Maximum-Likelihood Estimation for the Two-Dimensional Discrete Boolean Random Set and Function Models Using Multidimensional Linear Samples

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The Boolean model is a random set process in which random shapes are positioned according to the outcomes of an independent point process. In the discrete case, the point process is Bernoulli. Estimation is done on the two-dimensional discrete Boolean model by sampling the germ–grain model at widely spaced points. An observation using this procedure consists of jointly distributed horizontal and vertical runlengths. An approximate likelihood of each cross observation is computed. Since the observations are taken at widely spaced points, they are considered independent and are multiplied to form a likelihood function for the entire sampled process. Estimation for the two-dimensional process is done by maximizing the grand likelihood over the parameter space. Simulations on random-rectangle Boolean models show significant decrease in variance over the method using horizontal and vertical linear samples, each taken at independently selected points. Maximum-likelihood estimation can also be used to fit models to real textures. This method is generalized to estimate parameters of a class of Boolean random functions.

1. INTRODUCTION

The Boolean model is a fundamental random set used in image analysis and microscopy to model images of overlapping objects [12, 14, 15]. While much effort has been devoted to estimation for the continuous case [13, 18], our work focuses on the discrete case, which is appropriate for pixel-based methods used in image processing.

The Boolean model is a random shape process coupled with an independent point process. The outcome of the model consists of random shapes together with a collection of points. The random shapes are translated to corresponding points. What one observes, however, is the union of the translated shapes. Indeed, this allows one to model overlapping objects. The random shape process is called the grain process and the point process is called the germ process. We call the union process a germ–grain model to distinguish the observed process from the generating set and point processes. In the two-dimensional (2D) discrete setting, we restrict the random shape process to a finite collection of bounded subsets of the discrete plane. The shapes will also be horizontally and vertically convex; that is, a vertical or horizontal line intersecting the shape anywhere will produce only a single line segment. This is required to use our previous work on the one-dimensional (1D) model. Points covered by a random shape are called black, and white otherwise.

In previous work, we derived the likelihood function of the one-dimensional directional model. Under certain conditions described in [10], 2D models induce 1D models on intersecting lines. Observations of induced 1D models are observations of the 2D model and the 1D likelihood contains information about the 2D model [9]. These likelihood functions provide maximum-likelihood estimates via numerical optimization. In this work, we estimate the parameters of the 2D model with “cross-window” observations instead of runlengths on horizontal and vertical lines. The advantage of using a cross is that we capture joint vertical and horizontal information. Sampling is done at widely spaced points so that approximately independent observations are obtained. At a sampling point, the lengths of the horizontal and vertical runlengths containing the point are measured. We compute the approximate likelihood for each sample. The likelihood functions for the sampling points are multiplied and used as a grand likelihood function for the entire image. Maximization over
parameters in the model provides the estimate. The sampling method is similar to that used in [1] where observations are first-contact distances to the random set taken at widely spaced points. As with many other continuous estimation methods, the method of [1] relies on expressions relating the germ–grain model to the Boolean model via Minkowski functionals [20], which are tractable for regular geometric random shapes such as disks with random radii.

A discrete 1D directional Boolean model consists of a Bernoulli process along with a random variable producing segments. The Bernoulli process is a sequence of binary random variables indicating whether or not a point is marked. A point is marked with probability \( p (q = 1 - p) \). At each marked point, a segment with length \( k \) possessing distribution \( C(k) \) is placed there, emanating to the right. Let \( \{ \xi_i \} \) be the outcome of the Bernoulli process and let \( \{ k_i \} \) be the outcome of the segment-length process. The event of a point not being marked into the shape is \( \text{inducement through intersection of the grains produces} \). The difficulty is that the method of inducement through intersection of the grains produces not a Bernoulli process on the line, but a complicated point process that depends on the random process. The practical situation is not so bleak, because under reasonable conditions on the shape process (from an applied standpoint), inducement occurs. It is shown in [10] that a sufficient condition for a 2D discrete directional Boolean model to induce a 1D discrete directional Boolean model on a horizontal intersecting line is that the union of the left borders of the sets in \( \mathcal{S} \) be one pixel thick.

2. FUNDAMENTAL EVENTS

We first introduce the fundamental covering events of the 1D discrete directional Boolean model. Proofs of all formulas and supporting assertions are provided in [4].

The events are described with respect to an interval \( 1 \) to \( m \). If a segment emanates from \( i \) (with probability \( p \)) and its length is \( k \) (distributed as \( C(k) \)), then the set \( [i, i + k - 1] \) is covered by the process. The marking probability \( p \) and the segment-length distribution can be combined into a single distribution to describe the process: \( F(k) = q + pC(k) \), \( k = 0, 1, 2, \ldots \), where \( q = 1 - p \) and \( C(0) \) is defined to be 0. If \( X_i \) has distribution \( F \), then \( (X_i = 0) \) is the event that there is no segment at \( i \) and \( (X_i = k > 0) \) is the event that a segment of length \( k \) emanates from \( i \). Owing to stationarity of the Boolean model (the random variables \( X_i \) are i.i.d.), probabilities of covering events depend on the length of the interval but not its position. Events will be denoted by reference to particular intervals. Let \( (i, j) \) denote some unspecified event on the interval \( [i, j] \). If \( i = j \), denote the event by \( A(i) \) and if \( j < i \), denote it by \( A(O) \). \( A(O) \) may not represent any actual event, but instead serves as a place holder to start a recursion.

\( \mathbf{E}(1, m) \) is the event that the interval \( [1, m] \) is covered by segments emanating in \( [1, m] \) and none of these segments extend beyond \( m \). Since random variables \( X_i \) are independent, \( \mathbf{E}(1, m) \) does not depend on outcomes occurring before 1 or after \( m \) and

\[
P(\mathbf{E}(1, m)) = \sum_{j=1}^{m} (F(m) - F(j - 1)) \prod_{i=1}^{j-1} F(i - 1)P(\mathbf{E}(j + 1, m)),
\]

where \( P(\mathbf{E}(O)) = 1 \) and \( P(\mathbf{E}(i)) = F(1) - F(0) \). \( \mathbf{W}(1, m) \) is the event that no point in the interval \( [1, m] \) is covered, so that the interval is fully white. If \( F \) possesses a finite mean, \( P(\mathbf{W}(1, m)) = q^m \prod_{j=1}^{m} F(j) \).

\( \mathbf{D}(1, m) \) is the event that \( [1, m] \) is covered by segments emanating within \( [1, m] \) and

\[
P(\mathbf{D}(1, m)) = 1 - F(m) + \sum_{j=1}^{m} (F(m) - F(j - 1)) \prod_{i=1}^{j-1} F(i - 1)P(\mathbf{D}(j + 1, m)),
\]

where \( P(\mathbf{D}(O)) = 1 \) and \( P(\mathbf{D}(i)) = 1 - F(0) \) since \( D(i) = (X_i > 0) \). \( \mathbf{G}(1, m) \) is the event that \( [1, m] \) is covered by segments emanating before or at \( m \), but no segment extends beyond \( m \). In the simplest case, \( P(\mathbf{G}(1)) = P(\mathbf{W}(1))/p/q \). Finally, the event that a point \( i \) is covered in any way whatsoever is denoted by \( \mathbf{H}(i) \) and \( P(\mathbf{H}(i)) = 1 - P(\mathbf{W}(i)) \).
We impose conditions on the grain process for induction to occur (that is, when a grain intersects a horizontal line, the intersection can be considered a line segment emanating to the right from the left-most point of intersection): if the union of the left (top) borders of the grains is one pixel thick, then the 2D process induces a horizontal (vertical) 1D directional Boolean model [10]. For instance, if the shape process consists of rectangles with random widths and heights, and with centers placed in the upper-left hand corner, then the grain process induces. If \( C_{WH} \) is the joint width and height distribution, respectively, it can be shown [9] that the induced horizontal and vertical distributions are

\[
P(X_i \leq w; p, \theta, \mu) = \prod_{h=0}^{w} F_{W}(w; p, \theta, \mu) + F_{H}(h; p, \theta, \mu) - F_{WH}(w, h; p, \theta, \mu),
\]

\[
P(Y_j \leq h; p, \theta, \mu) = \prod_{w=0}^{h} F_{W}(w; p, \theta, \mu) + F_{H}(h; p, \theta, \mu) - F_{WH}(w, h; p, \theta, \mu),
\]

where \( p \) is the marking probability in the 2D process, \( \theta \) and \( \mu \) are parameters of the joint width and height distribution and \( F_{WH}(w, h; p, \theta, \mu) = q + pC_{WH}(w, h; \theta, \mu) \). The marking probabilities of the horizontal and vertical processes are \( p_X = 1 - P(X_i = 0; p, \theta, \mu) \) and \( p_Y = 1 - P(Y_j = 0; p, \theta, \mu) \), respectively.

A black runlength is a consecutive run of black points commenced by a white point and terminated by a white point; a white runlength is a sequence of white points commenced by a black point and terminated by black point. Letting \( K \) and \( V \) denote the black and white runlength random variables, it is shown in [8] that black runlengths have density \( P(K = m) = P(E(1,m))q^m/p, m = 1, 2, \ldots \), white runlengths have a geometric density \( P(V = m) = q^{m-1}p, m = 1, 2, \ldots \), and black and white runlength random variables are independent.

### 3. CROSS-WINDOWED LIKELIHOOD FUNCTION

Runlength probabilities can be generalized for cross observations. A point (x, y) within a realization of a 2D germ–grain model will lie in the intersection of a vertical and a horizontal runlength (Fig. 1). The observation can be summarized by five pieces of information, the color (black or white) and the lengths in each of the four directions (left, right, up, down). The random variables for this observation will be denoted by \( K = (K_\bar{x}, K_{\bar{y}}, K_{\bar{y}}, K_{\bar{x}}) \) for a black observation and \( V = (V_{\bar{x}}, V_{\bar{y}}, V_{\bar{y}}, V_{\bar{x}}) \) for white. Without loss of generality, suppose \( (x, y) = (0, 0) \), and let a black observation be \( k = (k_\bar{x}, k_{\bar{y}}, k_{\bar{y}}, k_{\bar{x}}) \). In the horizontal \((\bar{x})\) direction, we have a white pixel at \(-k_{\bar{x}} - 1\), black pixels in the interval \([k_{\bar{x}}, k_{\bar{x}}]\), a white pixel at \( k_{\bar{x}} + 1\), and the probability of this covering event is computed given that the origin is covered. Stationarity and \( W_X(-k_{\bar{x}} - 1) \cap E_X(-k_{\bar{x}}, k_{\bar{x}}) \cap W_X(k_{\bar{x}} + 1) \subset H_X(0) \) yield

\[
P(K_{\bar{x}} = k_{\bar{x}}, K_{\bar{y}} = k_{\bar{y}}) = P(W_X(-k_{\bar{x}} - 1) \cap E_X(-k_{\bar{x}}, k_{\bar{y}}) \cap W_X(k_{\bar{x}} + 1))(1 - P(W_X(0))),
\]

for \( k_{\bar{x}}, k_{\bar{y}} \geq 0 \). The vertical case is handled analogously.

For a white observation \( v = (v_{\bar{x}}, v_{\bar{y}}, v_{\bar{y}}, v_{\bar{x}}) \), we again consider the horizontal part of the cross observation. We start with a black pixel at \( v_{\bar{x}} - 1 \) followed by white pixels in the interval \([v_{\bar{x}}, v_{\bar{x}}]\) and terminated by a black pixel and the probability of this event is computed given that the origin is white. By stationarity and \( G_X(-v_{\bar{x}} - 1) \cap W_X(-v_{\bar{x}}, v_{\bar{x}}) \cap D_X(v_{\bar{x}} + 1) \subset W_X(0) \), for \( v_{\bar{x}}, v_{\bar{y}} \geq 0 \),

\[
P(V_{\bar{x}} = v_{\bar{x}}, V_{\bar{y}} = v_{\bar{y}}) = P(G_X(-v_{\bar{x}} - 1) \cap W_X(-v_{\bar{x}}, v_{\bar{x}}) \cap D_X(v_{\bar{x}} + 1))(1 - P(W_X(0))),
\]

The true likelihood of \( (K = k) \) requires the probability of the bidirectional covering event; an approximate likelihood for the complete horizontal and vertical observation is formed by the product of the horizontal and vertical probabilities,

\[
P(K = k) = q_Xq_Y P(E_X(-k_{\bar{x}}, k_{\bar{x}})) P(E_Y(-k_{\bar{y}}, k_{\bar{y}})) P(W_X(0))(1 - P(W_X(0)))
\]
and

\[ P(\mathbf{V} = \mathbf{v}) = \frac{q^{\frac{v_1^2 + v_2^2}{2}}}{X^3} p^{\frac{v_1^2 + v_2^2}{2}} q^{\frac{v_1^2 + v_2^2}{2}} p^{\frac{v_1^2 + v_2^2}{2}}. \]  

(8)

Suppose that multiple samples are taken over an image with observations \( k_i, i = 1, \ldots, m \), and \( v_j, j = 1, \ldots, n \). The grand likelihood function for the entire sample is formed by the product of the likelihoods for each cross:

\[
L(k_1, \ldots, k_m; v_1, \ldots, v_n; \Theta) = \prod_{i=1}^{m} P(K = k_i; \Theta) \prod_{j=1}^{n} P(V = v_j; \Theta).
\]  

(9)

If the 2D Boolean model depends on a parameter vector \( \Theta \), then the maximum likelihood estimate is the parameter value maximizing \( L \) in Eq. (9).

4. ESTIMATION FOR RANDOM-RECTANGLE MODEL

To compare this estimator to previous work, we generated the same random-rectangle processes as those in [9] and discussed in Section 2. Widths and heights of rectangles were independent with shifted Poisson distributions (Poisson variate plus one). Fifty realizations of 512 × 512 images were generated for each process with marking.

FIG. 2. Realization of a random rectangle process.
probabilities ranging from $p = 0.01$ to 0.10 in increments of 0.01 (producing vacancies from 0.84 to 0.16), mean width $\mu = 6.0$, and mean height $\theta = 3.0$. Figure 2 shows a typical simulated image with 50% vacancy ($p = 0.04$). We also simulated a model in which the rectangle height is a shifted Poisson variate plus the width so that the widths and heights of rectangles are dependent (cross-dependent model with $\theta = \mu = 3.0$). Sampling was done along horizontal scanlines spaced vertically 10 pixels apart and within each scanline at points spaced 10 pixels apart. Pixels on adjacent scanlines were staggered 5 pixels. This procedure resulted in 2,576 samples for each image. The likelihood function in Eq. (9) was implemented using the induced distributions in Eqs. (3) and (4) to compute Eqs. (7) and (8) and was maximized by numerical methods. Maximum-likelihood estimates for the particular image for Fig. 2 are $\hat{p} = 0.04$, $\hat{\theta} = 3.11$, and $\hat{\mu} = 5.89$. Figures 3, 4, and 5 show the estimated bias for the cross-estimation procedure compared to the linear sampling method of [9]. Bias for both methods are similar, except that, when the vacancy is very low and the samples are dominated by long black runs, the cross-sample method has higher bias. Figures 6, 7, and 8 show the coefficient of variation for the cross-sample estimator and the linear-sample estimator. The cross-windowed sampling technique results in a significant decrease in all coefficients of variation, at least by half and by as much as one tenth. By sampling in a cross window, we capture information about how the horizontal and vertical runlengths vary jointly. This carries more shape information than a single horizontal or vertical runlength. It is also clear that width and height dependency has little effect on the quality of the estimator except when vacancy is at the extremes.

Idealized models such as the random rectangle model can be used to fit other processes as an approximation. For example, Diggle used a random disk model to fit spatial data of heather in a forest using a least-squares fit of point covariance [3; 7, p. 301]. To see how we can use this cross-window estimation technique on a real texture, we fit a 166 × 226 pixel section of a thresholded image of gravel (left image of the stereo pair “Rocks and Gravel” from University of Southern California IRIS image database, captured by W. A. Hoff; see also [11]) with a random rectangle Boolean model. The maximum-likelihood esti-
functions and the notion of a primary random grain is replaced by a primary random function (see [14, 16, 17] for mathematical descriptions of the model). Taking the maximum of random translations yields a model of rough surfaces. Although the model is quite general, as in the Boolean random set model, one typically restricts the primary function to parameterized shapes such as half-spheres [2, 5] or, in our case, random-height frusta. In the digital case, the primary function takes values in some interval $[0, M]$, where $0$ represents black (absence of objects) and $M$ is the maximum brightness. A Bernoulli point process in the discrete plane provides germs where random functions with bounded supports are placed. As in the 2D Boolean random set model, the random process can be viewed as a set of independent random variables indexed by the discrete grid. At each coordinate, a trial determines whether or not a function appears and, independently of this marking trial, other random variables determine the size and shape of the function to be placed there. The maximum of all outcomes produces a random surface analogous to the germ-grain process in the set model. In image processing, surface values are gray levels. Estimation involves the intensity parameter governing translation locations in the plane, as well as the parameters of the distributions governing the random function. Estimation is more difficult than for the Boolean random set model, since not only does planar overlapping obscure spatial distribution, but vertical information is obscured by translations of the primary function covering other translations, many completely. We shall consider a model whose primary function is a random frustum. In the present paper, we adapt the multidimensional linear sampling approach to perform maximum-likelihood estimation for Boolean random functions, the goal being to construct a cross-window likelihood function for the Boolean random function similar to the one for the 2D Boolean random set.

Just as in the random set model, we estimate the random function model by linear samples. Under appropriate conditions of a 2D random set, the random process induced on an intersecting line is a 1D Boolean model. For the random function case, we need a 1D Boolean model induced by a horizontal line intersecting with its umbra. Let us first consider how a 1D random function induces a 1D Boolean model. Table 1 shows how a typical primary function emanating from $x = 2$ induces a line segment at $x = 4$ on the line $z = 3$.

The “center” of the line segment is its leftmost point (smallest $x$-coordinate). Let $\sigma(4, 3)$ be the set of points on the line $z = 1$ where a primary function can emanate and intersect the line $z = 3$ in such a way that the smallest $x$-coordinate coincides with the point $(x, z) = (4, 3)$. In Table 1 with an isosceles right-triangle primary function, there is only one such point: $(x, z) = (2, 1)$. The segment-length random variable, $X_{4,3}$, is a function of outcomes of
random functions emanating from the single point in \( \mathcal{A}(4, 3) \). In general, \( \mathcal{A}(x, z) \) need not consist of only a single point.

Because a 1D Boolean model is characterized by a sequence of i.i.d. segment-length random variables, we must ensure that the random variables \( \{X_{i,3}\}, i = \ldots, -1, 0, 1, \ldots \) are i.i.d. As for \( \{X_{i,3}\} \) having a common distribution, this follows by stationarity of the random variables governing the Boolean random function. A sufficient condition for independence of \( X_{i,3} \) and \( X_{i',3} \) is that \( \mathcal{A}(i, 3) \) and \( \mathcal{A}(i', 3) \) be disjoint if \( i \neq i' \). Disjointness of the \( \mathcal{G} \)-sets at level \( z \) implies that each \( \mathcal{G} \)-set is a distinct singleton, and conversely. A sufficient condition for the \( \mathcal{G} \)-sets at each level to be distinct singletons is that the union of the left borders of the primary function outcomes be one-pixel thick. This follows by the inducement analysis that appears in [10] where it is also shown how to achieve disjointness through model construction. Segment-length random vari-

\[ \begin{array}{cccccccccccc}
5 & 5 \\
4 & 4 & 4 & 4 \\
3 & 3 & 3 & 3 & 3 & 3 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
\end{array} \]

**TABLE 1**

Simple Primary Function
ables on parallel lines are quite likely to be dependent. In the example of Table 1, \(X_{3,3}\) and \(X_{5,4}\) share the same \(\mathcal{G}\)-set.

A 2D Boolean random function can induce a 2D Boolean random set model in a manner similar to the induction of 1D models from 1D functions. The *umbra* of a function is its graph and points beneath it (see below for a mathematical definition). If we consider a plane \(Z = z\), its intersection with the primary-function umbra produces grains in the plane. If the placement of the grains is done according to a Bernoulli process, then a 2D Boolean model is induced by the Boolean random function via intersection. We seek to identify i.i.d. random sets in the plane \(Z = z\) just as we sought i.i.d. random variables on the line in the 1D case. The umbra of each translated primary function can be thought of as a stack of grains (we assume the unbrae are suitably convex). Each grain in the stack has a center from which it will be considered to emanate if it intersects the plane. At a given point \((x, y, z)\) in the plane \(Z = z\), one must consider all the positions in the plane \(Z = 1\) where a function can emanate to intersect the plane \(Z = z\) with the center of the grain coinciding with \((x, y, z)\). The set of these points we denote by \(\mathcal{G}(x, y, z)\). If the sets \(\mathcal{G}(i_2, j_2, z)\) and \(\mathcal{G}(i_1, j_1, z)\) are disjoint for \((i_1, j_1) \neq (i_2, j_2)\), then the germ process in the plane \(Z = z\) is Bernoulli and a 2D Boolean model is induced. Once again, the \(\mathcal{G}\)-sets for each level are singleton if they are disjoint. A sufficient condition for the \(\mathcal{G}\)-sets to be distinct singletons at a given level is that the shapes induced by the function at that level by functions emanating from a single point share a common center. This condition can be verified by inspection. If disjointness is not satisfied, a similar model which does induce can be constructed using a method described in [10]. The center for each grain at each level can be properly chosen for inducement, provided the number of outcomes is bounded. To complete the analysis, one must also compute the probabilities of the grains. Once we have the Boolean model in a plane, we can invoke all the previous analysis to determine the 1D Boolean models induced on lines parallel to the \(Z = 1\) plane, in the \(x\) and \(y\) directions.

Given inducement, one can analyze the Boolean random function by analyzing the 1D Boolean random sets it induces; however, now we must include a \(z\)-value with the observation. This is accomplished by volume sampling in three dimensions. Assuming a square image plane \([1, N] \times [1, N]\), sample points are selected within the box \(B = [1, N] \times [1, N] \times [1, M]\). Let \(T\) denote the random gray-value of the Boolean random function and let \(U[T]\) be the set of all points \((x, y, z)\) in \(B\) such that \(T(x, y) \geq z\). \(U[T]\) is called the umbra of \(T\) and volume points in \(U[T]\) lie beneath the surface defined by \(T\). If the point \(\mathbf{p} = (x, y, z)\) is in \(U[T]\) and we take \(x\), \(y\), and \(z\)-direction runlengths \(U[T]\), which setwise we consider to be black, then (as in the 2D Boolean random set model) we obtain a vector of four black spatial runlengths,

\[
K(\mathbf{p}) = (K_{\bar{x}}(\mathbf{p}), K_{\bar{y}}(\mathbf{p}), K_{\bar{y}}(\mathbf{p}), K_{\bar{y}}(\mathbf{p})),
\]

and a vertical black runlength \(T_b(\mathbf{p})\), which is actually the...
function value at \((x, y), T_b(p) = T(x, y)\). If, on the other hand, \(p = (x, y, z) \notin U[T]\), and we take \(x, y, z, z\)-direction runlengths in \(U[T]\), which setwise we consider to be white, then (as in the 2D Boolean random set model) we obtain a vector of four white spatial runlengths,

\[
V(p) = (V_{x}(p), V_{y}(p), V_{w}(p), V_{v}(p)),
\]

and a vertical white runlength \(T_w(p) = M - T(x, y)\). The situations for both \(p \in U[T]\) and \(p \notin U[T]\) are depicted in Fig. 10 for the case of a 1D Boolean random function.

The approximate probabilities for \(K(p)\) and \(V(p)\) are computed via Eqs. (7) and (8), using the segment-length distributions induced at \(p\). The conditional probabilities relating to \(T_b\) and \(T_w\) are determined by the probability distribution \(F_T\) of \(T\). \(F_T\) is the first order probability distribution of the random function and, owing to stationarity, is independent of the point \((x, y)\). Given a sampling point \(p = (x, y, z) \in U[T]\) (\(p\) is black), the probability mass for \(T_b\) is given by

\[
P(T_b = t_b) = P(T = T_b | T \geq z).
\]

Given a sampling point \(p = (x, y, z) \notin U[T]\) (\(p\) is white), the probability mass for \(T_w\) is

\[
P(T_w = t_w) = P(T = M - T_w | T < z).
\]

To arrive at a likelihood function, we simply multiply the probabilities, even though the random variables are not independent. Doing this obligates us to investigate the statistical properties of the resulting maximum-likelihood estimator by simulations. Keeping in mind the conditioning in Eqs. (12) and (13), we compute the probabilities of black and white three-dimensional cross-observations. Given \(p = (x, y, z) \in U[T]\),

\[
P(K(p) = k, T_b = t_b) \approx \frac{P(K(p) = k) f_T(t_b)}{1 - F_T(z - 1)},
\]

for \(t_b = z, \ldots, M\). Given \(p = (x, y, z) \notin U[T]\),

\[
P(V(p) = v, T_w = t_w) \approx \frac{P(V(p) = v) f_T(t_w)}{F_T(z - 1)},
\]

for \(t_w = 0, \ldots, z - 1\). Assuming widely spaced observations throughout an observed image (\(m\) black and \(n\) white), the grand likelihood function is given by

\[
L(k_1, t_{b,1}, \ldots, k_m, t_{b,m}, v_1, t_{w,1}, \ldots, v_n, t_{w,n}) = \prod_{i=1}^{m} P(K = k_i, T_b = t_{b,i}) \prod_{j=1}^{n} P(V = v_j, T_w = t_{w,j}).
\]

### 6. FRUSTUM BOOLEAN RANDOM FUNCTION

Consider first a primary function given a pyramid with random height \(U\) and square base \((2U - 1) \times (2U - 1)\), where \(U \in \{1, \ldots, M\}\). We define a set of independent random variables \(U_{i,j}\) governing the placement and size of the pyramids. If \(U_{i,j} = 0\), then no pyramid is positioned at \((i, j)\); if \(U_{i,j} = u > 0\), then a pyramid of peak gray-value
u is placed with its lower left corner at \((i, j)\). If \(p\) is the marking probability, then \(P(U_{ij} = 0) = 1 - p\); \(P(U_{ij} = u) = pP(U = u)\) for \(u = 1, \ldots, M\). Pyramid bases emanate in the \(x\) and \(y\) directions. Now suppose a random rectangle is generated and each level of the pyramid is dilated by it. The result is a random frustum primary function governed by three parameters: the intensity \(p\), a width parameter \(\theta\), and a length parameter \(\mu\). A random surface results from taking the maximum over all translated random frusta.

We now derive the 2D Boolean models produced by the frustum random function on each gray-level plane \(Z = z, z = 1, \ldots, M\). A pyramid of height \(u\) consists of a stack of \(u\) odd-sized squares with sizes: \((2u - 1) \times (2u - 1), (2u - 3) \times (2u - 3), \ldots, 1 \times 1\). If we take the center of each square to be its smallest \((x, y)\) coordinate, then grains produced by a random pyramid dilated with a random rectangle of size \(i \times j\) and intersecting the plane \(Z = z\) can be seen to be rectangles of size \(2(u - z) + i \times 2(u - z) + j\) emanating in the \(x\) and \(y\) directions (see Fig. 11). Without loss of generality, consider the point \((0, 0, z)\). There is only one point in the plane \(Z = 1\) that can produce a rectangle at that point at that level: \((-z + 1, -z + 1, 0)\). For this model it is easy to see that, in general \(\alpha(x, y, z) = \{(x - z + 1, y - z + 1, 0)\}\). For any given gray level \(z\), a 2D Boolean model is induced because distinct points on the plane \(Z = z\) have disjoint \(\omega\)-sets. The grain processes are different at each level. Let \(F(w, h; z)\) be the distribution of the widths and lengths of the random rectangles produced at level \(z = 1, \ldots, M\) and \((I, J)\) be the random rectangle size random variables for the dilating rectangle.

\[
P(U \leq u) = 1 - p + \frac{p}{M} I_{[1,M]}(u),
\]

for \(u = 0, 1, \ldots, M\). We assume that the maximum gray value \(M\) is given and estimate \((p, \theta, \mu)\), where \(\theta\) and \(\mu\) are the parameters of the width and height distributions, respectively.

To investigate the bias and coefficient of variation (cv) for the estimator of \((p, \theta, \mu)\), we use a range of images with \(p = 0.04\) to \(p = 0.08\), \(M = 15\), \(\theta = 3\), and \(\mu = 6\) (see Fig. 12). We chose this range because these marking probabilities produce interesting textures. For each value of \(p\), we have generated 50 realizations of \(512 \times 512\) images and estimated \((p, \theta, \mu)\) two ways. The first estimator is based on the distribution of heights or gray levels [5].

### Table 4
Estimated Bias of the Length Estimators

<table>
<thead>
<tr>
<th>(p)</th>
<th>(\hat{\rho}_{\text{ln}})</th>
<th>(\hat{\rho}_{\text{mle}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.04</td>
<td>0.386</td>
<td>0.201</td>
</tr>
<tr>
<td>0.05</td>
<td>3.110</td>
<td>0.302</td>
</tr>
<tr>
<td>0.06</td>
<td>1.862</td>
<td>0.340</td>
</tr>
<tr>
<td>0.07</td>
<td>0.168</td>
<td>0.276</td>
</tr>
<tr>
<td>0.08</td>
<td>-1.013</td>
<td>0.233</td>
</tr>
</tbody>
</table>

### Table 5
Estimated cv of the Intensity Estimators

<table>
<thead>
<tr>
<th>(p)</th>
<th>(\hat{\rho}_{\text{ln}})</th>
<th>(\hat{\rho}_{\text{mle}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.04</td>
<td>0.179</td>
<td>0.028</td>
</tr>
<tr>
<td>0.05</td>
<td>0.072</td>
<td>0.029</td>
</tr>
<tr>
<td>0.06</td>
<td>0.184</td>
<td>0.026</td>
</tr>
<tr>
<td>0.07</td>
<td>0.128</td>
<td>0.019</td>
</tr>
<tr>
<td>0.08</td>
<td>0.068</td>
<td>0.014</td>
</tr>
</tbody>
</table>

### Table 6
Estimated cv of the Width Estimators

<table>
<thead>
<tr>
<th>(p)</th>
<th>(\hat{\rho}_{\text{ln}})</th>
<th>(\hat{\rho}_{\text{mle}})</th>
</tr>
</thead>
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<td>0.04</td>
<td>0.600</td>
<td>0.092</td>
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<tr>
<td>0.05</td>
<td>0.301</td>
<td>0.122</td>
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<tr>
<td>0.06</td>
<td>0.920</td>
<td>0.132</td>
</tr>
<tr>
<td>0.07</td>
<td>0.748</td>
<td>0.108</td>
</tr>
<tr>
<td>0.08</td>
<td>0.453</td>
<td>0.095</td>
</tr>
</tbody>
</table>
uses the same points as the next estimator. The other estimator is the maximum-likelihood estimator with sample points in the plane spaced \(2(M + \theta)\) pixels horizontally and \(2(M + \mu)\) vertically, yielding 174 samples for each image. The \(z\)-level at each sample point was chosen to be the median gray level over the sample points plus an offset of \(-2, -1, 0, 1, \) or \(2\) gray levels. The median was chosen to yield a balanced selection of white and black runlengths. The offset for the \(i^{th}\) sample point is \(3 - i \mod 5\). This sampling method is designed to avoid long black runlengths which we know from our previous work on 1D estimation introduces a bias in the estimates [4]. Estimates were computed by numerically maximizing Eq. (15) over \((p, \theta, \mu)\). The average of the estimates from 50 images was used to estimate the bias for each estimator. Tables 2, 3, and 4 show the results. The absolute value of the bias of the cross-mle is consistently less than or equal to that of the height-based estimator. Tables 5, 6, and 7 show that the cv for the cross-mle remains stable throughout the range of \(p\) while the cv’s for the height-estimator vary wildly and are much greater.

### 7. CONCLUSION

An approximate maximum-likelihood estimator for the discrete 2D Boolean model based on cross-windowed observation statistics at widely-spaced samples of a germ-grain model produces a superior estimator than one using runlength statistics from linear samples. Estimation based on this method can also be used to model textures by fitting regular geometric figures to real data. This method can be extended to provide an approximate maximum-likelihood estimator for the Boolean random function model. The estimator has been applied to estimate parameters of a random-height frustum model.

<table>
<thead>
<tr>
<th>(p)</th>
<th>(\rho_{h0})</th>
<th>(\rho_{mle})</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.152</td>
<td>0.061</td>
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<tr>
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<td>0.460</td>
<td>0.066</td>
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<tr>
<td>0.07</td>
<td>0.374</td>
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<tr>
<td>0.08</td>
<td>0.227</td>
<td>0.048</td>
</tr>
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### REFERENCES