Partial synchronization in oscillator arrays with asymmetric coupling

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Abstract

A new form of cluster synchronization is explored in cellular arrays of chaotic oscillators. Previously it was shown in the literature that symmetries of the coupling topology with uniform interaction weights lead to several coexisting clusters of synchronized cells. In this study a new phenomenon is presented where highly asymmetric interaction weights can give rise to cluster synchronization regimes with partial synchronization. In addition, cluster or partial synchronization regimes corresponding to asymmetric interaction patterns can break the underlying symmetries of the network topology and boundary conditions at the expense of some residual synchronization error.
1 Introduction

Synchronization of oscillator networks is a prevalent phenomenon in nature. Despite its widespread presence, synchronization is used only in a few specific fields of engineering, e.g. communication with chaotic lasers.

Understanding the principles of synchronization phenomena occurring in networks of coupled chaotic oscillators is very difficult due to the non-linear, high-dimensional nature of these systems. Studies mainly focusing on deriving conditions for synchronization include [Pogromsky et al., 2002], where global symmetries of a network of identical, diffusively coupled oscillators were used to classify linear invariant manifolds corresponding to cluster synchronization regimes. In [Belykh et al., 2001] the link between graph topology and stability of global synchronization was clarified. In [Belykh et al., 2003] it was shown that stability of cluster synchronization regimes depends on the individual oscillator dynamics and on the choice of state variables used in the coupling. However the existence of synchronized clusters depends only on the coupling topology, boundary conditions and the number of oscillators. Another issue is the dependence of the stability of synchronization manifolds on the mismatch of individual oscillator parameters. Cluster synchronization regimes were reported to persist with 10-15% mismatch between parameters of individual oscillators [Belykh et al., 2003]. A common aspect of the above mentioned studies is that the coupling was the same for all oscillators.

For networks with asymmetric connectivity - i.e. where coupling varies from cell to cell - methods based on calculating the eigenvalue of the connectivity matrix [Pecora & Carroll, 1998] for determining coupling values needed for synchronization may be difficult to apply. In [Belykh et al., 2001] a graph theory based method was introduced to estimate the value of the coupling coefficient needed for global synchronization of a network. This was further elaborated for asymmetrically coupled networks in [Belykh et al., 2006] with the restriction that every node has the same input and output weight sum. In [Ma & Liu, 2006] a method for constructing a coupling scheme for arbitrarily selected n-cluster synchronization was presented for networks with non-nearest neighbor connections.

The above mentioned studies focused on providing a means to estimate coupling coefficients of the network in order to ensure complete synchronization of the cells. However, synchronization can be exhibited in a variety of additional forms including phase-[Rosenblum et al., 1996], lag-[Rosenblum et al., 1997] and generalized [Abarbanel et al., 1996] synchronization. In addition, perfect and imperfect synchronization can also be differentiated within these forms of synchronization. This high complexity motivated the development of new methods that can measure synchronization in a wide variety of forms. One such measure of synchronization was proposed in [Shabunin et al., 2005], where a normalized coherence function based on the cross Fourier spectrum of two chaotic oscillators were calculated from time series. This metric has direct physical meaning and highly mature spectrum evaluation methods exist for its efficient and robust calculation. However, extending it to describe synchronization phenomena between more than two oscillators may be a complex task.

In [Suykens et al., 1999] it was shown that two Lur’e systems which may even have different
qualitative behaviour (e.g. limit cycle versus chaos or stable points versus chaos) can be synchronized to each other up to a small synchronization error. Chua’s circuits, multi-scroll circuits and networks consisting of such cells with linear coupling belong to this class of Lur’e systems.

In this letter we report a new kind of synchronization in oscillator networks with local, diffusive coupling where the coupling can be different for each cell. The reason to allow any interaction pattern - restricted to contain non-negative weight values - is that new kinds of cooperative behavior may be possible that were not observed in previous studies when coupling was kept identical for all cells.

This paper is organized as follows. In Section 2 the oscillator network model is specified. In Section 3 a new phenomenon is presented where highly asymmetric interaction weights can give rise to cluster synchronization regimes with partial synchronization. In Section 4, methods used to find cluster synchronization regimes are presented. Conclusions are given in Section 5.

2 Array of oscillatory cells

Consider an array composed of autonomous nonlinear oscillators:

\[
\dot{x}_{r,c} = F(x_{r,c}; \Theta_{r,c}) + \sum_{k,l \in S_{\rho}(r,c)} A_{r,c} P(x_{k,l} - x_{r,c}), \quad r, c = 1, \ldots, M \in \mathbb{N}_0
\]

where \(x_{r,c} \in \mathbb{R}^d\) represents the state vector of the \(r, c\)th oscillator; \(F : \mathbb{R}^d \times \mathbb{R}^q \rightarrow \mathbb{R}^d\) is a nonlinear function depending on parameters \(\Theta_{r,c} \in \mathbb{R}^q\) that define the cells; \(A \in \mathbb{R}^{M \times M}\) is the spatially varying coupling weight matrix that defines the interaction pattern of the array; \(P = (p_{i,j}) \in \mathbb{R}^{d \times d}\) with \(p_{i,j} \in \{0, 1\}\) determines which state variables couple the oscillators and \(S_{\rho}(r,c)\) defines a fixed neighborhood topology with neighborhood radius \(\rho\).

In the experiments shown in this letter, individual cells were modified Chua’s oscillators generating \(n\)-scroll attractors. \(n\)-scroll generating chaotic oscillators were introduced in [Suykens & Vandewalle, 1993] generalizing the original Chua’s circuit by introducing additional breakpoints in the nonlinear resistor. In the current experiments hyperbolic tangent non-linearity was used [Özoguz et al., 2002], with state equations:

\[
\frac{d}{dt} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\Theta_{r,c} & -\Theta_{r,c} \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\Theta_{r,c} f(x^1) \end{pmatrix}
\]

where \(f(x^1) = \sum_{j=-V}^{W} (-1)^{j-1} \tanh(q(x^1 - 2j)), \quad V = 1, W = 3, q = 2\). The qualitative behavior of an oscillator cell is show on Fig. 1. All cells in all experiments exhibit similar qualitative properties.

In this letter, the phenomena of cluster and partial synchronization are studied on square arrays with four-connected local topology \((\rho = 1)\) and zero-flux boundary conditions. Time evolution of each array configuration was simulated in Matlab with time horizon large enough to analyze its long-term stationary behavior \((T = 9000\) with relative error tolerance \(10^{-6}\)). Bifurcation parameter \(\Theta_{r,c}\) may be
3 Cluster synchronization in arrays with unorganized interaction pattern

Ten mutually exclusive, coexisting cluster synchronization regimes can be observed on Fig. 2. Pairs of cells synchronize with respect to the principal diagonal of the square array but cells on the diagonal remain desynchronized. No straightforward spatial relationship can be noticed in the interaction pattern $A$ that could explain the symmetry in the spatial layout of synchronizing cell clusters. The clusters are not synchronized to each other despite the non-zero couplings between them. This is a surprising phenomenon that may be related to the findings of [Belykh et al., 2003] where a similar phenomenon was presented for an array where all coupling coefficients were identical.

According to [Belykh et al., 2003], topological products of synchronization regimes possible in a 1D chain of oscillators define all possible set of synchronization regimes in a 2D array. E.g. a possible regime is that cells synchronize in vertical stripes, i.e. a regime defined as the product of global synchronization in the vertical direction and global desynchronization in the horizontal direction. Cluster synchronization regimes in a 1D chain with zero flux boundary conditions are symmetric to the middle of the chain. In a similar manner, in a 2D array, spatial layout of cluster synchronization regimes can arise from symmetries along the principal or secondary diagonal, the middle of the rows or columns or any combination of these axes.

Putting no restriction on the interaction pattern, the possible spatial layouts of cluster synchronization regimes in a square array was investigated. Fig. 3 shows all stubs of possible cluster synchronization regimes in a $3 \times 3$ array, not including situations that can be obtained by rotating these stubs with $90^\circ$, $180^\circ$ or $270^\circ$. Interaction patterns learned using the approach described in the Methods section may synchronize either just these pairs or larger cell clusters that contain these stubs. Stubs of Fig. 3 may be part of a larger cluster synchronization regime following the spatial symmetry related rules outlined above. The sole exception is the pair of cells shown on Fig. 3g that is not symmetric to any axis.

Fig. 4 shows array configurations giving rise to cluster synchronization regimes that contain the specified stubs as specified on Fig.3a - 3f. Also, on Fig. 2 the stub used in the learning process (see Methods section for details) was symmetric with respect to the principal diagonal and it is contained in the cluster synchronization regime shown on Fig. 2. However, for the stub shown on Fig. 3g, the only synchronization regime we found was when all cells in the array synchronized.

Figure 5a-5b shows that the error between synchronizing cells is not zero. Such type of synchronization was already studied extensively in [Suykens et al., 1999] where it was shown that qualitatively different dynamical systems can synchronize with some small residual error. Examples shown in this
letter confirms this phenomenon extending it into the more complex context of cluster synchronization occurring in networks defined in Eq. (1).

The cluster synchronization regimes shown in this letter are persistent for random perturbations up to 20% of both the coupling and the bifurcation parameters. Persistence of the results were also confirmed for each experiment for a large number of random initial conditions.

4 Methods

Inspired by [Suykens et al., 1994] and [Suykens et al., 1999] the problem of finding cluster synchronization regimes was cast into an optimization problem:

$$\min_{A, \Theta} U(\{X_{stac}(t_0), X_{stac}(t_1), ..., X_{stac}(t_n)\})$$

where $$U(\cdot)$$ denotes the cost function and $$X_{stac} = (x_1(t), x_2(t), ..., x_d(t)) : \mathbb{R} \rightarrow \mathbb{R}^{M^2d}$$ is a stationary solution of Eq. (1) for a given initial condition and $$t_0 < t_1 < ... < t_n < \infty$$. Additional inequality constraints may apply to $$A$$ and $$\Theta$$. $$U(\cdot)$$ is constructed in such a way that dynamical properties of individual cells or cell populations fulfill the desired requirements. These requirements may include different types of synchronization and stability criteria, desired Lyapunov spectrum or embedding dimension of the attractor, etc. We assume that $$F_{r,c}$$ can provide a rich enough set of dynamical behaviors.

An important advantage of the proposed approach compared to other studies is that the only requirement on the vector field defining individual oscillators is that they permit the solution to exist and be unique. In this study we used oscillators with continuous vectors fields. On the other hand, in some cases this liberty may result in an optimization problem that is very hard to solve if it is possible at all. The global optimization framework used to learn network configurations corresponding to cluster synchronization regimes is generic, i.e. no assumption was made about network size or topology.

The choice of optimization algorithm is an intricate task. In this study the time evolution of the network was used to drive the learning process. Integrating the network for a given parameter set lasts long, thus a key issue is to choose an optimization algorithm for learning desired behavior that requires a low number of function evaluations to find the global optimum. In [Xavier-de-Souza et al., 2006] a new global optimization algorithm called Coupled Simulated Annealing (CSA) was introduced. In CSA the annealing temperatures of several Simulated Annealing processes are interconnected in order to improve performance in convergence speed and to increase the probability of exploring all basins of attraction in a given number of cost function evaluations. The number of cost function evaluations per individual optimizers decreases exponentially when the number of optimizers is increased linearly. This makes CSA a good candidate for the current problem since interactions between solutions decrease the number of cost function evaluations to reach a given energy threshold.

In order to learn cluster synchronization regimes, a task of primary importance is to define a proper cost function which assigns the desired behavior of the network to the global optimum.
The cost function has to embed a metric that

- is able to capture different forms of synchronization (complete-, phase-, lag-, generalized),
- provides a measure in a way that does not need any human interpretation, i.e. can be embedded in the cost function,
- is computationally feasible.

The following metric that fulfills these requirements was used in our study:

\[
U_{std} = \sum_{t_j=1}^{n} \sum_{i=1}^{d} \log \left( \frac{1}{N_S} \sum_{k \in \Pi_S} \left( x^i_k(t_j) - \overline{x}_S(t_j) \right)^2 \right)^{\frac{1}{2}} \left( \frac{1}{N_R} \sum_{k \in \Pi_R} \left( x^i_k(t_j) - \overline{x}_R(t_j) \right)^2 \right)^{\frac{1}{2}}
\]  

(4)

where \( x^i_S(t_j) \in \Pi_S \) represents a number of snapshots taken from the time evolution of cells to be synchronized, \( x^i_R(t_j) \in \Pi_R \) represents those that are not to be synchronized. \( \Pi_S, \Pi_R \subset \{1, ..., M\} \times \{1, ..., M\} \) denote cell populations with \( N_S \) standing for the total number of cells to be synchronized and \( N_R \) for the number of cells to be desynchronized and \( \Pi_S \cap \Pi_R = \emptyset \). \( \overline{x}_S(t_j) \) is the spatial average for each time instance of the target population of cells to be synchronized and \( \overline{x}_R(t_j) \) is the average for the rest of the cells.

The numerator of Eq. (4) is the distance of state variables of all cells to be synchronized with respect to the mean of their individual state variables, evaluated in corresponding time instances. Minimizing Eq. (4) leads to complete synchronization of cells to be synchronized. However, to achieve partial synchronization, the rest of the cells must be as desynchronized as possible from each other and from the synchronizing cells. The denominator stands for the same kind of distance as the numerator, but it is calculated for the cells to be desynchronized. The role of the denominator is to make state variables of desynchronizing cells as uncorrelated as possible with any other cell.

Given a specific cluster synchronization regime to be learnt, Eq. (4) assigns the global and all cluster synchronization regimes to local optima. The cost value corresponding to these local optima decreases as the position of synchronizing and desynchronizing cells better correspond with the imposed cluster synchronization regime. On Fig. 2 and Fig. 4, cells imposed to synchronize in the cost function are marked with dashed rectangles. However, synchronized behavior also appears within other cell clusters for the same array configuration.

In all experiments presented, learning was performed as follows. Only the coupling matrix \( A \) was included in the learning process with inequality constraints \( 0 \leq A_{r,c} \leq 30 \). Bifurcation parameters \( \Theta_{r,c} \) were fixed, chosen from a normal distribution with mean 0.25 (the nominal value that produces the n-scroll attractor) and standard deviation 0.01. Time evolution of the array was calculated using adaptive step-size solver ode45 in Matlab with relative error tolerance \( 10^{-4} \) on a time horizon \( T = 600 \). Time instants \( t \in [450, 600] \) were used in calculating the cost value for each probed array configuration.
Initial condition was set to the same value for all variables, with value 0.1 for all cells, except for one of the cells in the pair on which synchronization was imposed in the cost function where it was set to 0.5. The reason for such initial conditions was to ensure that the resulting synchronization is not due to the identical value of initial conditions.

To obtain the configuration shown on Fig. 2 CSA with initial temperature 2.0 and 150 individual optimizers with 15000 population iterations was used. To obtain the configurations shown on Fig. 4 CSA with initial temperature 1.4 and 36 individual optimizers with 2500 population iterations was used.

5 Conclusions

In [Belykh et al., 2003] symmetries of the coupling topology were exploited to find cluster synchronization regimes of identical oscillators with identical coupling weights. In this study, the oscillators were not identical and the coupling coefficients were highly asymmetric, still the symmetries similar to those shown in [Belykh et al., 2003] can be observed in the resulting cluster synchronization regimes. However, at the expense of some residual synchronization error, the asymmetry of the interaction pattern between cells can give rise to cluster synchronization regimes that break the underlying symmetries of the network topology and boundary conditions. This is a new phenomenon to be analyzed more in detail in future studies.

Global optimization techniques were successfully applied to find cluster synchronization regimes in chaotic oscillator arrays. A drawback of this approach is the high computational cost for large arrays that strongly motivates VLSI implementation of chaotic oscillator arrays. Hardware implementation could speed up experimentation and rule out the potential errors numerical simulations can introduce.

In future studies, it would be interesting to see what are the ingredients in a highly asymmetric coupling configuration that are essential in order to give rise to a specific cluster synchronization regime. A more involved question is to see how cluster synchronization is related to information processing in oscillator networks of living organisms.

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References


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Figure 1: Dániel Hillier et al
![Figure 2: Dániel Hillier et al](image)

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Figure 2: Dániel Hillier et al
Figure 3: Dániel Hillier et al
Figure 4: Dániel Hillier et al
Figure 5: Dániel Hillier et al
Figure 1:
Figure 1a shows the attractor of cell (2,1) corresponding to the configuration of Fig. 4a. Figure 1b shows the attractor of the same cell when coupling is removed from the network. All cells in all experiments exhibit similar qualitative behavior.

Figure 2:
Ten mutually exclusive, coexisting cluster synchronization regimes in a 5×5 array of chaotic oscillators. Cells belonging to the same synchronizing cluster are marked with the same grayscale level. White cells marked with X do not synchronize to any other cell. Pairs of cells synchronize with respect to the principal diagonal of the array. However, no straightforward spatial relationship can be noticed in the interaction pattern A that could explain the symmetry in the spatial layout of cluster synchronization regimes. The clusters are not synchronized to each other despite the non-zero couplings between them. A cell is influenced by its nearest neighbors (four connected local topology) with coupling weight value $A_{r,c}$ shown in each cell, where index $r$ denotes the row, $c$ denotes the column of a cell.

Figure 3:
Stubs of possible cluster synchronization regimes in a 3×3 array, not including situations that can be obtained by rotating these stubs with 90°, 180° or 270°. Interaction patterns learned using the approach described in the Methods section may synchronize either just these pairs or larger cell clusters that contain these stubs. All stubs may be part of a synchronization regime following an appropriate combination of axial symmetries as reported in [Belykh et al., 2003]. The sole exception is the pair of cells shown on Fig. 3g. These two cells can never synchronize in the same cluster if we assume the clusters must follow some combination of axial symmetries.

Figure 4:
Six different cluster synchronization regimes learnt using Eq. (4). Cells belonging to the same synchronizing cluster are marked with the same grayscale level. White cells marked with X do not synchronize to any other cell. Dashed rectangles indicate the cells that were leant to synchronize during optimization. A cell is influenced by its nearest neighbors (four connected local topology) with coupling weight value $A_{r,c}$ shown in each cell.

Figure 5:
On Fig. 5a the magnitude of state vector of a cell is denoted by $|x|$. Magnitude of state vector of cell (2,1) was plotted against the same quantity of all other cells. Cells in the cluster marked with the darkest shade in Fig. 4a - i.e. cells (2,1) and (1,2)) - synchronize as the 45° lines indicate. Fig. 5b shows the magnitude of the difference between the state vector of cell (2,1) and the others $e_{r,c} = ||x_{2,1} - x_{r,c}||_2$. All synchronizing clusters in all experiments have qualitatively similar error plot.