Product and quotient of correlated beta variables

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\begin{abstract}
Let $U, V, W$ be independent random variables having a standard gamma distribution with respective shape parameters $a, b, c$, and define $X = U/(U + W), Y = V/(V + W)$. Clearly, $X$ and $Y$ are correlated each having a beta distribution, $X \sim B(a, c)$ and $Y \sim B(b, c)$. In this article we derive probability density functions of $XY, X/Y$ and $X/(X + Y)$.
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\section{Introduction}

The random variable $X$ is said to have a beta distribution, denoted by $X \sim B(a, b)$, if its probability density function (p.d.f.) is given by
\begin{equation}
{B(a, b)}^{-1}x^{a-1}(1-x)^{b-1}, \quad 0 < x < 1,
\end{equation}
where $a > 0, b > 0$, and $B(a, b)$ is the beta function defined by
\begin{equation}
B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.
\end{equation}
It is well known that if $U$ and $V$ are independent random variables having a standard gamma distribution with respective shape parameters $a$ and $b$, then $U/(U + V) \sim B(a, b)$. Several univariate generalizations of this distribution are given in Gordy [1], McDonald and Xu [7], Nagar and Zarrazola [8] and Ng and Kotz [9]. For an extensive review on univariate and matrix variate beta distributions the reader is referred to Johnson, Kotz and Balakrishnan [4] and Gupta and Nagar [3]. Beta distribution is widely used in statistical modeling of bounded random variables. Applications of densities of the product and ratio of independent beta variates in the field of reliability can be found in Pham-Gia [11] and Pham-Gia and Turkkan [12].

Although, a wealth of results on distributions of products and quotients of independent random variables is available in the literature, little appears to have been done to derive densities of products and quotients of correlated random variables.

Let $U, V, W$ be independent random variables having a standard gamma distribution with respective shape parameters $a, b, c$, and define
\begin{equation}
X = \frac{U}{U + W}, \quad Y = \frac{V}{V + W}.
\end{equation}
Clearly, $X$ and $Y$ each has a beta distribution, $X \sim B(a, c)$ and $Y \sim B(b, c)$. However, they are correlated so that $(X, Y)$ has a bivariate beta distribution. Libby and Novick [5] derived the joint density of $X$ and $Y$ as

$$
\frac{x^{a-1}y^{b-1}(1-x)^{b+c-1}(1-y)^{a+c-1}}{B(a, b, c)(1-xy)^{a+b+c}}, \quad 0 < x, y < 1,
$$

where

$$
B(a, b, c) = \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a + b + c)}.
$$

Olkin and Liu [10] studied several properties of the above distribution and showed that it is positively likelihood ratio dependent. They also pointed out that this distribution may serve as the prior for a correlated binomial distribution.

In this article we derive the probability density functions of $XY$, $X/Y$ and $X/(X+Y)$.

2. Main results

In this section we derive densities and moments of $XY$, $X/Y$ and $X/(X+Y)$ where the joint density of $X$ and $Y$ is given by (4). Since expressions for densities involve the Gauss hypergeometric function $\, _2F_1$, we first give its definition. The Gauss hypergeometric function $\, _2F_1$ is defined by the Euler integral (Luke [6]),

$$
\, _2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 v^{a-1}(1-v)^{c-a-1}(1-vz)^{-b} dv,
$$

where $\text{Re}(a) > 0$, $\text{Re}(c-a) > 0$ and $|\text{arg}(1-z)| < \pi$. By expanding $(1-vz)^{-b}$, $|vz| < 1$, in power series and integrating $v$, the series expansion for the Gauss hypergeometric function $\, _2F_1$ is obtained as

$$
\, _2F_1(a, b; c; z) = \sum_{r=0}^{\infty} \frac{(a)_r(b)_r}{(c)_r} \frac{z^r}{r!},
$$

where $z$ is a complex variable, $a$, $b$ and $c$ can take arbitrary real or complex values (provided that $c \neq 0, -1, -2, \ldots$) and $(a)_n = a(a+1) \cdots (a+n-1) = (a)_{n-1}(a+n-1)$ for $n = 1, 2, \ldots$, and $(a)_0 = 1$. If either $a$ or $b$ is zero or a negative integer, the series terminates after a finite number of terms, and its sum is then a polynomial in $z$. Except for this case, the radius of convergence of the hypergeometric series is 1.

We now turn to our problem of deriving densities of $XY$, $X/Y$ and $X/(X+Y)$.

**Theorem 2.1.** Let the joint density of $X$ and $Y$ be given by (4). Then, the p.d.f. of $Z = XY$, denoted by $f(z)$, is given by

$$
f(z) = \frac{B(a + c, b + c)z^{a-1}(1-z)^{c-1}}{B(a, b, c)} \, _2F_1(a + c, a + c; a + b + 2c; 1 - z), \quad 0 < z < 1.
$$

**Proof.** Transforming $Z = XY$, $Y = Y$ with the Jacobian $J(x, y \to z, y) = 1/y$ we obtain the joint p.d.f. of $Z$ and $Y$ as

$$
\frac{z^{a-1}(y-z)^{b+c-1}(1-y)^{a+c-1}}{B(a, b, c)(1-z)^{a+b+c}y^{a+c}}, \quad 0 < z < y < 1.
$$

To find the marginal p.d.f. of $Z$, we integrate (9) with respect to $y$ to get

$$
\frac{z^{a-1}}{B(a, b, c)(1-z)^{a+b+c}} \int_z^1 y^{-(a+c)}(y-z)^{b+c-1}(1-y)^{a+c-1} dy.
$$

In (10), the change of variable $w = (1-y)/(1-z)$ yields

$$
\frac{z^{a-1}(1-z)^{c-1}}{B(a, b, c)} \int_0^1 w^{a+c-1}(1-w)^{b+c-1}[1 - (1-z)w]^{-(a+c)} dw = \frac{B(a + c, b + c)z^{a-1}(1-z)^{c-1}}{B(a, b, c)} \, _2F_1(a + c, a + c; a + b + 2c; 1 - z),
$$

where the last step has been obtained by using (6).

Fig. 1 illustrates the shape of the p.d.f. for selected values of $a, b$ and $c$. 

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Consider the transformation (4) over 0 let the joint density of 6 integrating where 0 < W. Proof. Theorem 2.2. and obtains the where Re h obtaining the where Re(α) > 0, Re(σ) > 0, Re(γ + σ − α − β) > 0 and 3F2 is the generalized hypergeometric function (Luke [6]), one obtains the 4th moment of Z as

\[ E(Z^4) = \frac{B(a + c, b + c)}{B(a, b, c)} \int_0^1 z^{a+c-1} (1-z)^{b-1} 2F_1(a+c, a+b+2c; 1-z) \, dz \]

\[ = \frac{B(a + c, b + c)B(a + h, c)}{B(a, b, c)} \, 3F_2(a + c, a + c, a + b + 2c, a + c + h; 1). \]

**Theorem 2.2.** Let the joint density of X and Y be given by (4). Then, the p.d.f. of W = X/Y is given by

\[ w^{\alpha-1}(b + 2a + 2r - 1) \Gamma(a + b + c + r) \sum_{r=0}^{\infty} \frac{B(a + b + 2r, a + c) \Gamma(a + b + c + r) w^r}{\Gamma(a + b + c)} \frac{1}{r!} F_1(a + b + 2r, 1 - b - c; 2a + b + c + 2r; w), \quad 0 < w \leq 1, \]

and

\[ w^{\beta-1} \sum_{r=0}^{\infty} \frac{B(a + b + 2r, b + c) \Gamma(a + b + c + r) w^r}{\Gamma(a + b + c)} \frac{1}{r!} F_1(a + b + 2r, 1 - a - c; a + 2b + c + 2r; \frac{1}{w}), \quad w > 1. \]

**Proof.** Consider the transformation W = X/Y, Y = Y whose Jacobian is j(x, y → w, y) = y. Thus, using (4), we obtain the joint p.d.f. of W and Y as

\[ w^{\alpha-1}(1 - wy)^{b+c-1} \frac{(1 - y)^{d+c-1} B(a, b, c)(1 - wy)^{a+b+c}}{B(a, b, c)(1 - wy)^{a+b+c}}, \]

where 0 < y < 1 for 0 < w ≤ 1, and 0 < y < 1/w for w > 1. For 0 < w ≤ 1, the marginal p.d.f. of W is obtained by integrating (12) over 0 < y < 1. Thus, the p.d.f. of W, for 0 < w ≤ 1, is obtained as
\[
\frac{w^{a-1}}{B(a, b, c)} \int_0^1 y^{b+c-1}(1 - wy)^{b+c-1} (1 - y)^{a+c-1} dy \\
= \frac{w^{-1}}{B(a, b, c)} \sum_{r=0}^{\infty} \frac{\Gamma(a + b + c + r)}{\Gamma(a + b + c)} \int_0^1 y^{a+b+2r-1}(1 - wy)^{b+c-1} (1 - y)^{a+c-1} dy,
\]
where the last line has been obtained by expanding \((1 - wy)^2\) in power series. Now, evaluation of the above integral using (6) gives the desired result. For \(w > 1\), the density of \(W\) is given by
\[
\frac{w^{-1}}{B(a, b, c)} \int_0^1 y^{a+b-1}(1 - wy)^{b+c-1} (1 - y)^{a+c-1} dy \\
= \frac{w^{-1}}{B(a, b, c)} \sum_{r=0}^{\infty} \frac{\Gamma(a + b + c + r)}{\Gamma(a + b + c)} \int_0^1 y^{a+b+2r-1}(1 - wy)^{b+c-1} (1 - y)^{a+c-1} dy,
\]
where last two lines have been obtained by substituting \(v = wy\) and expanding \((1 - v^2/w)^{-a+b+c}\) in power series. Finally, application of (6) yields the p.d.f. of \(W\) for \(w > 1\).

By using (6) and (11) and the joint density of \(X\) and \(Y\), we obtain the \(h\)th moment of \(W = X/Y\) as
\[
E(W^h) = \frac{1}{B(a, b, c)} \int_0^1 \int_0^1 \frac{x^{a+b-1} y^{b+c-1}(1 - x)^{b+c-1} (1 - y)^{a+c-1}}{(1 - xy)^{a+b+c}} dx dy \\
= \frac{B(a + h, b + c)B(b - h, a + c)}{B(a, b, c)} \text{E}_2(a + h, a + b + c, b - h; a + b + c + h, a + b + c - h; 1),
\]
where \(\text{Re}(a + h) > 0\) and \(\text{Re}(b - h) > 0\).

**Theorem 2.3.** Let the joint density of \(X\) and \(Y\) be given by (4). Then, the p.d.f. of \(T = X/(X + Y)\) is given by
\[
\frac{e^{-t}(1 - t)^{-a-1} \sum_{r=0}^{\infty} B(a + b + 2r, a + c) \Gamma(a + b + c + r)}{B(a, b, c)B(1 - t)^{a+b+c}} \left( \frac{t}{1 - t} \right)^r \text{E}_2(a + b + 2r, 1 - b - c; 2a + b + c + 2r; \frac{t}{1 - t})
\]
for \(0 < t \leq 1/2\), and
\[
\frac{e^{-t}(1 - t)^{b-1} \sum_{r=0}^{\infty} B(a + b + 2r, b + c) \Gamma(a + b + c + r)}{B(a, b, c)B(1 - t)^{a+b+c}} \left( \frac{1 - t}{t} \right)^r \text{E}_2(a + b + 2r, 1 - a - c; a + 2b + c + 2r; \frac{1 - t}{t})
\]
for \(1/2 < t < 1\).

**Proof.** Making the transformation \(T = X/(X + Y), V = X + Y\) with the Jacobian \(J(x, y \rightarrow t, v) = v\) in (4), we find the joint p.d.f. of \(T\) and \(V\) as
\[
\frac{e^{-t}(1 - t)^{b-1} v^{a+b-1}(1 - tv)^{b+c-1}[1 - (1 - t)v]^{a+c-1}}{B(a, b, c)[1 - (1 - t)v]^{a+b+c}}.
\]

Now, to evaluate the p.d.f. of \(T\), we integrate (13) with respect to \(v\). For \(0 < t \leq 1/2\, the density \(h(t)\) of \(T\) is derived as
\[
h(t) = \frac{e^{-t}(1 - t)^{b-1} \int_0^1 (1 - t)^{-a-1} \sum_{r=0}^{\infty} B(a + b + 2r, a + c) \Gamma(a + b + c + r)}{B(a, b, c)B(1 - t)^{a+b+c}} \left( \frac{t}{1 - t} \right)^r \text{E}_2(a + b + 2r, 1 - b - c; 2a + b + c + 2r; \frac{t}{1 - t})
\]
Writing \([1 - (1 - t)v]^{a+b+c}\) in power series and substituting \(w = v(1 - t)\) in the above expression, we obtain
\[
h(t) = \frac{e^{-t}(1 - t)^{-a-1} \sum_{r=0}^{\infty} B(a + b + 2r, b + c) \Gamma(a + b + c + r)}{B(a, b, c)B(1 - t)^{a+b+c}} \left( \frac{1 - t}{t} \right)^r \text{E}_2(a + b + 2r, 1 - a - c; a + 2b + c + 2r; \frac{1 - t}{t})
\]
Now, evaluation of the above integral using (6) yields the desired result. For \(1/2 < t < 1\, we have
\[
h(t) = \frac{e^{-t}(1 - t)^{b-1} \sum_{r=0}^{\infty} B(a + b + 2r, a + c) \Gamma(a + b + c + r)}{B(a, b, c)B(1 - t)^{a+b+c}} \left( \frac{t}{1 - t} \right)^r \text{E}_2(a + b + 2r, 1 - b - c; 2a + b + c + 2r; \frac{t}{1 - t})
\]
Finally, evaluating the above integral, we get the density of \(T\) for \(1/2 < t < 1\).

By using the joint density of \(X\) and \(Y\), we obtain the \(h\)th moment of \(T = X/(X + Y)\) as
\[
E(T^h) = \frac{1}{B(a, b, c)} \int_0^1 \int_0^1 \frac{x^{a+b-1} y^{b-1}(1 - x)^{b+c-1} (1 - y)^{a+c-1}}{(x + y)^h (1 - xy)^{a+b+c}} dx dy.
\]
Now, expanding \((1-xy)^{-(a+b+c)}\) in power series, substituting \(u = 1 - x\) and \(v = 1 - y\), the above expression is re-written as

\[
E(T^h) = \frac{1}{B(a, b, c)} \sum_{r=0}^{\infty} \frac{(a+b+c)_r}{r!} \int_0^1 \int_0^1 (1-u)^{a+r+b-1}(1-v)^{b+r-1}u^{b-1}v^{a+c-1} \, du \, dv.
\]

Finally, evaluating the above integral using the result \((\text{Gradsteyn and Ryzhik [2, Eq. 9.184(2)]})\),

\[
F_2(\alpha, \beta, \beta'; \gamma, \gamma'; x, y) = \frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma'-\beta)\Gamma(\gamma'-\beta')} \int_0^1 \int_0^1 u^{\beta-1}v^{\beta'-1}(1-u)^{\gamma-\beta-1}(1-v)^{\gamma'-\beta'-1} \, du \, dv,
\]

where \(\text{Re}(\beta) > 0, \text{Re}(\beta') > 0, \text{Re}(\gamma - \beta) > 0, \text{Re}(\gamma' - \beta') > 0\) and \(F_2\) is Appell’s second hypergeometric function, the \(h\)th moment of \(T = X/(X+Y)\) is derived as

\[
E(T^h) = \frac{\Gamma(a+c)\Gamma(b+c)}{2^h\Gamma(a)\Gamma(b)\Gamma(c)} \sum_{r=0}^{\infty} \frac{\Gamma(a+r+h)\Gamma(b+r)}{\Gamma(a+b+c+r+h)\Gamma(a+b+c+r)} F_2\left(h, b+c, a+c; a+b+c+r+h, a+b+c+r; \frac{1}{2}, \frac{1}{2}\right).
\]

3. Computations

The computation of the percentage points of \(Z = XY\) has been carried out by using \(\alpha = F(z) = \int_0^z f(v) \, dv\) where \(f(v)\) is given by \((8)\). First, \(f(v)\) is simplified by expanding the Gauss hypergeometric function in the series form. Then, the CDF \(F(z)\) is obtained by integrating term by term the simplified density function. Then, \(z\) is computed for \(\alpha = 0.01, 0.025, 0.05, 0.1\) and various values of \(a, b\) and \(c\). These percentage points are given in Table 1. We have used MATHEMATICA 4.0 to carry out these computations. To compute \(z\) for given value of \(\alpha = F(z)\), we have used the bisection method. Similar tabulations could be easily obtained for other values of \(a, b\) and \(c\). The densities of \(W = X/Y\) and \(T = X/(X+Y)\) also involve the Gauss hypergeometric function and hence the percentage points can be computed similarly.

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