection problem. If determining whether a set of elements is independent in a matroid is considered as an elementary step, then the matroid intersection problem can be solved in \( O(|E|^3) \), where \( E \) is the ground set of the matroid. In our case, \( |E| \) is at most \( |L| \).

The major work in procedures \textsc{Find} \( x' \in \Omega(L') \) and \textsc{Augment} is the construction of an auxiliary graph. Given a canonical family, the time to construct an auxiliary graph is \( O(|L|^3) \). So, the algorithm runs in \( O(|L|^5) \) steps.

Appendix

In order to be able to compute efficiently the closure of the current 2-lattice matching, the algorithm maintains an extreme 2-lattice matching in each iteration. In Step 3b, however, the construction does not apparently lead to an extreme point. By exploiting the characterization of extreme points in [2] and constructing the solution to the matroid intersection problems in a particular way, we show how to obtain an extreme 2-lattice matching in this case.

Recall that the maximum 2-lattice matching \( x' + \frac{1}{2} 1_{\ell'^*} \) constructed in Step 3b is obtained by solving at most two matroid intersection problems. If \( x' \) is constructed from the solution to a single matroid intersection problem, it is clearly extreme. If \( x' \) is constructed from the solutions to two matroid intersection problems, we obtain the second intersection \( I_2 \) from the first intersection \( I_1 \) so as to ensure that we can construct a canonical family for \( x' + \frac{1}{2} 1_{\ell'^*} \) from a canonical family for the extreme 2-lattice matching \( x_1 \) induced by \( I_1 \). So, the algorithm constructs \( I_2 \) in the following way.

Let \( f \in \ell'^* \cap \text{cl}(x_1) \). Since \( \text{cl}(x_1) = I_1 \lor T^* \), \( \beta(I_1/T^* \lor f) = |I_1| - 1 \). It follows that \( I_1 \) contains a unique circuit \( C \) in \( \mathcal{M}_2(L', \{ f \}) \). Let \( x_0 \in C \). Then, \( I_0 = I_1 \setminus \{ x_0 \} \) is a \((\beta(S^*) - 1)\)-intersection in \( \mathcal{M}_1(L') \) and \( \mathcal{M}_2(L', \{ f \}) \). Since the maximum cardinality of an intersection in \( \mathcal{M}_1(L') \) and \( \mathcal{M}_2(L', \{ f \}) \) is \( \beta(S^*) \), there is an augmenting path \( P \), such that \( I_2 = I_0 \oplus P \equiv (I_0 \setminus P) \cup (P \setminus I_0) \) is a \( \beta(S^*) \)-intersection in \( \mathcal{M}_1(L') \) and
\(M_2(L', \{f\})\). We choose \(P\) so that either \(x_0\) is the first element of \(P\) or \(P\) contains no element of the circuit \(C\). A minor modification of the matroid intersection procedure [15] finds an augmenting path with this property.

Procedure **\(\text{FIND} \ x' \in \Omega(L')\)** provides the details of Step 3b in the algorithm **MAXIMUM**.

**Procedure:** **\(\text{FIND} \ x' \in \Omega(L')\)**

**Input:**
- An extreme point \(x \in \Omega_{\text{ext}}(L')\) and the dominant cover \((S^*, T^*)\) of \(L'\).
- A line \(\ell^* \subseteq L \setminus L'\) such that \(\ell^* \land T^* = \emptyset\) and \(v(L' \cup \{\ell^*\}) = v(L') + \frac{1}{2}\).

**Output:** \(x' \in \Omega(L')\) such that \(x^* = x' + \frac{1}{2}I_{\ell^*} \in \Omega_{\text{ext}}(L' \cup \{\ell^*\})\).

**Step 1:** Choose an element \(e \in \ell^*\) and find a \(\beta(S^*)\)-intersection \(I_1\) in \(M_1(L')\) and \(M_2(L', \{e\})\). Construct the corresponding extreme 2-lattice matching \(x^1 \in \Omega_{\text{ext}}(L')\) such that \(e \notin \text{cl}(x^1)\).

**Step 2:** Let \(f = \ell^* \land \text{cl}(x^1)\) and let \(C\) be the unique circuit of \(M_2(L', \{f\})\) in \(I_1\). Choose \(x_0 \in C\) and let \(I_0 = I_1 \setminus \{x_0\}\). Find an augmenting path \(P\) such that \(I_2 = I_0 \oplus P\) is a \(\beta(S^*)\)-intersection in \(M_1(L')\) and \(M_2(L', \{f\})\) as follows.

**Step 2a:** Look for an augmenting path by labeling only from the source \(x_0\). If we find a path \(P\), go to Step 3. Otherwise, go to Step 2b.

**Step 2b:** Look for \(P\) with \(C \cap P = \emptyset\)

Find an augmenting path \(P\) by labeling from all sources. Go to Step 3.

**Step 3:** Construct the extreme 2-lattice matching \(x^2 \in \Omega_{\text{ext}}(L')\) such that \(f \notin \text{cl}(x^2)\) corresponding to the \(\beta(S^*)\)-intersection \(I_2 = I_0 \oplus P\) in \(M_1(L')\) and \(M_2(L', \{f\})\). Then \(x' = \frac{1}{2}x^1 + \frac{1}{2}x^2\) is a 2-lattice matching in \(\Omega(L')\) such that \(x^* = x' + \frac{1}{2}I_{\ell^*} \in \Omega_{\text{ext}}(L' \cup \{\ell^*\})\).

Lemma A.1 shows that if we obtain \(x^*\) in this way, we can construct a canonical family proving that it is in fact extreme.

**Lemma A.1.** Procedure **\(\text{FIND} \ x' \in \Omega(L')\)** constructs an extreme 2-lattice matching \(x^* \in \Omega_{\text{ext}}(L' \cup \{\ell^*\})\).

**Proof.** We first argue that the augmenting path \(P\) constructed in Step 2 of Procedure **\(\text{FIND} \ x' \in \Omega(L')\)** either has first element \(x_0\) or contains no element of \(C\). We then show how to construct a canonical family for \(x^*\) in each of these two cases.

Suppose there is no augmenting path with first element \(x_0\), but there is an augmenting path \(P = \{y_1, y_2, \ldots, y_p\}\) such that \(P \cap C \neq \emptyset\). Let \(y_k \in P \cap C\). Then, \(P' = \{x_0, y_k, \ldots, y_p\}\) is an augmenting path with first element \(x_0\); a contradiction. We may conclude then that the augmenting path \(P\) constructed in Step 2 either has first element \(x_0\) or contains no element of \(C\).

Let \(P = \{y_1, x_1, y_2, x_2, \ldots, x_{p-1}, y_p\}\) where \(y_i \in L'(T^*) \setminus I_0\) and \(x_i \in I_0\) and let \(w = [e_1, e_2, \ldots, e_l]\) be a maximal word in \(\Gamma(x^1)\). For \(i = 1, \ldots, p\), let \(\tilde{\ell}(y_i)\) be an element such that \(\ell(y_i) \lor \tilde{\ell}(y_i)\) spans the line \(y_i\). Construct the sequence \(w'\) as follows.
If \( y_1 = x_0 \), let \( w' = [(y_p, t(y_{p-1}), \ldots, t(y_2), e_1, \ldots, e_t, f, \tilde{r}(y_2), \tilde{r}(y_3), \ldots, \tilde{r}(y_p)]. \)
If \( y_1 \neq x_0 \), let \( w' = [(y_p, t(y_{p-1}), \ldots, t(y_1), e_1, \ldots, e_t, f, \tilde{r}(y_1), \tilde{r}(y_2), \ldots, \tilde{r}(y_p)]. \)

We show that \( \mathcal{S}(w') \) is a canonical family for \( x^* \) thereby proving that \( x^* \) is an extreme 2-lattice matching in \( \Omega(L' \cup \{f\}) \) and \( w' \) is a maximal word in \( L(x^*). \)

Let \( \mathcal{S}(w') = \{T_1, T_2, \ldots, \} \), let \( \mathcal{S}(w) = \{S_1, S_2, \ldots, S_t\} \) and for \( i = 1, 2, \ldots, t \), let \( E_i = \{e_1, e_2, \ldots, e_t\}. \) We consider first the case in which \( y_1 = x_0. \)

In this case, we show that \( T_i \in \Gamma(x^*) \) for \( i = 1, 2, \ldots, p - 1 \) by showing that \( T_i \in \Gamma(x^* \cap \Gamma(x^2)) \) and hence that

\[
\alpha(T_i)x^* \geq \alpha(T_i)[\frac{1}{2}x^2 + \frac{1}{2}x^2] = \beta(T_i).
\]

Since \( x^* \) is a 2-lattice matching, we must have \( \alpha(T_i)x^* \leq \beta(T_i) \) and so \( T_i \in \Gamma(x^*). \)

To see that \( T_i \in \Gamma(x^2) \), observe that

\[
T_i = L_1(x^*) \cup \{t(y_p), t(y_{p-1}), \ldots, t(y_{p-i+1})\},
\]

\[
L_1(x^*) \subseteq L_1(x^2) \setminus P \quad \text{and} \quad \{y_p, y_{p-1}, \ldots, y_{p-i+1}\} \subseteq I_2 \subseteq L_1(x^2). \]

It follows that \( \alpha(T_i)x^2 \geq 2|L_1(x^*)| + i \geq \beta(T_i) \). Since \( x^2 \) is a 2-lattice matching, we must have \( \alpha(T_i)x^2 \leq \beta(T_i) \) and so \( T_i \in \Gamma(x^2). \)

To see that \( T_i \in \Gamma(x^1) \), we show that \( \{t(x_{p-1}), t(x_{p-2}), \ldots, t(x_{p-i})\} \subseteq T_i \), from which it follows that

\[
T_i \supseteq L_1(x^1) \cup \{t(x_p), t(x_{p-2}), \ldots, t(x_{p-i})\}.
\]

Since \( L_1(x^1) \subseteq L_1(x^2) \setminus P \) and \( \{x_{p-1}, x_{p-2}, \ldots, x_{p-i}\} \subseteq I_1 \subseteq L_1(x^1) \), it follows that

\[
\alpha(T_i)x^1 \geq 2|L_1(x^1)| + i \geq \beta(T_i). \]

Since \( x^1 \) is a 2-lattice matching, we must have \( \alpha(T_i)x^1 \leq \beta(T_i) \) and so \( T_i \in \Gamma(x^1). \)

We prove that \( \{t(x_{p-1}), t(x_{p-2}), \ldots, t(x_{p-i})\} \subseteq T_i \) by induction on \( i \). For \( j = 2, \ldots, p \), let \( C_j \) be the unique circuit of \( \mathcal{M}_1(L') \) in \( I_0 \cup \{y_j\} \) and let \( C_j' \) be the unique circuit of \( \mathcal{M}_2(L', \{f\}) \) in \( I_0 \cup \{y_{j-1}\} \). Since \( P \) is short-cut free (see [15] for a complete definition),

\[
C_j \subseteq I_0 \cup \{y_j\} \setminus \{x_1, x_2, \ldots, x_{j-2}\} \quad \text{for} \quad j = 2, 3, \ldots, p
\]

and

\[
C_j \subseteq I_0 \cup \{y_j\} \setminus \{x_{j+1}, x_{j+2}, \ldots, x_{p-1}\} \quad \text{for} \quad j = 1, 2, \ldots, p - 1.
\]

We see that \( t(x_{p-1}) \in T_1 \) as follows. By the definition of the augmenting path \( P \), \( x_{p-1} \in C_p \subseteq I_0 \cup \{y_p\} \setminus \{x_1, x_2, \ldots, x_{p-2}\} \). So,

\[
x_{p-1} \in \sigma_1(I_0 \cup \{y_p\} \setminus \{x_1, x_2, \ldots, x_{p-1}\}),
\]

where \( \sigma_1(\cdot) \) denotes the span in \( \mathcal{M}_1(L') \). It follows that

\[
t(x_{p-1}) \in \sigma(\{t(\ell) : \ell \in I_0 \cup \{y_p\} \setminus \{x_1, x_2, \ldots, x_{p-1}\}\})
\]

\[
\subseteq \sigma(t(\ell) : \ell \in (I_1 \cap I_2) \cup \{y_p\})
\]

\[
\subseteq \sigma(\{t(\ell) : \ell \in (L_1(x^1) \cap L_1(x^2)) \cup \{y_p\}\})
\]

\[
\subseteq L_1(x^1) \cup \{t(y_p)\}
\]

\[
= T_1.
\]
Assume, inductively, that \( \{t(x_{p-1}), t(x_{p-2}), \ldots, t(x_{p-j})\} \subseteq T_j \) for some \( j \in \{1, 2, \ldots, p-2\} \). We see that \( t(x_{p-(j+1)}) \in T_{j+1} \) as follows. Since \( x_{p-(j+1)} \in C_{p-j} \subseteq I_0 \cup \{y_{p-j}\} \setminus \{x_1, x_2, \ldots, x_{p-(j+2)}\} \),

\[ x_{p-(j+1)} \in \sigma(I_0 \cup \{y_{p-j}\} \setminus \{x_1, x_2, \ldots, x_{p-(j+1)}\}). \]

It follows that

\[ t(x_{p-(j+1)}) \in \sigma(\{t(\ell) : \ell \in I_0 \cup \{y_{p-j}\} \setminus \{x_1, x_2, \ldots, x_{p-(j+1)}\}\}) \]

\[ \subseteq \sigma(\{t(\ell) : \ell \in (I_1 \cap I_2) \cup \{y_{p-j}, x_{p-1}, x_{p-2}, \ldots, x_{p-j}\}\}) \]

\[ \subseteq \sigma(\{t(\ell) : \ell \in (L_1(x^1) \cap L_1(x^2)) \cup \{y_{p-j}, x_{p-1}, x_{p-2}, \ldots, x_{p-j}\}\}) \]

\[ \subseteq L_1(x^*) \cup \{t(y_{p-j}), t(x_{p-1}), t(x_{p-2}), \ldots, t(x_{p-j})\} \]

\[ \subseteq T_j \cup \{t(y_{p-j})\} = T_{j+1}. \]

Since, \( \{t(x_{p-1}), t(x_{p-2}), \ldots, t(x_{p-j})\} \subseteq T_j \subset T_{j+1} \), we have that \( \{t(x_{p-1}), t(x_{p-2}), \ldots, t(x_{p-(j+1)})\} \subseteq T_{j+1} \) as desired. And so, by induction, \( \{t(x_{p-1}), t(x_{p-2}), \ldots, t(x_{p-j})\} \subseteq T_j \).

Note that since \( y_1 = x_0 \), \( y_1 \in I_1 \cap I_2 \subseteq L_1(x^*) \). It follows that \( t(y_1) \in T_{p-1} = L_1(x^*) \setminus \{t(y_{p-1}), \ldots, t(y_2)\} \). Thus, \( S^* = \sigma(\{t(y) : y \in I_2\}) \subseteq T_{p-1} \) and \( L_1(x^*) \cup S^* \subseteq T_{p-1} \).

But \( L_1(x^*) \cup S^* = L_1(x^*) \cup \{t(y) : y \in I_2\} \supseteq L_1(x^*) \cup \{t(y_{p-1}), \ldots, t(y_2)\} = T_{p-1} \).

So, \( T_{p-1} = L_1(x^*) \cup S^* \).

We next show that \( T_{p-1+t} = T_{p-1} \cup E_t = L_1(x^*) \cup S^* \cup E_t \) is in \( \Gamma(x^*) \). Since, \( L_1(x^*) \cup S^* \cup E_t \supseteq L_1(x^*) \cup T^* \) and both \( L_1(x^*) \) and \( T^* \) are in \( \Gamma(x^1) \cap \Gamma(x^2) \), we have by Lemma 2.1 that \( L_1(x^*) \cup T^* \in \Gamma(x^1) \cap \Gamma(x^2) \) and so also in \( \Gamma(x^*) \).

Consider \( i \in \{1, 2, \ldots, t\} \). We show that \( T_{p-1+i} = L_1(x^*) \cup S^* \cup E_i \in \Gamma(x^*) \) by showing that \( T_{p-1+i} \in \Gamma(x^1) \cap \Gamma(x^2) \).

To show that \( L_1(x^*) \cup S^* \cup E_i \in \Gamma(x^1) \), we show that \( L_1(x^*) \cup S^* \cup E_i = (L_1(x^*) \cup S^* \cup E_i) \cap S_i \). It follows then, from Lemma 2.1, that \( L_1(x^*) \cup S^* \cup E_i \subseteq \Gamma(x^1) \).

We show that \( L_1(x^*) \cup S^* \cup E_i = (L_1(x^*) \cup S^* \cup E_i) \cap S_i \) as follows. Since \( L_1(x^*) \cup S^* \subseteq L_1(x^1) \), \( L_1(x^*) \cup S^* \cup E_i \subseteq (L_1(x^*) \cup S^* \cup E_i) \cap S_i \). We see that \( (L_1(x^*) \cup S^* \cup E_i) \cap S_i \subseteq L_1(x^*) \cup S^* \cup E_i \) as follows. Let \( Z_1 = \{\bar{\ell}(\ell) : \ell \in L_1(x^1) \setminus L_1(x^2)\} \), then \( Z_1 \subseteq D \cup E_i \) is a base of \( S_i \), where \( D \) is a base of \( (L_1(x^1) \cap L_1(x^2)) \cup S^* \). Consider an element \( p \in (L_1(x^*) \cup S^* \cup E_i) \cap S_i \). Then, there is a unique circuit \( C \subseteq Z_1 \cup D \cup E_i \cup \{p\} \).

Since \( p \in L_1(x^*) \cup S^* \cup E_i \), \( C \cap Z_1 = \emptyset \). Also, since \( p \in S_i \), \( C \cap \{e_{i+1}, e_{i+2}, \ldots, e_t\} = \emptyset \).

Thereby, \( C \subseteq D \cup E_i \cup \{p\} \subseteq \sigma(D \cup E_i) = L_1(x^*) \cup S^* \cup E_i \).

A similar argument shows that \( T_{p-1+i} \in \Gamma(x^2) \). Therefore, \( T_{p-1+i} = L_1(x^*) \cup S^* \cup E_i \in \Gamma(x^1) \cap \Gamma(x^2) \) as desired.

Note that \( T_{p-1+i} = L_1(x^*) \cup S^* \cup E_i = L_1(x^*) \cup T^* \in \Gamma(x^*) \). We see that \( T_{p+t} = L_1(x^*) \cup T^* \cup \{f\} \in \Gamma(x^*) \) as follows.

Since \( P \) is short-cut free,

\[ x_1 \in C_{i}^2 \subseteq I_0 \cup \{y_1\} \setminus \{x_2, x_3, \ldots, x_{p-1}\} \]

It follows that \( x_1 \in \sigma_2(I_0 \cup \{y_1\} \setminus \{x_1, x_2, \ldots, x_{p-1}\}) \). Since \( y_1 = x_0 \) and \( I_1 \setminus \{x_1, x_2, \ldots, x_{p-1}\} = I_1 \cap I_2 \), we have that \( x_1 \in \sigma_2(I_1 \cap I_2) \), i.e., \( x_1 \subseteq T^* \cup (I_1 \cap I_2) \cup \{f\} \). Since \( I_1 \)
is independent in $\mathcal{M}_2(L', \{e\})$, $x_1 \notin T^* \cup (I_1 \cap I_2)$. It follows that

$$\begin{align*}
\alpha(T^* \cup (I_1 \cap I_2) \cup \{f\})x_1 & \geq \alpha(T^* \cup (I_1 \cap I_2))x_1 + 1 \\
& = \beta(T^* \cup (I_1 \cap I_2)) + 1 \\
& = \beta(T^* \cup (I_1 \cap I_2) \cup \{f\})
\end{align*}$$

and so, $T^* \cup (I_1 \cap I_2) \cup \{f\} \in \Gamma(x^1)$. Now, although $T^* \cup (I_1 \cap I_2) \cup \{f\}$ is not in $\Gamma(x^2)$, we do know that

$$\begin{align*}
\alpha(T^* \cup (I_1 \cap I_2) \cup \{f\})x_2 & \geq \alpha(T^* \cup (I_1 \cap I_2))x_2 \\
& = \beta(T^* \cup (I_1 \cap I_2)) \\
& = \beta(T^* \cup (I_1 \cap I_2) \cup \{f\}) - 1.
\end{align*}$$

Thus,

$$\begin{align*}
\alpha(T^* \cup (I_1 \cap I_2) \cup \{f\})x^* = \alpha(T^* \cup (I_1 \cap I_2) \cup \{f\})(\frac{1}{2}x^1 + \frac{1}{2}x^2 + \frac{1}{2}I_{f^*}) \\
& \geq \beta(T^* \cup (I_1 \cap I_2) \cup \{f\});
\end{align*}$$

proving that $T^* \cup (I_1 \cap I_2) \cup \{f\} \in \Gamma(x^*)$.

We see that $L_1(x^*) \cup T_{p+t+1} = T^* \cup (I_1 \cap I_2) \cup \{f, y_2\} \in \Gamma(x^*)$ as follows. Clearly,

$$\begin{align*}
\alpha(T^* \cup (I_1 \cap I_2) \cup \{f, y_2\})x^2 & \geq \alpha(T^* \cup (I_1 \cap I_2))x^2 \\
& = \beta(T^* \cup (I_1 \cap I_2)) \\
& = \beta(T^* \cup (I_1 \cap I_2) \cup \{f, y_2\}) - 1.
\end{align*}$$

Further, since

$$\begin{align*}
x_2 & \in C^2_2 \subseteq I_0 \cup \{y_2\} \setminus \{x_3, \ldots, x_{p-1}\},
\end{align*}$$

it follows that $x_2 \in \sigma_2(I_0 \cup \{y_2\} \setminus \{x_3, \ldots, x_{p-1}\})$ and hence that

$$\begin{align*}
x_2 & \subseteq T^* \cup (I_0 \cup \{y_2\} \setminus \{x_3, \ldots, x_{p-1}\}) \cup \{f\} \\
& = T^* \cup [(I_1 \cap I_2) \cup \{x_1, y_2\} \setminus \{x_0\}] \cup \{f\} \\
& \subseteq T^* \cup [(I_1 \cap I_2) \cup \{x_1, y_2\}] \cup \{f\}.
\end{align*}$$

Since $x_1 \subseteq T^* \cup (I_1 \cap I_2) \cup \{f\}$, we have that $x_2 \subseteq T^* \cup (I_1 \cap I_2) \cup \{f, y_2\}$. It follows that $T_{p+t+1} = T^* \cup (I_1 \cap I_2) \cup \{f, y_2\} \in \Gamma(x^1)$ and hence in $\Gamma(x^*)$.

We prove by induction on $i$ that $T_{p+i+1} = T^* \cup (I_1 \cap I_2) \cup \{f, y_2, y_3, \ldots, y_i\} \in \Gamma(x^*)$ as follows.

Assume, inductively, that $\{x_1, x_2, \ldots, x_j\} \subseteq T^* \cup (I_1 \cap I_2) \cup \{f, y_2, y_3, \ldots, y_j\}$ for some $j \in [2, \ldots, p-2]$. We see that $x_{j+1} \subseteq T^* \cup (I_1 \cap I_2) \cup \{f, y_2, y_3, \ldots, y_j, y_{j+1}\}$ as follows. Since $x_{j+1} \in C^2_{j+1} \subseteq I_0 \cup \{y_{j+1}\} \setminus \{x_{j+2}, x_{j+3}, \ldots, x_{p-1}\}$,

$$\begin{align*}
x_{j+1} & \in \sigma_2(I_0 \cup \{y_{j+1}\} \setminus \{x_{j+2}, x_{j+3}, \ldots, x_{p-1}\}).
\end{align*}$$
It follows that
\[
x_{j+1} \subseteq T^* \lor I_0 \cup \{y_{j+1}\} \setminus \{x_{j+1}, x_{j+2}, \ldots, x_{p-1}\} \lor \{f\}
\]
\[
\subseteq T^* \lor (I_1 \cap I_2) \lor \{y_{j+1}, x_1, x_2, \ldots, x_j\} \lor \{f\}
\]
\[
\subseteq T^* \lor (I_1 \cap I_2) \lor \{x_1, x_2, \ldots, x_j, y_2, y_3, \ldots, y_{j+1}\} \lor \{f\}.
\]
Since, \(\{x_1, x_2, \ldots, x_j\} \subseteq T^* \lor (I_1 \cap I_2) \lor \{f, y_2, y_3, \ldots, y_{j+1}\}\), we have that \(\{x_1, x_2, \ldots, x_j, y_{j+1}\} \subseteq T^* \lor (I_1 \cap I_2) \lor \{f, y_2, y_3, \ldots, y_{j+1}\}\). It follows that
\[
\alpha(T^* \lor (I_1 \cap I_2) \lor \{f, y_2, y_3, \ldots, y_{j+1}\})^2
\]
\[
= \beta(T^* \lor (I_1 \cap I_2) \lor \{f, y_2, y_3, \ldots, y_{j+1}\})
\]
\[
= \beta(T^* \lor (I_1 \cap I_2) \lor \{f, y_2, y_3, \ldots, y_{j+1}\}) - 1,
\]
it follows that
\[
T_{p+t+j}^* = T^* \lor (I_1 \cap I_2) \lor \{f, y_2, y_3, \ldots, y_{j+1}\} \in \Gamma(x^*).
\]
To see that
\[
T_{i+2,p-1}^* = T^* \lor (I_1 \cap I_2) \lor \{f, y_2, y_3, \ldots, y_p\} = T^* \lor I_2 \lor f \in \Gamma(x^*),
\]
note that the maximum cardinality of an intersection in \(\mathcal{H}(L')\) and \(\mathcal{H}_2(L', \{\ell^*\})\) is \(\beta(S^*) - 1\). It follows that \(\beta(I_2 \lor T^* \lor \ell^*) = \beta(I_2 \lor T^* \lor f)\). Therefore,
\[
\alpha(T_{i+2,p-1}^*)x^2 \geq \alpha(T^* \lor I_2)x^2
\]
\[
= \beta(T^* \lor I_2)
\]
\[
= \beta(T^* \lor I_2 \lor f) - 1.
\]
Similarly, since \(\{x_1, \ldots, x_{p-1}\} \subseteq T_{i+2,p-1}^*,\) it follows that \(I_1 \subseteq T_{i+2,p-1}^*\) and so
\[
\alpha(T_{i+2,p-1}^*)x^1 = 2 \sum x^1(\ell)
\]
\[
= 2 \sum x^2(\ell)
\]
\[
= \beta(T^* \lor I_2)
\]
\[
= \beta(T^* \lor I_2 \lor f) - 1.
\]
Finally, since \(\ell^* \subseteq T^* \lor I_2 \lor f\),
\[
\alpha(T_{i+2,p-1}^*)x^* = \alpha(T_{i+2,p-1}^*)\frac{1}{2}(x^1 + x^2 + 1_{\ell^*})
\]
\[
= \beta(T^* \lor I_2 \lor f).
\]
For the case of \(y_1 \neq x_0\), an argument analogous to the case of \(y_1 = x_0\) shows that
\(T_i \in \Gamma(x^*),\) for \(i = 1, \ldots, p + t\).

We see that \(T_{p+t+i} = T^* \lor (I_1 \cap I_2) \lor \{f\} \in \Gamma(x^*)\) as follows. Recall that \(P\) is an augmenting path such that \(C \cap P = \emptyset\), where \(C\) is the circuit of \(\mathcal{H}_2(L', \{f\})\) in \(I_1\) and \(x_0 \in C\). So, \(x_0 \in C \subseteq \sigma_2(I_1 \cap I_2)\). It follows that \(x_0 \subseteq (I_1 \cap I_2) \lor T^* \lor \{f\} = T_{p+t+i}^*\). Since \(I_1\) is independent in \(\mathcal{H}_2(L', \{e\}), x_0 \notin (I_1 \cap I_2) \lor T^*.\) Now, arguments similar to those used in the case of \(y_1 = x_0\) show that \(T_{p+t+i}^* \in \Gamma(x^*)\).

Arguments similar to those used in the case \(y_1 = x_0\) show that \(T_{i+p+i} \in \Gamma(x^*)\) for \(i = 2, \ldots, p\) as well.
Finally, we observe that $x^*$ is an extreme 2-lattice matching by observing that the auxiliary graph corresponding to $\mathcal{F}$ is a collection of vertex disjoint odd cycles. In particular, when $y_1 = x_0$, $x_i$ is incident to $T_{p-i}$ and $T_{p+i-1}$, $y_{i+1}$ is incident to $T_{p-i+1}$ and $T_{p+i-1}$ for $i \in [1,2,\ldots, p-1]$, and $\ell^*$ is incident to $T_{p+i}$ and $T_{2p+i}$. When $y_1 \neq x_0$, $x_i$ is incident to $T_{p-i}$ and $T_{p+i+1}$, $y_{i+1}$ is incident to $T_{p-i+1}$ and $T_{p+i+1}$ for $i \in [0,1,2,\ldots, p-1]$, and $\ell^*$ is incident to $T_{p+i+1}$ and $T_{2p+i+1}$. Therefore, the lines in $(I_1 \oplus I_2) \cup \{\ell^*\}$ form an odd cycle. Also, observe that the components formed by lines in $L_1(\mathcal{H})$ remain the same as in the auxiliary graph corresponding to $\mathcal{F}(w)$. Therefore, $x^*$ is extreme. 

References