Numerical and perturbative computations of solitary waves of the Benjamin–Ono equation with higher order nonlinearity using Christov rational basis functions

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Abstract

Computation of solitons of the cubically-nonlinear Benjamin–Ono equation is challenging. First, the equation contains the Hilbert transform, a nonlocal integral operator. Second, its solitary waves decay only as $O(1/|x|^2)$. To solve the integro-differential equation for waves traveling at a phase speed $c$, we introduced the artificial homotopy $H(u_{xx})/c + (1 - \delta)u^2 + \delta u^3 = 0, \delta \in [0, 1]$ and solved it in two ways. The first was continuation in the homotopy parameter $\delta$, marching from the known Benjamin–Ono soliton for $\delta = 0$ to the cubically-nonlinear soliton at $\delta = 1$. The second strategy was to bypass continuation by numerically computing perturbation series in $\delta$ and forming Padé approximants to obtain a very accurate approximation at $\delta = 1$. To further minimize computations, we derived an elementary theorem to reduce the two-parameter soliton family to a parameter-free function, the soliton symmetric about the origin with unit phase speed. Solitons for higher order Benjamin–Ono equations are also computed and compared to their Korteweg–deVries counterparts. All computations applied the pseudospectral method with a basis of rational orthogonal functions invented by Christov, which are eigenfunctions of the Hilbert transform.

1. Introduction

The Benjamin–Ono equation, sometimes called the Benjamin–Davis–Ono equation, was independently derived by Benjamin [6] and Davis and Acrivos [20] through singular perturbation theory as an approximation to weakly nonlinear water waves in deep water:

$$u_t + uu_x + H(u_{xx}) = 0 \quad [\text{Benjamin–Ono equation}],$$

where a subscripted coordinate denotes differentiation with respect to that coordinate and $H$ is the Hilbert transform defined by

$$H[f](x) \equiv \frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{f(y)}{y-x} \, dy,$$

where $\mathcal{P}$ denotes the Cauchy principal value. The solitary waves are [35]
where the phase speed is \( c \) and \( \phi \) is an arbitrary constant.

The Benjamin–Ono equation is integrable and may be solved by the inverse scattering method [7, 1, 21, 26, 27, 30, 32, 31]. Unfortunately, when the equation is generalized by replacing the quadratic nonlinearity by a cubic or higher nonlinearity, the resulting generalized Benjamin–Ono equation is not integrable nor is the soliton known in explicit form for any \( m > 1 \):

\[
    u_t + u^m u_x + H(u_{xx}) = 0 \quad \text{GBO Equation},
\]

where \( m > 0 \) is a constant. Bona and Kalisch [8] have numerically studied the quartic nonlinearity (\( m = 3 \)). Previous theoretical investigations, reviewed in their article, were unable to prove that solution is nonsingular for \( m \neq 2 \).

In this work, we specialize to solitary waves propagating steadily at a speed \( c \). Introducing the translating coordinate

\[
    X \equiv x - ct,
\]

the generalized Benjamin–Ono equation becomes the ordinary integro-differential equation (OIDE)

\[
    H(u_{xx}) - cu_x + u^m u_x = 0, \quad X \in [-\infty, \infty] \quad \text{GBO Traveling Wave OIDE},
\]

subject to the boundary condition \( u \to 0 \) as \( |X| \to \infty \).

2. Parameter reduction

For a given nonlinearity exponent \( m \), it is sufficient to compute the soliton for unit phase speed with its maximum at \( X = 0 \) because the general two-parameter soliton can be trivially obtained from this special case by the following.

**Theorem 2.1.** If \( u(X;m) \) solves

\[
    H(u_{xx}) - cu_x + u^m u_x = 0, \quad X \in [-\infty, \infty],
\]

subject to the boundary condition \( u \to 0 \) as \( |X| \to \infty \) for \( c = 1 \), then

\[
    \nu(X,c,\phi,m) \equiv c^{1/m} u(c|X - \phi|; m)
\]

is also a solution (with the same \( c \) in both solution and differential equation) for arbitrary constants \( c > 0 \) and \( \phi \).

**Proof.** Substitution of \( \nu(X,c,\phi,m) \) into the OIDE followed by cancellation of a common factor of \( c \). \( \Box \)

If \( m \) is even, solitons of both signs occur, but it is sufficient to compute only solitary waves of elevation because of the following.

**Theorem 2.2.** If \( m \) is even, then if \( u(X;m) \) is a traveling wave solution to the generalized Benjamin–Ono equation, so also is

\[
    \nu(X,c,\phi,m) = -u(X,c,\phi,m)
\]

for arbitrary \( \phi \) and positive \( c \).

**Proof.** substitution of \( (-u) \) into the OIDE; for even \( m \), the signs cancel. \( \Box \)

The solitary waves of the ordinary Benjamin–Ono equation are symmetric about their peak. We will assume that the same is true of generalized Benjamin–Ono solitons as well. Our successful computations verify that this assumption is true for at least some classes of GBO solitary waves. There is no theoretical proof at present that excludes the existence of unsymmetrical solitons.

3. Christov rational orthogonal functions

Grosch and Orszag [22] and Boyd [9–11] have shown the usefulness of rational orthogonal functions which are the images of a Fourier cosine series under a change of coordinate:

\[
    TB_n(x; L) = \cos(nt),
\]

\[
    x = L \cot(t), \quad L = \text{constant}.
\]

Norbert Wiener defined a different set of complex-valued orthogonal rational functions as transforms of Laguerre functions. These were developed further by Higgins [23] and applied successfully to time integration of the Benjamin–Ono equation by James and Weideman [24, 40] and to computation of the complex error function [39].
where $L$ is the map parameter.

However, no simple form for $\hat{\mathcal{H}}[T_{B_n}(x; L)]$ is known. The Wiener–Higgins–Weideman functions are eigenfunctions of the Hilbert transform, but have disadvantage of being complex-valued even though the solutions of the Benjamin–Ono equation and its generalizations are real-valued.

We therefore chose a basis devised by Christov [19,4,18]. The relationship between the three species of rational functions is discussed in [12,33,34].

The Christov functions are defined by a change-of-coordinate via

$$x = L \cot(s/2), \quad L = \text{constant},$$

$$CC_{2n}(x) = \cos(ns) - \cos((n + 1)s),$$

$$SC_{2n+1}(x) = \sin((n + 1)s) - \sin(ns),$$

where $L > 0$ is a user-choosable constant, the “map parameter”. We shall discuss the choice of $L$ in the next subsection.

Evaluating the Hilbert transform in the Christov basis is easy because the Hilbert transform of a Christov function is just another Christov function.

**Theorem 3.1** (Hilbert transform of Christov functions).

$$\hat{\mathcal{H}}[CC_{2n}(x)] = -SC_{2n+1}(x),$$

$$\hat{\mathcal{H}}[SC_{2n+1}(x)] = CC_{2n}(x).$$

**Proof.** Boyd [12] has shown that the Christov functions can be written in terms of the Higgins functions $\phi_n$ as

$$CC_{2n}(x) = \frac{1}{2} \{ \phi_n(x) - \phi_{n-1}(x) \}, \quad SC_{2n+1}(x) = \frac{1}{2i} \{ \phi_n(x) + \phi_{n-1}(x) \}, \quad n = 0, 1, \ldots, \infty,$$

where

$$\phi_n(x) = \frac{(x - iL)^n}{(x + iL)^n+1}.$$  \hfill (13)

Weideman, whose definition of $\phi_n$ differs from ours only by an irrelevant multiplicative factor of $i$, shows that

$$\hat{\mathcal{H}}[\phi_n(x)] = i \text{Sgn}(n) \phi_n(x),$$

where Sgn is the usual sign function except that Sgn (0) = 1 instead of the standard sgn (0) = 1. It then follows that

$$\hat{\mathcal{H}}[CC_{2n}(x)] = \frac{1}{2} \{ \hat{\mathcal{H}}[\phi_n(x)] - \hat{\mathcal{H}}[\phi_{n-1}(x)] \} \quad n \geq 0,$$

$$= \frac{i}{2} \{ \phi_n(x) + \phi_{n-1}(x) \},$$

$$= -SC_{2n+1}(x).$$  \hfill (17)

The proof of the second identity is similar. \hfill $\Box$

The derivatives inside the Hilbert Transform in the generalized Benjamin–Ono equation cause no difficulty because of the following:

**Theorem 3.2** (Hilbert Transform of a Derivative). If $f(x)$ is a function and $g(x)$ is its Hilbert transform, $\mathcal{H}[f(x)] = g(x)$, then the Hilbert transform of $df/dx$ is $dg/dx$. In other words, differentiation (to all orders) commutes with the Hilbert Transform operator. ([44, Proposition 6 on pg. 296].)

Consequently, $\hat{\mathcal{H}}[u_{ss}]$ can be evaluated as $\partial_{ss}(\hat{\mathcal{H}}[u])$ if convenient as done in our code.

Derivatives of the Christov functions can be easily evaluated by first differentiating the trigonometric form in $s$ and then applying the chain rule to convert $s$-derivatives to $x$-derivatives.

$$\frac{dCC_{2n}}{dx}(x) = -2 \frac{\sin^2(s/2)}{L} \{ -n \sin(ns) + (n + 1) \sin((n + 1)s) \},$$

$$\mathcal{H} \frac{d^2CC_{2n}}{dx^2}(x) = -4 \frac{\sin^3(s/2)}{L^2} \{ \sin(s/2) \{ -(n + 1)^2 \sin([n + 1]s) + n^2 \sin(ns) \} + \cos(s/2) \{ (n + 1) \cos([n + 1]s) - n \cos(ns) \} \}.  \hfill (19)$$
The Christov functions of even degree are symmetric about the origin whereas the \( SC_{2n+1}(x) \) are antisymmetric. To remove the translational degree of freedom and simultaneously exploit the assumed symmetry of the solitary wave with respect to its peak, we restrict the basis to the symmetric functions \( CC_{2n}(x) \):

\[
u(X) \approx \sum_{j=0}^{N-1} a_j CC_2(X), \tag{20}\]

where \( N \) is the user-chosen truncation of the infinite series. The symmetry of the basis guarantees that \( du/dX(0) = 0 \).

The collocation points for the pseudospectral method are similarly restricted to lie on the half-line \( X \in [0, \infty] \). The optimum points are the images of a uniformly spaced grid in \( s \), which has long been known to be optimum for Fourier computations:

\[
s_j = \frac{2j - 1}{2N} \longrightarrow X_j = L\cot(s_j/2) \quad j = 1, 2, \ldots, N. \tag{21}\]

The nonlinear ordinary integro-differential equation is then discretized by demanding that the residual of the integro-differential equation should be zero at each of the \( N \) collocation points where \( N \) is the number of terms in the truncated series of \( CC_2(x) \). The Christov functions individually satisfy the boundary conditions that \( u \to 0 \) as \( |x| \to \infty \), so it is not necessary to reserve rows of the discretized system of nonlinear equations to impose boundary conditions.

The \( N \) residual collocation conditions are solved by Newton’s iteration. This requires an initialization \( u^{\text{init}}(X) \). We shall describe methods for generating this function in the next section. Interpolation is used to convert the values of \( u^{\text{init}}(X_j) \) to the initial Christov coefficients \( a_j^{\text{init}} \). Define the elements of the square interpolation matrix by \( M_{ij} \equiv CC_{2j-2}(X_i) \). Denoting the vector of initial Christov coefficients by \( \vec{a}^{\text{init}} \) and the values of \( u^{\text{init}}(X) \) on the interpolation grid by \( \vec{u}^{\text{init}} \), the coefficients of the Christov interpolant are found by solving the matrix equation.

\[
\vec{M} \vec{a}^{\text{init}} = \vec{u}^{\text{init}}. \tag{22}\]

The first author’s book [14] reviews the pseudospectral method in more greater detail than is appropriate here. The exigencies of Newton’s iteration for soliton-finding are catalogued in [13,15,16].

3.1. The Choice of the map parameter \( L \)

Fig. 1 shows the sensitivity of the Christov basis to the choice of the map parameter \( L \). A one-hundred basis function solution was truncated to include only \( N_T \) CC functions and the maximum error on \( X \in [0, \infty] \) for the shortened series was computed. For a given truncation \( N_T \), there is an optimum \( L \) which increases with increasing truncation as catalogued in Table 1. (This is a trend often observed with rational Chebyshev series, too.) However, using \( L \) which is half or double the optimum still yields high accuracy for sufficiently large \( N \). The choice of \( L \) is not very critical. Table 2 shows that high accuracy is achieved even for small \( N \).
When the soliton for the ordinary Benjamin–Ono equation is used as the first guess for the solution to the cubic Benjamin–Ono equation, Newton’s iteration for the Christov-discretized equation fails. A useful strategy is to make the problem more complicated by adding an artificial parameter, \( \delta \in [0, 1] \), to interpolate between these two cases:

\[
H(u_{XX}) - u_X + (1 - \delta)u_{XX} + \delta \delta^2 u_X = 0.
\]

This is a very general trick: to solve a problem \( \mathfrak{H}(u) = 0 \) by solving an inflated problem that interpolates between the target at \( \delta = 1 \) and a problem with a known solution, \( \mathfrak{H}_0(u) = 0 \) at \( \delta = 0 \),

\[
(1 - \delta)\mathfrak{H}_0(u) + \delta \mathfrak{H}(u) = 0.
\]

This inflation trick is often called a “homotopy” method by numerical analysts.

The Benjamin–Ono soliton for \( \delta = 0 \) is successful as an initialization for Newton’s iteration for sufficiently small \( \delta \). Once a solution has been obtain for \( \delta = \delta_1 \), one can use it as a first guess for \( \delta = 2\delta_1 \), and thus proceed in small steps in \( \delta \) to \( \delta = 1 \). This
stepwise numerical March is the “continuation” method described well in texts such as [36]. For this problem, six steps failed, but ten steps was successful.

In theory, the continuation method will always succeed for a sufficiently small step size in \( \delta \), but both of us bear battle scars from applying continuation in the real world as explained in [16]. We therefore experimented with perturbative continuation as described in the next section.

5. Perturbation series in an artificial parameter

5.1. Unconventional Perturbation Parameters: 3.99 dimensions, the 1/N expansion and all that

The introduction of an artificial parameter \( \delta \) may seem unorthodox, but there is a very long tradition of inserting a parameter where none is obvious, and employing either perturbation series or numerical continuation in the artificial parameter. Sometimes a very unobvious choice of perturbation parameter is best. In the theory of critical phenomena, an important tool is perturbation in the difference between the dimension \( d \) and 4, which led to a famous paper with the amusing title “phase transitions in 3.99 dimensions” [42,41]. In the words of the physicist and Fields Medalist Ed Witten, “After decades in which the study of critical phenomena was thwarted by the absence of an expansion parameter, [Nobel Laureate] Kenneth Wilson and Michael Fisher suggested that to introduce a parameter, one should regard the number of spatial dimensions not in which the study of critical phenomena can be understood by perturbation theory in \( d \). Even at \( d = 4 \), the original three-dimensional problem, this perturbation theory is quite successful” [43].

A number of other problems simplify when the dimension \( N \) is infinite, and the perturbation parameter is the reciprocal of \( N \). As Witten explains, the \( 1/N \) series was first invented in atomic physics, but the Nobel Laureate Gerald t’Hooft extended it to quantum chromodynamics (QCD) in 1974 by pretending there are \( N \) quarks instead of the three quarks of standard QCD.

Similarly, Bender, Milton, Pinsky and Simmons treated the exponent of nonlinearity, which in most applications is restricted to an integer, as a continuous small parameter [5]. Thus, a polynomial equation may be rewritten as

\[
u^5 + au + b = 0 \rightarrow u^{1+\delta} + au + b = 0
\]

and solved as a perturbation series in powers of \( \delta \). This article has been cited more than a hundred times and has spawned many applications including differential equations as well as algebraic equations.

Solving the inflated “homotopy” problem by perturbation theory is often called the “linear delta expansion” by physicists [2,3].

5.2. The Korteweg–deVries Example

The cubic KdV, also called the “Modified KdV” (MKdV) in the literature, can be connected to the usual KdV through

\[u_{xx} - u + (1 - \delta)u^2 + \delta u^3 = 0.
\]

This is a useful illustration because the cubic KdV soliton is known in explicit form and all orders of the perturbation series can be expressed as polynomials.

The initial and final points on \( \delta \in [0,1] \) are

\[u(X;\delta = 0) = (3/2)\text{sech}^2(X/2) \quad \text{[KdV]}, \]
\[u(x;\delta = 1) = \sqrt{2}\text{sech}(X) \quad \text{[MKdV]}.
\]

A valuable trick is to make the change of coordinate

\[z \equiv \tanh(X/2) \rightarrow X = 2 \arctanh(z).
\]

This allows one to convert each perturbative order into a finite polynomial in \( z \). This technique is illustrated in detail with a table of helpful identities and an application to shocks of the KdV-Burgers equation in Appendix B of [13]. Useful auxiliary formulas needed here include

\[\text{sech}^2(X/2) = 1 - z^2; \quad \text{sech}(X) = \frac{1 - z^2}{1 + z^2}.
\]

In \( z \), the KdV homotopy ODE becomes

\[(1 - z^2)((1 - z^2)u_{zz} - 2zu_z) - 4u + 4(1 - \delta)u^2 + 4\delta u^3 = 0
\]

with the limits

\[u(x;\delta = 0) = (3/2)(1 - z^2), \quad u(x;\delta = 1) = \sqrt{2}\frac{1 - z^2}{1 + z^2}.
\]

Substituting
\[ u(x, \delta) = \frac{3}{2}(1 - z^2) + \delta u_1(z) + \delta^2 u_2(z) + \ldots \]  

(33)

into the ODE, matching powers of \( \delta \), and solving the linear inhomogeneous problem at each order by matching powers of \( z \) gives

\[
\begin{align*}
    u(z, \delta) &= \frac{3}{2} - 3/2z^2 + \left(-\frac{3}{16} - \frac{3}{2}z^2 + \frac{27}{16}z^4\right)\delta + \left(\frac{15}{64} + \frac{51}{128}z^2 + \frac{81}{64}z^4 - \frac{243}{128}z^6\right)\delta^2 \\
    &+ \left(\frac{327}{1024} + \frac{141}{256}z^2 - \frac{25}{256}z^4 + \frac{243}{128}z^6 + \frac{2187}{256}z^8\right)\delta^3 \\
    &+ \left(\frac{1905}{4096} + \frac{333}{1024}z^2 + \frac{513}{1024}z^4 + \frac{729}{8192}z^6 + \frac{2187}{4096}z^8 - \frac{19683}{8192}z^{10}\right)\delta^4 \\
    &+ \left(-\frac{23343}{32768} + \frac{1293}{1024}z^2 - \frac{54027}{1024}z^4 + \frac{243}{32768}z^6 + \frac{10935}{1024}z^8 + \frac{177147}{65536}z^{10}\right)\delta^5 \\
    &+ \left(\frac{148575}{131072} + \frac{1065471}{524288}z^2 + \frac{365067}{262144}z^4 + \frac{524288}{131072}z^6 - \frac{45927}{524288}z^{10} - \frac{531441}{262144}z^{12} - \frac{1594323}{524288}z^{14}\right)\delta^6,
\end{align*}
\]

(34)

A symbolic manipulation language such as Reduce or Mathematica is very good at manipulating polynomials; the complete Maple code is listed in Table 3.

Unfortunately, the perturbation series converges too poorly to be useful at \( \delta = 1 \). However, if we form Padé approximants from the power series, the sixth order approximation—a ratio of one cubic polynomial in \( \delta \) divided by another of the same degree—has a maximum pointwise error at \( \delta = 1 \) (the cubic-KdV soliton) of only 0.0009. The [3/3] Padé approximant is

\[
\begin{align*}
    u(X, \delta = 1) &\approx 1/2 \left( -\frac{2268721629}{512}z^6 + \frac{2076339987}{512}z^4 + \frac{136729053}{512}z^2 + \frac{55652589}{512} \right) \\
    &\left( \frac{108945405}{64}z^4 + \frac{5481351}{32}z^2 + \frac{1229823}{32} + \frac{100442349}{64}z^6 \right)^{-1},
\end{align*}
\]

(35)

where \( z = \tanh (X/2) \) as in (29). Of course, this rational function of hyperbolic functions is much more complicated than the exact Modified KdV soliton (28), but the success of the perturbation theory here encourages applications to other examples where the higher order solution is unknown.

In many physics problems, the rate of convergence can be improved by introducing an additional artificial parameter. Here, for example, a reviewer suggested solving

\[
u_{ax} - u + (1 - \delta)uz^2 + \delta uz^3 = 0,
\]

(36)

where \( z \) is an arbitrary parameter. Since the true answer is independent of \( z \), a logical strategem is to choose whichever \( z \) yields the slowest variation of the approximation. This is known as the Principle of Minimum Sensitivity [38,23]. The perturbative expansion without PMS was sufficiently good that we chose not pursue it here.

5.3. Perturbative extrapolation in \( \delta \) for the cubic Benjamin–Ono equation

Matching powers in \( \delta \), we find that at \( O(\delta^6) \), we must solve a linear differential of the form

\[
\mathcal{H}(u_{n,xx}) - u_{ax} + u_0u_{nx} + u_nu_{0x} = r_n(X).
\]

(37)

Table 3

<table>
<thead>
<tr>
<th>restart; umKdV := sqrt(2)*(1-z^2)/(1+z^2) ; norder := 6;</th>
</tr>
</thead>
<tbody>
<tr>
<td>uKdV := (3/2)*(1-z^2);</td>
</tr>
<tr>
<td>for j from 1 by 1 to norder do</td>
</tr>
<tr>
<td>u[j] := sum[a[i,j]<em>z^i</em>(2^k), k=0..(j+1)]; od;</td>
</tr>
<tr>
<td>u := uKdV + sum(u[j]p^&quot;delta&quot;*j, j=1..norder);</td>
</tr>
<tr>
<td>resid := (1-z^2)^2*(1-z^2)<em>diff(u,z,z)<em>z^2</em>diff(u,z) - 2</em>z^2<em>diff(u,z) - 4</em>u + 4*(1-2<em>delta)^2</em>u^2 + 4<em>delta^2</em>u^2*delta;</td>
</tr>
<tr>
<td>rescoll := collect(resid,delta);</td>
</tr>
<tr>
<td>for j from 1 by 1 to norder do res[j] := coeff(rescoll,delta,j); print(res[j]); eqset := []; for k from 1 by 1 to (j+1) do r[k] := coeff(res[j],z^2*(k-1)); eqset := eqset union r[k]; plot(k,r); od;</td>
</tr>
<tr>
<td>varset := []; for k from 0 by 1 to (j+1) do varset := varset union a[i,k]; od;</td>
</tr>
<tr>
<td>eqset, plot(eqset,varset); assign(eqset);</td>
</tr>
<tr>
<td>print(uPade); od;</td>
</tr>
<tr>
<td>uPade := convert(series(u0,delta),ratpoly); delta := 1; plot(uMKdV, uPade, z = 0..1);</td>
</tr>
</tbody>
</table>
where

\[
\begin{align*}
  r_1 &= -\left\{ u_0^2 u_{0,X} - u_0 u_{0,X} \right\} \\
  r_2 &= -\left\{ u_0^2 u_{1,X} + 2 u_0 u_{1,X} - u_0 u_{1,X} - u_1 u_{0,X} + u_1 u_{1,X} \right\}
\end{align*}
\]

and so on. The linear equations at each order cannot be solved explicitly, but the Christov pseudospectral method is just as effective as for the full nonlinear problem. (Indeed, the perturbation series can be regarded as a quasi-Newton method in which the linearization is not performed about the current iterate, but always about \( \delta = 0 \); this is called the “chord” or “Shamanski” method [37].)

Fig. 2 shows that the perturbation series has a radius of convergence of roughly \( \delta = 1/2 \). However, at \( \delta = 1 \)—the cubic Benjamin–Ono soliton—the [3/3] Padé approximant has an error of only about 0.03. This is more than sufficient so that Newton’s iteration will converge to the cubic BO soliton from this initialization.

Although this is not an important consideration for a one-space-dimensional soliton, the perturbative extrapolation is much cheaper than numerical continuation in floating point operations because only a single matrix need be computed and Cholesky-factored. Numerical continuation requires roughly \( n_s n_t \) such matrix computations where \( n_s \) is the number of continuation steps and \( n_t \) is the number of Newton iterations at each step.

6. The Cubic Benjamin–Ono soliton

The left panel of Fig. 3 compares the cubic Benjamin–Ono soliton of unit phase speed (solid) with its counterpart for the classical Benjamin–Ono equation. The cubic-BO soliton decays from its maximum of 3.282 at the origin to half of the maximum at \( |X| = 0.414 \). The decay for larger \( |X| \) is much slower, asymptotically proportional to roughly \( 4.15/|X|^2 \). (The asymptotic decay proportional to \( 1/|X|^2 \) is rigorously proved in a later section.)

It is generically the case that expansions in rational orthogonal functions (or Hermite functions or any other spectral scheme for the unbounded interval) will converge subgeometrically in the language of [14], that is, the coefficients \( a_n \) will fall proportional to \( \exp(-q n^r) \) for some positive constants \( q \) and \( r \) where \( r < 1 \). The reason is that functions on the unbounded interval are typically singular at infinity; \( r = 1/2 \) or \( r = 2/3 \) is common (Chapters 2 and 17 of [14]).

The right panel shows a geometric rate of convergence, that is, the coefficients fall roughly proportional to \( \exp(-1/3|n|) \).

Li and Bona [29] proved that, for a wide class of wave equations that includes the cubic Benjamin–Ono equation as a special case, the soliton is analytic everywhere within an infinitely long strip of some finite but unspecified width which is symmetric about the real axis. The geometric rate of convergence makes a stronger statement. If the spectral coefficients fall proportional to \( \exp(-q n) \) where \( q \) is real and positive, then this implies the soliton is singular at \( \Im(s) = \pm q \) where for the Christov basis, the Fourier-to-physical coordinate mapping is \( x = L \cot(q/2) \); the images of these lines of constant \( \Im(s) \) in the Fourier plane are circles in the complex \( x \)-plane with centers at \( c \) and radius \( r \) [11]:

\[
c = \pm i L \coth(q), \quad r = L/ \sinh(q).
\]

(38)

The soliton is singular at one or more points on both circles, and analytic everywhere on the region exterior to both circles. Here, \( q \approx 1/3 \) implies \( c \approx \pm i 4.66 \) and \( r \approx 4.42 \).

Fig. 3. Left: (solid curve) Unit phase speed cubic BO soliton, solving \((u^2 - 1)u_x + 7u_{ux} = 0\). The dashed curve shows the soliton for the ordinary, quadratically-nonlinear Benjamin–Ono equation. Only positive \( X \) is shown because both curves are symmetric about the origin. The solitary wave decays from its maximum of 3.282 to half of the maximum at \( X = \pm 0.414 \); the decay for larger \( |X| \) is much slower. Right: The Christov coefficients \( a_j \).
7. Analysis of an inhomogeneous linear integro-differential equation

If we integrate the integro-differential equation $H(u_{xx}) - u_x = -u^m u_x$ with respect to $X$, recalling that the $X$-derivative commutes with the Hilbert transform, it becomes

$$H(u_X) - u = -\frac{1}{m + 1} u^{m+1},$$

where the constant of integration is zero because $u$ vanishes for large $|X|$. This is of the form of the linear inhomogeneous equation

$$H(u_X) - u = f(X),$$

where $f(X)$ is symmetric with respect to the origin. It turns out that one can learn much about the nonlinear problem by studying this linear equation, leaving $f(X)$ unspecified for the moment.

**Theorem 7.1 (Properties of Linear OIDE).** Let $u(x)$ denote the solution to

$$H(u_X) - u = f(X),$$

where $f(X)$ is symmetric with respect to the origin, that is, $f(X) = f(-X) \forall X$. Then

1. $$u(X) = -\frac{1}{\pi} \int_{0}^{\infty} \frac{F(k)}{1 + k} \cos(kX)dk,$$

where

$$F(k) = 2 \int_{0}^{\infty} f(X) \cos(kX)dX$$

is the Fourier transform of $f$.

2. $$u(X; m) \sim \frac{2}{\pi} \left\{ \int_{0}^{\infty} f(Y)dy \right\} \frac{1}{X^2} + O\left(\frac{1}{X^4}\right), \quad |X| \gg 1.$$ (44)

3. Define the Green’s function $G(X)$ as the solution to

$$H(G_X) - G = \delta(X),$$

where $\delta(X)$ is the Dirac delta function. Then

$$G(X) = -\frac{1}{\pi} \int_{0}^{\infty} \frac{1}{1 + k} \cos(kX)dk,$$

$$= -\frac{1}{\pi} \left\{ \left(\frac{\pi}{2} - \text{Si}(|X|)\right) \sin(|X|) - \text{Ci}(X) \cos(X) \right\}, \quad |X| \gg 1.$$ (47)

$$\approx \left( -\gamma - \ln(|X|) + \frac{3}{4} \pi |X| + \left( -\frac{3}{4} + \frac{1}{2} \gamma + \frac{1}{2} \ln(|X|) \right) X^2 - \frac{1}{12} \pi |X|^3 \right.\right.$$ (48)

$$\left. + \left( -\frac{1}{24} \gamma - \frac{1}{24} \ln(|X|) + \frac{25}{288} X^4 + \frac{1}{240} \pi |X|^5 + O(X^6) \right) \right).$$

4. $$u(X) = \int_{-\infty}^{\infty} f(X - Y)G(Y)dY.$$ (50)

**Proof.** By taking the Fourier transform of both sides and invoking the symmetry of $f$, one obtains the Fourier solution which is proposition one. Integration-by-parts establishes the identity, for any function $Q(k)$,

$$\frac{1}{\pi} \int_{0}^{\infty} Q(k) \cos(kX)dk = \frac{1}{\pi} \frac{dQ}{dk}(0) - \frac{1}{\pi} \frac{1}{X^2} \frac{d^2Q}{dk^2}(0) + \frac{1}{X^3} \frac{1}{\pi} \int_{0}^{\infty} \frac{d^4Q}{dk^4}(k) \cos(kX)dk$$

for any function $Q(k)$ whose fourth derivative is integrable. Substituting $Q(k) = F(k)/(1 + k)$ yields...
Theorem 8.1. Let \( u(X;m) \) denote a solution, decaying as \( |X| \to \infty \) and symmetric about \( X = 0 \), of
\[
H(u) = u'' + u
\]
8. A proof that generalized GBO solitons decay quadratically for large \(|X|\)

For large nonlinearity exponent \( m \), the function \( f(X) \), which is proportional to \( u^{m+1}(X) \), will be very narrow compared to the scale of \( u(X) \) itself, and has negligible amplitude. By making the replacement \( f(X) \to (1/\epsilon)f(X/\epsilon) \) where the parameter \( \epsilon \ll 1 \), we can build this narrowness into the inhomogeneous term of linear OIDE (41). For a symmetric \( f(X) \), the convolution solution can be written \( \int_{-\infty}^{\infty} f(Y - X)f(Y)dY = \int_{-\infty}^{\infty} G(X + Y)dY \). When the forcing is narrow,
\[
u(X;\epsilon) = \frac{1}{\epsilon} \int_{-\infty}^{\infty} f(Z/\epsilon)G(Z + X)dZ = \int_{-\infty}^{\infty} f(\zeta)G(\zeta + X)d\zeta
\]
by changing the integration variable to \( \zeta \equiv Z/\epsilon \). When \( \epsilon \ll 1 \),
\[
G(\zeta + X) \approx G(X) + \epsilon^2 \frac{dG}{dX}(X) + \frac{1}{2} \epsilon^2 \zeta^2 \frac{d^2G}{dX^2}(X) + \ldots
\]
\[
u(X;\epsilon) \sim G(X) \int_{-\infty}^{\infty} f(\zeta)d\zeta + \epsilon^2 \frac{1}{2} \frac{d^2G}{dX^2}(X) \int_{-\infty}^{\infty} f(\zeta)^2 d\zeta,
\]
where the symmetry of \( f(\zeta) \) implies that \( \int_{-\infty}^{\infty} f(\zeta)d\zeta = 0 \).

This approximation is not uniformly valid in space because \( G(X) \) is logarithmically singular at \( X = 0 \) whereas \( u(X;\epsilon) \) is not singular. Outside of a small neighborhood of the origin, however, the approximation \( u(X;\epsilon) \sim F(0)G(X) \) is accurate with an error \( O(\epsilon^2) \).

8. A proof that generalized GBO solitons decay quadratically for large \(|X|\)

Fig. 4. Cubic BO soliton and the Green’s function approximation and the constant/\(X^2\) approximation.
then
\[
  u(X; m) \sim \frac{2}{\pi} \frac{1}{m+1} \left\{ \int_0^\infty u^{m+1}(X) dX \right\} \frac{1}{X^2} + O\left(\frac{1}{X^2}\right), \quad |X| \gg 1. \tag{59}
\]

**Proof.** Integrating (58) with respect to \(X\), which is easy because both sides are perfect \(X\)-derivatives, gives a differential equation of the form of the linear OIDE (41) with \(f(X) = u^{m+1}/(m+1)\). The asymptotic behavior is then given by the second proposition of Theorem 7.1 after substituting this \(f(X)\). \(\Box\)

Theorem 5.1 of Chen and Bona [17] is similar but for a generalized “Benjamin-type” equation that includes arbitrary power-law nonlinearity and both a Hilbert transform dispersion and a derivative (KdV-like) of various orders. Their theorem does not technically apply here because their scaling does not allow the higher derivative term to vanish. However, the proof, which is also based on linear Green’s function reasoning, is very similar.

### 9. Green’s function approximation to generalized Benjamin–Ono solitons

The nonlinear term \(u^{m+1}/(m+1)\) varies on a narrower and narrower scale compared to \(u(X)\) as the exponent \(m\) increases. It follows that for large \(m\), the “narrow-forcing” approximation of \(u(X)\) by a constant times the Green’s function should be accurate.

![Fig. 5. Quintic BO soliton and the Green's function approximation and the constant/X^2 approximation.](image1)

![Fig. 6. Octic BO soliton and the Green's function approximation and the constant/X^2 approximation.](image2)
Figs. 4–6 confirm that (i) the approximation fails in a narrow region around the maximum at $X = 0$ and (ii) the Green’s function approximation is very accurate almost everywhere excluding a small neighborhood of the origin. For comparison, the dotted curves show the simpler approximation, proportional to $1/X^2$, which we rigorously justified in the previous section. For moderate $|X|$, the more complicated Green’s function approximation is very accurate.

It would be nice to have an analytic approximation for that narrow region where the Green’s function fails and the nonlinear term is important. We have not been sufficiently clever to find one. Perhaps the reader will fill this lacuna!

10. Analytic approximation to the cubically-nonlinear Benjamin–Ono soliton

By simple parameter-fitting, we found that the cubic Benjamin–Ono soliton of unit phase speed is well fit by the two-term approximation (Fig. 7)

$$u(X; m = 2) \approx \frac{1.0667}{1.3373 + X^2} + \frac{0.25639}{0.1032 + X^2}. \quad (60)$$

The maximum pointwise absolute error is only 0.014, which relative to the height of the soliton is a relative error of only 1 part in 250. To obtain similar accuracy from the Christov series requires roughly six terms.

![Fig. 7. Cubic BO soliton and a two-rational-function fit; varying all four parameters.](image)

![Fig. 8. Solitons of unit phase speed. Left: Generalized Benjamin–Ono. Right: same but for the Korteweg–de Vries generalizations where $u_{xx} + (u^{m+1} - c)u = 0$ is solved exactly for general $m$ by $u(X) = (m/2 + 1)^{1/m} \text{sech} \left( (m/2)X \right)^{1/m}$.](image)
The success of such a simple approximation confirms that the solitary wave is very smooth.

The need for two reciprocal-quadratics instead of one, as is the exact Benjamin–Ono soliton, is a lesson in structure. The narrow rational function is needed to approximate the narrow, tall spike for $|X| \leq 1.2$. The broad rational function is needed to furnish the slow, broad decay for large $|X|$.

11. Solitons of higher order nonlinearity

It is straightforward to compute Benjamin–Ono solitons of higher nonlinearity exponent $m$ as shown in Fig. 8 with the corresponding results for similarly generalized Korteweg–deVries solitons for comparison. As $m$ increases, the solitary wave of unit phase speed become narrower, and the region of strong curvature around the peak narrows as $O(1/m)$.

12. Summary

We have calculated solitary waves of generalized Benjamin Ono equations with cubic and higher nonlinearity using the pseudospectral method with a basis of rational Christov functions and Newton’s iteration. The Christov basis is preferable to other basis sets for an unbounded domain because the Hilbert transform merely changes one Christov function into another. For the cubic nonlinearity, the numerically computed Christov coefficients fall proportional to roughly $\exp(-n/3)$, a geometric rate of convergence that implies that the solitary wave is analytic on the entire real axis including infinity.

The assumption of a wave traveling at phase speed reduces the partial differential equation to an ordinary integro-differential equation. A simple rescaling theorem reduces the task, for a given nonlinearity exponent $m$, to computing the soliton which has its maximum at the origin and unit phase speed.

As the order $m$ of the nonlinearity increases, the soliton becomes shorter and narrower and increasingly well approximated, except in the neighborhood of the peak, by the Green’s function $G(X)$ of the linear part of the Benjamin Ono equation. This Green’s function is given by a simple analytical expression involving the sine and cosine and the sine and cosine integrals.

Numerical curve-fitting shows that the cubic Benjamin–Ono soliton can be approximated by the sum of two rational functions to an error no worse on the entire real axis than one part in 250 of its maximum.

Newton’s iteration requires a first guess. We generated this by inflating the problem through an artificial parameter $\delta \in [0,1]$ such that for $\delta$ equals zero, the inflated OIDE is the usual traveling wave version of the Benjamin–Ono equation and is the cubic Benjamin–Ono equation when $\delta$ is one, a “homotopy perturbation”. We then proceeded to march in small steps from the known Benjamin–Ono solitary wave at $\delta = 0$ to the cubic soliton, using the converged solution for the previous value of $\delta$ as the Newton’s initialization for the next value of $\delta$.

We also bypassed this numerical continuation by computing a perturbation series in $\delta$. We illustrated that this procedure, after applying Padé approximants to the power series in $\delta$, gave good approximations to the cubically nonlinear solitary wave both for the Korteweg–deVries and Benjamin–Ono equation families.

One obvious extension is to apply the methodology here to the “regularized” Benjamin–Ono equation of Kalisch and Bona [25]. However, the homotopy/delta expansion is very broadly applicable to nonlinear traveling waves. The Christov basis is useful wherever the Hilbert transform appears.

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References


