A method of finding automorphism groups of endomorphism monoids of relational systems

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Abstract

For any set \(X\) and any relation \(\rho\) on \(X\), let \(T(X, \rho)\) be the semigroup of all maps \(a : X \to X\) that preserve \(\rho\). Let \(S(X)\) be the symmetric group on \(X\). If \(\rho\) is reflexive, the group of automorphisms of \(T(X, \rho)\) is isomorphic to \(N_{S(X)}(T(X, \rho))\), the normalizer of \(T(X, \rho)\) in \(S(X)\), that is, the group of permutations on \(X\) that preserve \(T(X, \rho)\) under conjugation. The elements of \(N_{S(X)}(T(X, \rho))\) have been described for the class of so-called dense relations \(\rho\). The paper is dedicated to applications of this result.

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1. Introduction

For a mathematical structure \(\mathcal{A}\), let \(\text{End}(\mathcal{A})\) denote the monoid of endomorphisms of \(\mathcal{A}\). A large amount of effort in mathematical research has been devoted to the study of relations between the monoid \(\text{End}(\mathcal{A})\) and the structure \(\mathcal{A}\) itself. Of particular interest along this line of research has been the description of the automorphism group of \(\text{End}(\mathcal{A})\). For example, Schreier [35] and Mal’cev [25] described all automorphisms of \(\text{End}(X)\), where \(X\) is a set. Similar results have been obtained for various other structures such as orders, equivalence relations, graphs, and hypergraphs. (See [29], the survey paper, and [30]) More examples are provided, among others, by Glusk˘ın [14] (where the automorphisms of the endomorphism monoid of a vector space are described), Levi [19,20], Liber [23], Magill [24], Schein [34], Sullivan [37], and Šutov [38]. In [2], the authors described the automorphism group of \(\text{End}(X, \rho, R)\), where \(X\) is a set, \(\rho\) is an equivalence relation on \(X\), and \(R\) is a cross-section of \(X/\rho\). Recently, this general problem of describing \(\text{Aut}(\text{End}(\mathcal{A}))\) has attracted an even wider attention for its links to universal algebraic topology (see [26]). Examples of the research prompted by this new motivation are, among others, [9,27].

The purpose of this paper is to describe a method of finding the automorphism group of \(\text{End}(\mathcal{A})\) for certain relational systems \(\mathcal{A}\). The method is based on a result concerning dense relations (Theorem 3.1).
Therefore, to describe \( X \) transformations on \( T(\mathcal{X}) \) described in the previous paragraph that any automorphism of \( T(\mathcal{X}) \) follows from results of Schreier [35]. The set \( NS(X) (T(X,\rho)) \) of \( \mathcal{X} \) quasi-inner automorphisms of \( T(X,\rho) \) can be interpreted as \( T(X,\rho) \) on \( \mathcal{X} \), where \( \rho \) is a dense relation, and hence we obtain descriptions of their automorphism groups. This fact has universal application. Let \( \mathcal{A} \) be a mathematical structure. Whenever the monoid \( End(\mathcal{A}) \) of endomorphisms of \( \mathcal{A} \) can be interpreted as \( T(X,\rho) \) for some set \( X \) and some reflexive relation \( \rho \) on \( X \), we can determine the automorphism group of \( End(\mathcal{A}) \) provided we can describe the normalizer \( NS(X)(T(X,\rho)) \) of \( T(X,\rho) \) in the symmetric group \( S(X) \). Therefore, to describe \( Aut(T(X,\rho)) \) we only need to describe the elements of \( NS(X)(T(X,\rho)) \), that is, the permutations \( g \in S(X) \) such that \( g^{-1}T(X,\rho)g = T(X,\rho) \).

2. Preliminaries

Let \( S \) be a semigroup. Any bijection \( \phi : S \rightarrow S \) such that \( (ab)\phi = (a\phi)(b\phi) \) for all \( a, b \in S \) is called an automorphism of \( S \). (We write maps on the right, that is, \( a \phi \) rather than \( \phi(a) \), and compose them from left to right.) The group of automorphisms of \( S \) will be denoted by \( Aut(S) \).

Let \( X \) be an arbitrary non-empty set and let \( \mathcal{S} \) be a subsemigroup of \( T(X) \), where \( T(X) \) is the semigroup of full transformations on \( X \), that is, the semigroup of all maps from \( X \) to \( X \) under composition. Following [7], we will call an automorphism \( \phi \) of \( S \) quasi-inner if there is \( g \in S(X) \) such that \( a \phi = g^{-1}ag \) for every \( a \in S \). (In such a case, we say that \( \phi \) is the quasi-inner automorphism induced by \( g \).) Note that if \( g \in S(X) \), then \( \phi \) induced by \( g \) is an inner automorphism of \( S \). The set \( QuInn(S) \) of all quasi-inner automorphisms of \( S \) is a subgroup of \( Aut(S) \).

Given a subsemigroup \( \mathcal{S} \) of \( T(X) \), we denote by \( NS(X)(S) \) the normalizer of \( S \) in \( S(X) \), that is, the subgroup of the symmetric group \( S(X) \) consisting of all permutations \( g \) on \( S(X) \) such that \( g^{-1}Sg = S \). Subgroups \( G \) of \( S(X) \) such that \( G = NS(S)(S) \) for some transformation semigroup \( S \) have been studied by Levi [21,22]. Note that every element \( g \in NS(S)(S) \) induces a quasi-inner automorphism \( \phi^S \) of \( S \) (defined by \( a \phi^S = g^{-1}ag \) for every \( a \in S \)) and the group \( QuInn(S) \) is a homomorphic image of \( NS(S)(S) \) (via a homomorphism that maps \( g \) to \( \phi^S \)).

For a positive integer \( n \), let \( I_n = \{1, \ldots, n\} \). An \( n \)-tuple of elements of \( X \) is any function \( f : I_n \rightarrow X \). As customary, we may denote \( f \) by \((1f, \ldots, nf)\). An \( n \)-ary relation \( \rho \) on \( X \) is any set of \( n \)-tuples of elements of \( X \). By a relation on \( X \), we will mean an \( n \)-ary relation on \( X \) for some \( n \).

Let \( \rho \) be a relation on \( X \). We define \( T(X,\rho) \) to be the set of all \( a \in T(X) \) that preserve \( \rho \), that is,

\[
T(X,\rho) = \{ a \in T(X) \mid f \in \rho \Rightarrow fa \in \rho \},
\]

where \( fa : I \rightarrow X \) is the composition of \( f : I \rightarrow X \) and \( a : X \rightarrow X \). With the usual \( n \)-tuple notation, we have

\[
T(X,\rho) = \{ a \in T(X) \mid (x_1, \ldots, x_n) \in \rho \Rightarrow (x_1a, \ldots, x_na) \in \rho \}.
\]

It is clear that \( T(X,\rho) \) is a subsemigroup of \( T(X) \). It is in fact the endomorphism monoid of the structure \((X,\rho)\), that is, \( T(X,\rho) = End(X,\rho) \). When \( \rho \) is the universal relation on \( X \), that is, \( \rho \) consists of all \( n \)-tuples of elements of \( X \), then \( T(X,\rho) = T(X) \). We also have \( T(X,\rho) = T(X) \) when \( \rho \) is the identity relation on \( X \).

A relation \( \rho \) on \( X \) is said to be reflexive if it contains all constant functions \( f : I_n \rightarrow X \), that is, if \((x, x, \ldots, x) \in \rho \) for every \( x \in X \).

For the remainder of the paper \( X \) will denote a non-empty set, \( I_n = \{1, \ldots, n\} \), and \( \rho \) a reflexive \( n \)-ary relation on \( X \). Since the relation \( \rho \) is reflexive, any constant map preserves \( \rho \), and so \( T(X,\rho) \) contains all constant maps. Thus, the next theorem follows from results of Schreier [35].
Theorem 2.1. Let \( \rho \) be a reflexive \( n \)-ary relation on a set \( X \). Then

\[
\text{Aut}(T(X, \rho)) = \{ \phi^\rho \mid g \in N_{S(X)}(T(X, \rho)) \}.
\]

Let \( S \) be any subsemigroup of \( T(X) \). A function \( \alpha : N_{S(X)}(S) \to \text{Aut}(S) \) defined by \( g \alpha = \phi^g \) is always a group homomorphism. Indeed for all \( g, h \in N_{S(X)}(S) \) and \( a \in S \), we have

\[
a \phi^{gh} = (gh)^{-1}a(gh) = h^{-1}(g^{-1}ag)h = (a \phi^g) \phi^h = a(\phi^g \phi^h),
\]

and so \((gh)\alpha = (g\alpha)(h\alpha)\). If \( \alpha(S) = Q\text{Inn}(S) \) then \( \alpha \) is onto. If, in addition, \( \alpha \) is also one to one then \( \text{Aut}(S) \) is isomorphic to \( N_{S(X)}(S) \).

We know by Theorem 2.1 that \( \text{Aut}(T(X, \rho)) = Q\text{Inn}(T(X, \rho)) \). Thus, the group homomorphism \( \alpha : N_{S(X)}(T(X, \rho)) \to \text{Aut}(T(X, \rho)) \) is onto. In fact, it is also one to one. Indeed, suppose \( g \in \text{Ker}(\alpha) \), that is, \( a\phi^g = a \) for every \( a \in T(X, \rho) \). Let \( a_1 \) be the constant map with image \( \{ x \} \) (\( x \in X \)). Then \( a_1g = g^{-1}a_1g = a_1 \phi^g = a_1 \), and so \( xg = x \) (for every \( x \in X \)). Hence \( g = id_X \). \( \text{Ker}(\alpha) \) is trivial, and \( \alpha \) is one to one. We just proved the following theorem. (For groups \( G \) and \( H \), we write \( G \cong H \) when \( G \) is isomorphic to \( H \).)

Theorem 2.2. Let \( \rho \) be a reflexive \( n \)-ary relation on a set \( X \). Then

\[
\text{Aut}(T(X, \rho)) \cong N_{S(X)}(T(X, \rho)).
\]

3. Dense relations

In general, it may be difficult to describe the normalizer \( N_{S(X)}(T(X, \rho)) \). In this section, we obtain a usable description of \( N_{S(X)}(T(X, \rho)) \) for any dense relation \( \rho \). The class of dense relations, introduced in [3], includes the partial orders, binary relations that are reflexive and symmetric, and generalized equivalence relations.

Denote by \( \rho^* \) the set of all functions \( f : I_n \to X \) such that \( \sigma f \notin \rho \) for every \( \sigma \in S(I_n) \), where \( S(I_n) \) is the symmetric group on \( I_n \). That is,

\[
\rho^* = \{ f : I_n \to X \mid (\forall \sigma \in S(I_n)) \sigma f \notin \rho \}.
\]

A reflexive relation \( \rho \) on \( X \) is said to be dense if it satisfies the following two properties:

\[
(D_1) \text{ For every injective } f_1 \in \rho \cup \rho^* \text{ and every } f \in \rho, \text{ there is } a \in T(X, \rho) \text{ such that } f_1a = f.
\]

\[
(D_2) \text{ There is an injective } f_1 \text{ in } \rho.
\]

We say that a bijection \( g : X \to X \) is a \( p \)-automorphism of the relational system \((X, \rho)\) if there is a permutation \( \sigma \in S(I_n) \) such that for all \( f : I_n \to X \),

\[
f \in \rho \iff \sigma fg \in \rho.
\]

Denote by \( \text{Aut}_p(X, \rho) \) the set of \( p \)-automorphisms of \((X, \rho)\). It is easy to see that \( \text{Aut}_p(X, \rho) \) is a subgroup of \( S(X) \):

\[
\text{Aut}_p(X, \rho) \text{ clearly contains } id_X, \text{ it is closed under composition } (f \in \rho \iff \sigma_1fg_1 \in \rho \text{ and } f \in \rho \iff \sigma_2fg_2 \in \rho \text{ implies } f \in \rho \iff \sigma_2\sigma_1fg_1g_2 \in \rho), \text{ and it is closed under inverses } (f \in \rho \iff \sigma fg \in \rho \text{ implies } f \in \rho \iff \sigma^{-1}fg^{-1} \in \rho).
\]

Note that the group \( \text{Aut}_p(X, \rho) \) contains the group \( \text{Aut}(X, \rho) \) of automorphisms of \((X, \rho)\), that is, bijections \( g : X \to X \) such that for all \( f : I_n \to X \),

\[
f \in \rho \iff fg \in \rho.
\]

Suppose \( n = 2 \). The only elements of \( S(2) \) are \( id_2 \) (the identity permutation of \( I_2 \)) and the transposition \((1 2)\). It follows that if \( \rho \) is a binary relation then any \( p \)-automorphism \( g \) of \((X, \rho)\) is either an automorphism \((x, y) \in \rho \iff (xg, yg) \in \rho) \) or an anti-automorphism \((x, y) \in \rho \iff (yg, xg) \in \rho) \). Thus for every binary relation \( \rho \) on \( X \),

\[
\text{Aut}_p(X, \rho) = \text{Aut}(X, \rho) \cup \text{Aut}'(X, \rho),
\]

where \( \text{Aut}(X, \rho) \) is the group of automorphisms of \((X, \rho)\) and \( \text{Aut}'(X, \rho) \) is the set of anti-automorphisms of \((X, \rho)\). The sets \( \text{Aut}_p(X, \rho) \) and \( \text{Aut}(X, \rho) \cup \text{Aut}'(X, \rho) \) are endowed with the same operation, namely the composition in the symmetric group \( S(X) \). Thus, since we already established that \( \text{Aut}_p(X, \rho) \) is a subgroup of \( S(X) \), “=” in (3.1) means
Theorem 3.1. Let \( \rho \) be an arbitrary dense relation on \( X \). Then

1. \( \text{Aut}(T(X, \rho)) \cong \text{Aut}_p(X, \rho) \).
2. \( \text{Aut}(T(X, \rho)) \cong \text{Aut}(X, \rho) \cup \text{Aut}'(X, \rho) \) if \( \rho \) is a binary relation.

Statement (2) of Theorem 3.1 is a special case of (1) (see (3.1)). For completeness, we outline the proof of (1). By Theorem 2.2, it suffices to show \( N_{S(X)}(T(X, \rho)) \cong \text{Aut}_p(X, \rho) \). Since the proof of \( N_{S(X)}(T(X, \rho)) \cong \text{Aut}_p(X, \rho) \) is straightforward, we only prove the converse. Let \( g \in N_{S(X)}(T(X, \rho)) \). Since \( \rho \) is dense, there is an injective \( f \in \rho \).

We claim that there is a permutation \( \sigma \in S(I_n) \) such that \( \sigma f g \in \rho \). Indeed, either \( f g^{-1} \in \rho^* \) or \( f g^{-1} \notin \rho^* \). In the first case, select \( a \in T(X, \rho) \) such that \( f g^{-1} a = f \) (such an \( a \) exists by (D1)). Thus \( f \in \rho \Rightarrow f(g^{-1}a) \in \rho \Rightarrow f g \in \rho \Rightarrow \text{id}_{I_n} f g \in \rho \). If \( f g^{-1} \notin \rho^* \), then \( \delta^{-1} f g^{-1} \in \rho \) for some \( \delta \in S(I_n) \), so that there is \( a \in T(X, \rho) \) such that \( \delta^{-1} f g^{-1} a = f \). Then \( f g^{-1} a = \delta^{-1} f \) and we have \( f \in \rho \Rightarrow f(g^{-1} a) \in \rho \Rightarrow \delta^{-1} f g \in \rho \). Thus we have a desired \( \sigma : \sigma = \text{id}_{I_n} \) if \( f g^{-1} \in \rho^* \), and \( \sigma = \delta^{-1} \) if \( f g^{-1} \notin \rho^* \).

The next step in the proof is to show that \( \sigma f g \in \rho \) implies that \( \sigma f' g \in \rho \) for every \( f' \in \rho \). Again use (D1) to select \( a \in T(X, \rho) \) such that \( f a = f' \). Then \( \sigma f g \in \rho \Rightarrow \sigma f g(g^{-1} a) \in \rho \Rightarrow \sigma f a g \in \rho \Rightarrow \sigma f' g \in \rho \).

To conclude the proof that \( N_{S(X)}(T(X, \rho)) \subseteq \text{Aut}_p(X, \rho) \), observe that for an injective \( f \in \rho \), \( f_1 = \sigma f g \) is an injective element of \( \rho \) and \( \sigma^{-1} f_1 g^{-1} = f \in \rho \). Thus, by the foregoing argument, \( \sigma^{-1} f' g^{-1} \in \rho \) for every \( f' \in \rho \). It follows that for every \( f' : I_n \to X \), \( f' \in \rho \iff \sigma f' g \in \rho \), and \( g \) is a \( \rho \)-automorphism of \( (X, \rho) \). The result follows.

4. Partial orders and graphs

In this section, we show how two known results concerning partial orders and graphs with loops immediately follow from Theorem 3.1.

There has been a considerable amount of research concerning automorphisms and, more generally, isomorphisms of endomorphism monoids of directed graphs. By a directed graph (digraph), we mean a structure \( G = (X, \rho) \), where \( X \) is a set and \( \rho \) is a binary relation on \( X \). Gluskı́n [13] proved that if \( G_1 \) is a nontrivial quasi-order digraph and \( G_2 \) is a reflexive digraph, then \( \text{End}(G_1) \) and \( \text{End}(G_2) \) are isomorphic if and only if the digraphs \( G_1 \) and \( G_2 \) are isomorphic or anti-isomorphic. Vázenin [40] proved that Gluskı́n’s result remains true when \( G_1 \) is a reflexive digraph containing an edge that does not lie on any cycle of \( G_1 \). However, it is not true in general that the endomorphism monoid of a reflexive digraph \( G \) determines \( G \) up to isomorphism or anti-isomorphism. Ref. [1] contains (infinitely) many examples of reflexive digraphs \( G \) such that \( \text{Aut}(\text{End}(G)) \) is not isomorphic to \( \text{Aut}(G) \cup \text{Aut}'(G) \). The reader is referred to [29] for a survey of results concerning endomorphism monoids of digraphs.

A binary relation \( \leq \) on a set \( X \) is called a partial order on \( X \) if it is reflexive, anti-symmetric, and transitive. A poset is a pair \((X, \leq)\), where \( X \) is a set and \( \leq \) is a partial order on \( X \).

Lemma 4.1. Let \( \leq \) be a partial order on \( X \). If \( \leq \) is not the identity relation on \( X \) then \( \leq \) is dense.

Proof. Since \( \leq \) is not the identity relation, there are \( x, y \in X \) such that \( x \neq y \) and \( x \leq y \). Thus \( f_1 = (x, y) \in \leq \) is injective, and so \( \leq \) satisfies (D2).

To prove that \( \leq \) satisfies (D1), let \( f_1 = (x, y) \in \leq \cup \leq^* \) be injective and let \( f = (w, z) \in \leq \). That is, \( x \neq y \), either \( x \leq y \) (when \( f_1 \in \leq \)) or \( x, y \) are incomparable (when \( f_1 \in \leq^* \)), and \( w \leq z \). We need to construct \( a \in T(X, \leq) \) such that \( f_1 a = f \), that is, \( x a = w \) and \( y a = z \). Define \( a : X \to X \) by

\[
ra = \begin{cases} 
  w & \text{if } r \leq x, \\
  z & \text{otherwise},
\end{cases}
\]

where \( r \in X \). Since \( x \leq x \), we have \( x a = w \). If \( x \leq y \) then \( y \not\leq x \) (since \( x \neq y \) and \( \leq \) is antisymmetric). If \( x \) and \( y \) are incomparable then \( y \not\leq x \). Thus in either case \( y a = z \). Let \( r, s \in X \) with \( r \leq s \). If \( s \leq x \) then \( r \leq x \), and so \( r a = w \leq w = sa \).
If \( r \leq x \) and \( s \not\leq x \) then \( ra = w \leq z = sa \). Finally, if \( r \not\leq x \) and \( s \not\leq x \) then \( ra = z \leq z = sa \). Hence \( a \in T(X, \leq) \). The result follows. \( \Box 
\)

Let \((X, \leq)\) be a poset. An endomorphism of \((X, \leq)\) is a map \( a : X \to X \) that preserves the order, that is, for all \( x, y \in X \),
\[
x \leq y \Rightarrow xa \leq ya.
\]
(Order-preserving maps are known in the theory of partial orders as isotone functions [8, p. 2].) The semigroup of all endomorphisms of \((X, \leq)\) will be denoted by \( \text{End}(X, \leq) \).

It is clear that \( \text{End}(X, \leq) = T(X, \leq) \). Thus, if \( \leq \) is not the identity relation then \( \text{Aut}(\text{End}(X, \leq)) \cong \text{Aut}(X, \leq) \cup \text{Aut}^r(X, \leq) \) by Theorem 3.1 and Lemma 4.1. If \( \leq \) is the identity relation then both \( N_S(X)(T(X, \rho)) \) and \( \text{Aut}(X, \leq) \) are equal to \( S(X) \). Thus, by Theorem 2.2, the above congruence also holds for the identity relation, and we have the following result, which was proved by Gluskin [13, Theorem 1] and Schein [34, Corollary 5].

**Theorem 4.2.** Let \((X, \leq)\) be a poset. Then
\[
\text{Aut}(\text{End}(X, \leq)) \cong \text{Aut}(X, \leq) \cup \text{Aut}^r(X, \leq).
\]

A relation \( \rho \) on \( X \) (not necessarily reflexive) is called symmetric if for all \( f : I_n \to X \),
\[
f \in \rho \Rightarrow (\forall \sigma \in S(I_n)) \sigma f \in \rho.
\]

Note that if \( \rho \) is a binary relation then \( \rho \) is symmetric if and only if for all \( x, y \in X \), \((x, y) \in \rho \) implies \((y, x) \in \rho \).

We now return to our global assumption that \( \rho \) denotes a reflexive \( n \)-ary relation on \( X \). Suppose that \( \rho \) is symmetric. Then it is easy to see that \( \text{Aut}_\rho(X, \rho) = \text{Aut}(X, \rho) \), and so the following result follows from Theorem 3.1.

**Theorem 4.3.** Let \( \rho \) be a symmetric relation on \( X \). If \( \rho \) is dense then
\[
\text{Aut}(T(X, \rho)) \cong \text{Aut}(X, \rho).
\]

The conclusion of Theorem 4.3 is true for every (reflexive) non-identity binary symmetric relation on \( X \) since every such a relation is dense.

**Lemma 4.4.** Let \( \rho \) be a binary relation on \( X \) that is reflexive and symmetric. If \( \rho \) is not the identity relation on \( X \) then \( \rho \) is dense.

**Proof.** Since \( \rho \) is not the identity relation, there are \( x, y \in X \) such that \( x \neq y \) and \((x, y) \in \rho \). Thus \( f_1 = (x, y) \in \rho \) is injective, and so \( \rho \) satisfies \((D_2)\).

To prove that \( \rho \) satisfies \((D_1)\), let \( f_1 = (x, y) \in \rho \cup \rho^* \) be injective (that is, \( x \neq y \)) and let \( f = (w, z) \in \rho \). Since \( \rho \) is symmetric, we also have \((z, w) \in \rho \). Define \( a : X \to X \) by: \( xa = w \), \( ya = z \), and \( ua = w \) for every \( u \in X \setminus \{x, y\} \). It is clear that \( f_1 a = f \). Let \( r, s \in X \) with \((r, s) \in \rho \). By the definition of \( a \), \((ra, sa) \) is equal to \((w, w) \) or \((w, z) \) or \((z, w) \) or \((z, z) \). In either case, \((ra, sa) \in \rho \). Hence \( a \in T(X, \rho) \). The result follows. \( \Box \)

As an immediate application of Theorem 4.3 and Lemma 4.4, we obtain the group of automorphisms of \( \text{End}(G) \), where \( G \) is any graph (undirected, no multiple edges) with a loop at every vertex.

A graph is a pair \( G = (X, E) \), where \( X \) is a non-empty set and \( E \subseteq \{(x, y) \mid x, y \in X\} \). Elements of \( X \) are called vertices. An element \( \{x, y\} \in E \) is called an edge between \( x \) and \( y \). If \( \{x, x\} = \{x\} \in E \), we say that \( G \) has a loop at the vertex \( x \).

Let \( G = (X, E) \) be a graph. An endomorphism of \( G \) is a map \( a : X \to X \) such that for all \( x, y \in X \), \( \{x, y\} \in E \Rightarrow \{xa, ya\} \in E \). The semigroup of all endomorphisms of \( G \) will be denoted by \( \text{End}(G) \). An automorphism of \( G \) is a bijection \( g : X \to X \) such that for all \( x, y \in X \), \( \{x, y\} \in E \Leftrightarrow \{xg, yg\} \in E \). The group of automorphisms of \( G \) will be denoted by \( \text{Aut}(G) \).
Theorem 4.5. Let $G = (X, E)$ be a graph with a loop at every vertex. Then

$$\text{Aut}(\text{End}(G)) \cong \text{Aut}(G).$$

Proof. Suppose $E = \{\{x\} \mid x \in X\}$. Then $\text{End}(G) = T(X)$ and $\text{Aut}(G) = S(X)$. Thus, the result follows by the fact that $\text{Aut}(T(X))$ is isomorphic to $S(X)$ (which is a special case of Theorem 2.2).

Suppose $E \neq \{\{x\} \mid x \in X\}$, that is, there is at least one edge in $G$ that is not a loop. Define a binary relation $\rho$ on $X$ by $(x, y) \in \rho$ if $(x, y) \in E$. Since $G$ has a loop at every vertex, $\rho$ is reflexive. Moreover, $\rho$ is symmetric, $\rho$ is not the identity relation, $\text{End}(G) = T(X, \rho)$, and $\text{Aut}(G) = \text{Aut}(X, \rho)$. Thus, the result follows by Theorem 4.3 and Lemma 4.4. □

5. Generalized equivalence relations and hypergraphs

Many authors have studied isomorphisms of hypergraphs, see for example [4,6,32]. Molchanov [30] proved that any $p$-hypergraph $H$ (we define $p$-hypergraphs later in this section) is determined up to isomorphism by the endomorphism monoid of $H$.

In this section, we apply our dense-relation technique to the systems $(X, \rho)$, where $\rho$ is a generalized equivalence relation on $X$. An immediate corollary of our result on generalized equivalence relations (Theorem 5.2) is that for any $p$-hypergraph $H$, $\text{Aut}(\text{End}(H))$ is isomorphic to $\text{Aut}(H)$ (Corollary 5.3).

Hartmanis [15] generalized partitions to partitions of type $n$ ($n = 1, 2, \ldots$). Let $|X| \geq n$. A family $\mathcal{P}$ of subsets of $X$ is called a partition of type $n$ of $X$ if it satisfies the following two properties:

(P1) If $A \subseteq X$ with $|A| = n$ then there is exactly one $P \in \mathcal{P}$ such that $A \subseteq P$.

(P2) Every $P \in \mathcal{P}$ contains at least $n$ elements.

Following [15], we agree that if $|X| < n$ then $\{X\}$ is a partition of type $n$ of $X$. Note that a partition of type 1 is a partition in the usual sense, that is, a set of mutually disjoint, non-empty subsets of $X$ whose union is $X$.

Pickett [31] generalized equivalence relations to equivalence relations of type $n$ ($n = 1, 2, \ldots$). An $(n + 1)$-ary relation $\rho$ on $X$ is called an equivalence relation of type $n$ on $X$ if it satisfies the following three properties for all $x_0, x_1, \ldots, x_{n+1} \in X$:

(R) $(x_1, \ldots, x_n, x_1) \in \rho$.

(S) For every permutation $\sigma$ of $\{1, \ldots, n+1\}$,

$$(x_1, \ldots, x_{n+1}) \in \rho \Rightarrow (x_1\sigma, \ldots, x_{(n+1)\sigma}) \in \rho.$$

(T) If $x_1, \ldots, x_n$ are pairwise distinct then

$$(x_0, x_1, \ldots, x_n) \in \rho \quad \text{and} \quad (x_1, \ldots, x_n, x_{n+1}) \in \rho \Rightarrow (x_0, x_1, \ldots, x_{n-1}, x_{n+1}) \in \rho.$$

Note that when $n = 1$ (that is, when $\rho$ is a binary relation), the conditions (R), (S), and (T) mean that $\rho$ is, respectively, reflexive, symmetric, and transitive. Thus, an equivalence relation of type 1 is an equivalence relation in the usual sense.

There is a natural 1–1 correspondence between generalized partitions and generalized equivalence relations of the same type [31]. Let $\mathcal{P}$ be a partition of type $n$ of $X$. The corresponding equivalence relation $\rho$ of type $n$ on $X$ is defined by

$$(x_1, \ldots, x_{n+1}) \in \rho \quad \text{if} \quad x_1, \ldots, x_{n+1} \in P \quad \text{for some} \quad P \in \mathcal{P}.$$  

When $n = 1$, this gives the usual correspondence between equivalence relations and partitions.

For $n \geq 1$, denote by $\mu_n$ the $(n + 1)$-ary relation on $X$ consisting of all non-injective $(n + 1)$-tuples, that is,

$$\mu_n = \{(x_1, \ldots, x_{n+1}) \mid x_i = x_j \text{ for some } i, j \text{ such that } i \neq j\}.$$  \hspace{1cm} (5.1)

It is clear that $\mu_n$ satisfies the conditions (R), (S), and (T), that is, $\mu_n$ is an equivalence relation of type $n$. Moreover, it follows from (R) and (S) that every equivalence relation $\rho$ of type $n$ contains all non-injective $(n + 1)$-tuples. Thus, $\mu_n$
is the smallest (with respect to inclusion) equivalence relation of type \( n \). It corresponds to the partition \( \mathcal{M}_n \) of type \( n \) consisting of all subsets of \( X \) with \( n \) elements. Note that \( \mu_1 = \{(x, x) \mid x \in X\} \) and \( \mathcal{M}_1 = \{\{x\} \mid x \in X\} \).

With the exception of \( \mu_n \), all equivalence relations of type \( n \) are dense.

**Lemma 5.1.** Let \( \rho \) be an equivalence relation of type \( n \) on \( X \). If \( \rho \neq \mu_n \) then \( \rho \) is dense.

**Proof.** Since \( \rho \neq \mu_n \), there are \( x_1, \ldots, x_{n+1} \in X \) such that \( x_i \neq x_j \) for \( i \neq j \) and \( (x_1, \ldots, x_{n+1}) \in \rho \). Thus \( f_1 = (x_1, \ldots, x_{n+1}) \in \rho \) is injective, and so \( \rho \) satisfies \((D_2)\).

To prove that \( \rho \) satisfies \((D_1)\), let \( f_1 = (x_1, \ldots, x_{n+1}) \in \rho \cup \rho^* \) be injective (that is, \( x_i \neq x_j \) for \( i \neq j \)), and let \( f = (y_1, \ldots, y_{n+1}) \in \rho \). Define \( a : X \to X \) by \( x_ia = y_i \) for every \( i \in \{1, \ldots, n+1\} \), and \( ua = y_1 \) for every \( u \in X - \{x_1, \ldots, x_{n+1}\} \). It is clear that \( f_1a = f \). Let \( (z_1, \ldots, z_{n+1}) \in \rho \). If \( (z_1a, \ldots, z_{n+1}a) \) is not injective then \( (z_1a, \ldots, z_{n+1}a) \in \rho \) by \((R)\) and \((S)\). Suppose \( (z_1a, \ldots, z_{n+1}a) \) is injective. By the definition of \( a \), we have \( z_ia \in \{y_1, \ldots, y_{n+1}\} \) for every \( i \in \{1, \ldots, n+1\} \). It follows that \( (y_1, \ldots, y_{n+1}) \) is injective and \( (y_1, \ldots, y_{n+1}) = (y_{1\sigma}, \ldots, y_{(n+1)\sigma}) \) for some permutation \( \sigma \) of \( \{1, \ldots, n+1\} \). Thus \( (z_1a, \ldots, z_{n+1}a) \in \rho \) by \((S)\). Hence \( \rho \) is dense.®

In fact, we proved a stronger result: if \( \rho \) is an \((n+1)\)-ary relation on \( X \) such that \( \rho \) satisfies \((R)\) and \((S)\) and \( \rho \neq \mu_n \) then \( \rho \) is dense. This gives Lemma 4.4 as a special case.

With Theorem 4.3 and Lemma 5.1, we can describe the automorphism group of \( T(X, \rho) \) for any generalized equivalence relation \( \rho \).

**Theorem 5.2.** Let \( \rho \) be a generalized equivalence relation on \( X \). Then

\[ \text{Aut}(T(X, \rho)) \cong \text{Aut}(X, \rho). \]

**Proof.** Let \( n \) be the type of \( \rho \). Suppose \( \rho = \mu_n \). Then every \( a : X \to X \) preserves \( \rho \) (since if \( (x_1, \ldots, x_{n+1}) \) is not injective then \( (x_1a, \ldots, x_{n+1}a) \) is not injective). Thus \( T(X, \rho) = T(X) \) and \( \text{Aut}(X, \rho) = S(X) \), and so the result follows from the fact that \( \text{Aut}(T(X)) \) is isomorphic to \( S(X) \).

Suppose \( \rho \neq \mu_n \). Then \( \rho \) is dense (by Lemma 5.1) and symmetric (by \((S)\)). Thus \( \text{Aut}(T(X, \rho)) \) is isomorphic to \( \text{Aut}(X, \rho) \) by Theorem 4.3.®

As an application of Theorem 5.2, we determine the group of automorphisms of \( \text{End}(H) \), where \( H \) is any \( p \)-hypergraph.

A hypergraph is a pair \( H = (X, E) \), where \( X \) is a non-empty set (whose elements are called vertices) and \( E \) is a family of non-empty subsets of \( X \) (whose elements are called edges) [5]. Note that if every edge of a hypergraph \( H \) consists of one or two elements then \( H \) is a graph (see Section 4).

Let \( H = (X, E) \) be a hypergraph. An endomorphism of \( H \) is a map \( a : X \to X \) such that for every edge \( e \in E \) there is an edge \( e' \in E \) such that \( ea \subseteq e' \). The semigroup of all endomorphisms of \( H \) will be denoted by \( \text{End}(H) \).

An automorphism of \( H \) is a bijection \( g : X \to X \) such that for every \( e \subseteq X, e \in E \iff eg \in E \). The group of automorphisms of \( H \) will be denoted by \( \text{Aut}(H) \).

Let \( p \) be a positive integer such that \( p \geq 2 \). Following [30], we define a \( p \)-hypergraph to be a hypergraph \( H \) that satisfies the following three properties:

1. \((H_1)\) Any \( p \) vertices of \( H \) are contained in one and only one edge.
2. \((H_2)\) Every edge contains at least \( p + 1 \) vertices.
3. \((H_3)\) There exist \( p + 1 \) vertices of \( H \) that are not contained in the same edge.

An example of a 2-hypergraph is \( (X, E) = (\mathbb{R}^2, L) \), where \( \mathbb{R}^2 \) is the Euclidean plane and \( L \) is the set of Euclidean lines. (We note that every 2-hypergraph is an incidence geometry [28].) An example of a 3-hypergraph is \( (X, E) = (\mathbb{R}^2, C) \), where \( C \) is the set of Euclidean lines and circles. For an example of a finite 3-hypergraph, see [30, Example 2.2].

It follows from \((H_1)\) and \((H_2)\) that a \( p \)-hypergraph \( H = (X, E) \) is a partition of type \( p \) of \( X \). The corresponding equivalence relation of type \( p \) is given by

\[ \rho = \{(x_1, \ldots, x_{p+1}) \mid x_1, \ldots, x_{p+1} \in e \text{ for some } e \in E\}. \]

Note that \( \text{End}(H) = T(X, \rho) \) and \( \text{Aut}(H) = \text{Aut}(X, \rho) \). Thus, Theorem 5.2 gives us the following corollary.
Corollary 5.3. Let $H = (X, E)$ be a $p$-hypergraph. Then

$$\text{Aut(End}(H)) \cong \text{Aut}(H).$$

6. Ternary equivalence relations

In this section we deal with families of sets intersecting in at most one element. These families have been extensively investigated (see for example [10–12,16–18,33,36]). They are known under many different names such as nearly disjoint hypergraphs [17], families of nearly disjoint sets [36], etc. Since they are more general than the partitions of type 2 considered in Section 5, we will call them g-partitions of type 2.

We will prove that there is a 1–1 correspondence between these families and ternary relations that we call ternary equivalences, and that ternary equivalences are dense relations. Consequently, we will obtain a description of the automorphism group of $\text{End}(X, \rho)$, where $\rho$ is any ternary equivalence on $X$, and of $\text{End}(\mathcal{P})$, where $\mathcal{P}$ is any g-partition of $X$ of type 2.

A ternary relation $\rho$ on $X$ is called a ternary equivalence on $X$ if it satisfies the following five properties:

1. Reflexivity $(R)$: $(x, x, x) \in \rho$ for every $x \in X$.
2. Symmetry $(S)$: For all $x_1, x_2, x_3 \in X$ and every permutation $\sigma$ of $\{1, 2, 3\}$,
   $$(x_1, x_2, x_3) \in \rho \Rightarrow (x_{\sigma 1}, x_{\sigma 2}, x_{\sigma 3}) \in \rho.$$  
3. Transitivity $(T)$: For all $x, y, z, w \in X$ such that $x, y, z$ are pairwise distinct,
   $$(x, y, z) \in \rho \quad \text{and} \quad (y, z, w) \in \rho \Rightarrow (x, y, w) \in \rho.$$  
4.一部 $(U)$: For every $x \in X$, there is $y \in X$ with $y \neq x$ such that $(x, y, x) \in \rho$.
5. Equality $(V)$: For all $x, y, z \in X$,
   $$(x, y, x) \in \rho \Rightarrow (y, x, y) \in \rho.$$

The symbols $(R)$, $(S)$, and $(T)$ stand for reflexivity, symmetry, and transitivity, respectively. Note that if $|X| \geq 2$, then every equivalence relation of type 2 on $X$ (see Section 5) is a ternary equivalence on $X$.

We start by proving that the five axioms defining a ternary equivalence are independent. For $f \in \rho$, let $\Sigma f = \{\sigma f \mid \sigma \in S_{(3)}\}$ and $\Delta = \{(w, w, w) \mid w \in X\}$. For all relations defined below, the underlying set is $X = \{x, y, z\}$, where $x, y, z$ are pairwise distinct. To prove that $(V)$ is independent, consider

$$\rho_V = \Delta \cup \Sigma(x, y, y) \cup \Sigma(x, x, z) \cup \Sigma(z, z, x).$$

The relation $\rho_V$ obviously satisfies $(R)$ and $(S)$; it vacuously satisfies $(T)$ and it satisfies $(U)$ since $(x, z, x), (y, z, y), (z, x, z) \in \rho_V$. However, $(y, x, y) \in \rho_V$, but $(y, x, y) \notin \rho_V$ and hence axiom $(V)$ is not satisfied. To prove that $(U)$ is independent just take $\rho_U = \Delta$. The independence of $(T)$ follows from

$$\rho_T = \Delta \cup \Sigma(x, y, z) \cup \Sigma(x, x, z) \cup \Sigma(x, z, z) \cup \Sigma(y, y, z) \cup \Sigma(y, z, z).$$

This relation satisfies $(U)$ since $(x, z, x), (z, x, z), (y, z, y) \in \rho_T$, and it clearly satisfies $(R)$, $(S)$ and $(V)$. However, $(y, x, z), (z, x, z) \in \rho_T$, but $(y, x, x) \notin \rho_T$. To prove the independence of $(S)$ let

$$\rho_S = \Delta \cup \{(z, z, z), (y, y, y), (x, x, y), (x, y, x), (x, x, z), (z, z, z), (z, y, y)\}.$$

This relation obviously satisfies $(R)$ and $(V)$, but it does not satisfy $(S)$. Regarding $(T)$, we have $(z, y, x), (y, x, y) \in \rho_S$ and also $(z, y, y) \in \rho_S$. Since there is no other way of applying the transitivity, it follows that the relation satisfies $(T)$. It satisfies $(U)$ since $(x, z, x), (z, x, z), (y, x, y) \in \rho_T$. Finally, the independence of $(R)$ results from $\rho_R$ defined as follows:

$$\rho_R = \{(y, y, y), (z, z, z)\} \cup \Sigma(x, x, y) \cup \Sigma(x, y, y) \cup \Sigma(x, x, z) \cup \Sigma(x, z, z).$$
Axioms (S) and (V) are obviously satisfied, and (T) is vacuously satisfied. Axiom (U) follows from the fact that \((x, z, x), (z, x, z), (y, x, y) \in \rho_R\).

Now that we have established the independence of the defining axioms of ternary equivalence relations, we prove three easy properties of these relations.

**Lemma 6.1.** Let \(\rho\) be a ternary equivalence relation on \(X\). For all pairwise distinct \(x, y, z \in X\), if \((x, y, z) \in \rho\) then \((x, x, y) \in \rho\).

**Proof.** Suppose \((x, y, z) \in \rho\). Then, by symmetry, \((y, z, x) \in \rho\). By transitivity, \((x, y, z), (y, z, x) \in \rho\) implies \((x, y, x) \in \rho\), and so \((x, x, y) \in \rho\) by symmetry. \(\square\)

**Lemma 6.2.** Let \(\rho\) be a ternary equivalence relation on \(X\). Suppose \((u, v, w) \in \rho\). Then \((x, y, z) \in \rho\) for all \(x, y, z \in \{u, v, w\}\).

**Proof.** Suppose that \(u, v, w\) are pairwise distinct. If \(|\{x, y, z\}|=3\), then the result follows from (S). Suppose \(|\{x, y, z\}|=2\). By (S), we may assume without loss of generality that \(x=y\neq z\) and that \(w \notin \{x, z\}\). We want to prove that \((x, x, z) \in \rho\). If \((x, x, z) \in \rho\), then \((x, x, x) \in \rho\) since \(\rho\) is reflexive.

Now suppose that \(u, v, w\) are not pairwise distinct. By (S), we may assume that \(v=w\). Then we have \((u, v, v) \in \rho\) and \(x, y, z \in \{u, v\}\). In this case, \((x, y, z) \in \rho\) by (S) and (V) (if \(\{x, y, z\} = \{u, v\}\)) and by (R) (if \(\{x, y, z\} = \{u\}\) or \(\{x, y, z\} = \{v\}\)). \(\square\)

**Lemma 6.3.** Let \(\rho\) be a ternary equivalence relation on \(X\). Suppose \((x, y, z) \in \rho\) and \((y, z, w) \in \rho\) with \(y \neq z\). Then \((x, v, w) \in \rho\).

**Proof.** If \(x \notin \{y, z\}\), then \((x, y, w) \in \rho\) by (T). If \(x \in \{y, z\}\), then \(x, y, w \in \{y, z, w\}\), and so \((x, y, w) \in \rho\) by Lemma 6.2. \(\square\)

A family \(\mathcal{P}\) of subsets of \(X\) is called a \(g\)-partition of type 2 of \(X\) if it satisfies the following three properties:

1. \(\bigcup_{A \in \mathcal{P}} A = X\).
2. \(|A| \geq 2\) for every \(A \in \mathcal{P}\).
3. \(|A \cap B| \leq 1\) for all distinct \(A, B \in \mathcal{P}\).

Note that every partition of type 2 of \(X\) (see Section 5) is a \(g\)-partition of type 2 of \(X\).

We want to prove that there is a 1–1 correspondence between the ternary equivalences on \(X\) and \(g\)-partitions of type 2 of \(X\). Let \(\rho\) be a ternary relation on \(X\). For \(x, y \in X\) with \(x \neq y\), we denote by \(A^\rho_{x,y}\) the set \(\{z \in X \mid (x, y, z) \in \rho\}\).

**Lemma 6.4.** Let \(u, v \in X\) be such that \(u, v \in A^\rho_{x,y}\). Then \(A^\rho_{u,v} \subseteq A^\rho_{x,y}\).

**Proof.** We have \((x, y, u), (x, y, v) \in \rho\). Since \(x \neq y\), \((u, x, y), (y, x, v) \in \rho\) implies \((u, x, v) \in \rho\) by Lemma 6.3.

Suppose \(z \in A^\rho_{u,v}\), that is, \((u, v, z) \in \rho\). Since \(u \neq v\), \((u, x, v) \in \rho\) implies \((x, u, z) \in \rho\) by Lemma 6.3. If \(u \neq x\), then \((y, x, u), (u, x, z) \in \rho\) implies \((y, x, z) \in \rho\) by Lemma 6.3. Suppose \(u = x\). Then we have \((x, y, x), (x, y, v) \in \rho\). Note that \(x \neq v\) since \(u = x\) and \(u \neq v\). Thus \((y, x, v) \in \rho\) implies \((y, x, z) \in \rho\) by Lemma 6.3.

Thus in every possible case, \((y, x, z) \in \rho\). Hence \((x, y, z) \in \rho\), that is, \(z \in A^\rho_{x,y}\), and so \(A^\rho_{u,v} \subseteq A^\rho_{x,y}\). \(\square\)

**Theorem 6.5.** There is a 1–1 correspondence between \(g\)-partitions of type 2 of \(X\) and ternary equivalences on \(X\).

**Proof.** Let \(\rho\) be a ternary equivalence on \(X\). Define

\[\mathcal{P} = \{A \subseteq X \mid A = A^\rho_{x,y} \text{ for some } x, y \in X \text{ such that } A^\rho_{x,y} \neq \emptyset\}\.](6.1)

We claim that the family \(\mathcal{P}\) is a \(g\)-partition of type 2 of \(X\).
Let \( x \in X \). By \((U)\), there exists a \( y \in X \) with \( y \neq x \) such that \((x, y, x), (y, x, y) \in \rho\). Thus \( x \in A_{x,y}^\rho \), implying that \( \bigcup_{A \in \mathcal{P}} A = X \). Further, \((y, x, y) \in \rho\) implies \((x, y, y) \in \rho\), and so \( y \in A_{x,y}^\rho \). Since \( x \neq y \), we have \( |A_{x,y}^\rho| \geq 2 \). We proved that the family \( \mathcal{P} \) satisfies \((1)\) and \((2)\).

Now suppose that \( |A_{x,y}^\rho \cap A_{u,v}^\rho| \geq 2 \), say \( z, w \in A_{x,y}^\rho \cap A_{u,v}^\rho \) with \( z \neq w \). Then \((x, y, z), (x, y, w), (u, v, z), (u, v, w) \in \rho\). Since \( x \neq y \), \((z, x, y), (x, y, w) \in \rho\) implies \((z, x, w) \in \rho\) by Lemma 6.3. By a similar argument, \((z, u, w) \in \rho\).

We want to prove that \( x \in A_{u,v}^\rho \), that is, that \((u, v, x) \in \rho\). Since \( z \neq w \), we have by Lemma 6.3 that \((x, z, w), (z, w, u) \in \rho\) implies \((x, z, u) \in \rho\). If \( z \neq u \), then \((x, u, z), (u, z, v) \in \rho\) implies \((x, u, v) \in \rho\) by Lemma 6.3. Suppose \( z = u \). Note that then \( u \neq w \). Then we have \((x, u, y), (x, v, y), (u, v, u), (u, v, w) \in \rho\). Since \( x \neq y \), we have by Lemma 6.3 again that \((u, w, v) \in \rho\).

Thus, in any case, \((u, v, x) \in \rho\), and so \( x \in A_{u,v}^\rho \). In the same way we prove that \( y \in A_{u,v}^\rho \), and so \( A_{x,y}^\rho \subseteq A_{u,v}^\rho \). By symmetry, \( A_{y,z}^\rho \subseteq A_{u,v}^\rho \), and so \( A_{x,y}^\rho = A_{u,v}^\rho \). This proves that \( \mathcal{P} \) satisfies \((3)\).

Conversely, let \( \mathcal{P} \) be a \( g \)-partition of type 2 of \( X \). Define
\[
\rho = \{(x, y, z) \mid x, y, z \in A \text{ for some } A \in \mathcal{P}\}.
\]
(6.2)
We claim that \( \rho \) is a ternary equivalence on \( X \). Let \( x \in X \). Then, by \((1)\), \( x \in A \) for some \( A \in \mathcal{P} \), and so \((x, x, x) \in \rho\). Moreover, by \((2)\), \( |A| \geq 2 \) so that there exists \( y \in A \) with \( y \neq x \). Hence \((x, y, x) \in \rho\). We proved that \( \rho \) satisfies \((R)\) and \((U)\).

It is obvious that \( \rho \) is symmetric and that it satisfies \((V)\). It remains to prove that \( \rho \) is transitive. Let \((x, y, z), (y, z, w) \in \rho\) with \( x, y, z \) pairwise distinct. Then \( x, y, z \in A \) and \( y, z, w \in B \) for some \( A, B \in \mathcal{P} \). Thus \( |A \cap B| \geq 2 \) and hence \( A = B \). Therefore, \( x, y, w \in A \), and so \((x, y, w) \in \rho\).

It is easy to see that the functions defined in (6.1) and (6.2) are inverses of each other. The result follows. \( \square \)

Let \( \mu_2 \) be the ternary relation on \( X \) consisting of all non-injective triples \((x_1, x_2, x_3)\) of elements of \( X \) (see (5.1)).

**Lemma 6.6.** Let \( \rho \) be a ternary equivalence on \( X \) that is not included in \( \mu_2 \). Then \( \rho \) is dense.

**Proof.** Since \( \rho \) is not included in \( \mu_2 \), it must contain at least one injective triple, that is, it satisfies \((D_2)\).

Let \((x, y, z) \in \rho \cup \rho^* \) be injective and let \((u, v, w) \in \rho\). Let \( a : X \rightarrow \{u, v, w\} \) be any map such that \( xa = u \), \( ya = v \), and \( za = w \). Then clearly \((x, y, z) a = (u, v, w) \). Let \((x_1, x_2, x_3) \in \rho\). Then \( x_1 a, x_2 a, x_3 a \in \{u, v, w\} \), and so \((x_1 a, x_2 a, x_3 a) \in \rho\) by Lemma 6.2. Thus \( a \in T(X, \rho) \), and so \( \rho \) satisfies \((D_1)\). \( \square \)

**Lemma 6.7.** Let \( \rho \) be a ternary equivalence on a set \( X \) that is included in \( \mu_2 \). Then \( \text{Aut}(T(X, \rho)) \approx \text{Aut}(X, \rho) \).

**Proof.** Define a binary relation \( \rho_0 \) on \( X \) by
\[
(x, y) \in \rho_0 \iff (x, y, x) \in \rho.
\]
By \((R)\) and \((V)\), \( \rho_0 \) is reflexive and symmetric. We claim that \( T(X, \rho_0) \approx T(X, \rho) \). Let \( a \in T(X, \rho_0) \). Let \((u, v, w) \in \rho\). Since \( \rho \) is included in \( \mu_2 \), \( u, v, w \) are not distinct, that is, \( \{u, v, w\} = \{x, y\} \) for some \( x, y \in X \). Then \( x, y \in \{u, v, w\} \), and so \((u, v, x) \in \rho \) implies \((x, y, x) \in \rho \) by Lemma 6.2. Hence \((x, y) \in \rho_0 \), and so, since \( a \in T(X, \rho_0) \), \((xa, ya) \in \rho_0 \). By the definition of \( \rho_0 \), we now have \((xa, ya, xa) \in \rho \). Since \( \{u, v, w\} = \{x, y\} \), we have \( ua, va, wa \in \{xa, ya\} \), and so \((xa, ya, xa) \in \rho \) implies \((ua, va, wa) \in \rho \) by Lemma 6.2. Hence \( a \in T(X, \rho) \), and so \( T(X, \rho_0) \subseteq T(X, \rho) \).

Conversely, let \( a \in T(X, \rho) \). Then for all \( x, y \in X \),
\[
(x, y) \in \rho_0 \Rightarrow (x, y, x) \in \rho
\]
\[
\Rightarrow (xa, ya, xa) \in \rho
\]
\[
\Rightarrow (xa, ya) \in \rho_0.
\]
Thus, \( a \in T(X, \rho_0) \), and so \( T(X, \rho) \subseteq T(X, \rho_0) \). The claim has been proved.

Since \( \text{Aut}(T(X, \rho_0)) \) and \( \text{Aut}(X, \rho) \) are the groups of units of \( T(X, \rho_0) \) and \( T(X, \rho) \), respectively, it follows from the claim that \( \text{Aut}(X, \rho_0) \approx \text{Aut}(X, \rho) \). By Theorem 4.3 and Lemma 4.4, \( \text{Aut}(T(X, \rho_0)) \approx \text{Aut}(X, \rho_0) \). Hence \( \text{Aut}(T(X, \rho)) \approx \text{Aut}(X, \rho) \). \( \square \)
Theorem 6.8. Let \( \rho \) be any ternary equivalence on a set \( X \). Then
\[
\text{Aut}(T(X, \rho)) \cong \text{Aut}(X, \rho).
\]

Proof. If \( \rho \) is included in \( \mu_2 \), then the result follows from Lemma 6.7. If \( \rho \) is not included in \( \mu_2 \), then \( \rho \) is dense by Lemma 6.6, and so \( \text{Aut}(T(X, \rho)) \cong \text{Aut}(X, \rho) \) by Theorem 4.3. \( \square \)

Let \( P \) be any family of subsets of \( X \). An endomorphism of \( P \) is a map \( a : X \to X \) such that for every \( A \in P \), there is \( B \in P \) such that \( Aa \subseteq B \). An automorphism of \( P \) is a bijection \( g : X \to X \) such that for every \( A \subseteq X \), \( A \in P \iff Ag \in P \). We denote by \( \text{End}(P) \) and \( \text{Aut}(P) \) the endomorphism monoid of \( P \) and the automorphism group of \( P \), respectively.

Let \( P \) be a \( g \)-partition of type 2 of \( X \). Recall that the corresponding ternary relation on \( X \) is defined by
\[
\rho = \{(x, y, z) \mid x, y, z \in A \text{ for some } A \in P\}.
\]

It follows from Lemma 6.2 that \( \text{End}(P) = T(X, \rho) \) and \( \text{Aut}(P) = \text{Aut}(X, \rho) \). Thus Theorem 6.8 gives us the following corollary.

Corollary 6.9. Let \( P \) be any \( g \)-partition of type 2 of a set \( X \). Then
\[
\text{Aut}(\text{End}(P)) \cong \text{Aut}(P).
\]

7. Problems

We conclude with listing several problems suggested by the general approach used in the paper. In Section 3, we determined \( \text{Aut}(T(X, \rho)) \) for an arbitrary dense relation \( \rho \) on a set \( X \).

(1) Describe the \( n \)-ary dense relations on a set \( X \). The starting point might be a description of the binary dense relations on a finite set \( X \).

For various binary reflexive relations \( \rho \), \( \text{Aut}(T(X, \rho)) \cong \text{Aut}(X, \rho) \cup \text{Aut}^*(X, \rho) \) (see [30,34]). We proved (see Theorem 3.1) that for every dense relation \( \rho \), \( \text{Aut}(T(X, \rho)) \cong \text{Aut}_p(X, \rho) \) (which reduces to the above isomorphism in the case of binary relations).

(2) Describe the binary reflexive relations \( \rho \) on a set \( X \) such that \( \text{Aut}(T(X, \rho)) \cong \text{Aut}(X, \rho) \cup \text{Aut}^*(X, \rho) \). More generally, describe the \( n \)-ary reflexive relations \( \rho \) on \( X \) such that \( \text{Aut}(T(X, \rho)) \cong \text{Aut}_p(X, \rho) \). (This class includes the class of dense relations.)

In Section 4, we showed how our method yields a description of \( \text{Aut}(\text{End}(X, \leq)) \), where \( \leq \) is a partial order on \( X \).

(3) Describe \( \text{Aut}(\text{End}(X, \leq)) \), where \( \leq \) is a generalized partial order on \( X \) (see [39]).

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References


