Distance paired domination numbers of graphs

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Abstract

In this paper, we study a generalization of the paired domination number. Let $G = (V, E)$ be a graph without an isolated vertex. A set $D \subseteq V(G)$ is a $k$-distance paired dominating set of $G$ if $D$ is a $k$-distance dominating set of $G$ and the induced subgraph $\langle D \rangle$ has a perfect matching. The $k$-distance paired domination number $\gamma_{kp}^k(G)$ is the cardinality of a smallest $k$-distance paired dominating set of $G$. We investigate properties of the $k$-distance paired domination number of a graph. We also give an upper bound and a lower bound on the $k$-distance paired domination number of a non-trivial tree $T$ in terms of the size of $T$ and the number of leaves in $T$ and we also characterize the extremal trees.

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1. Introduction

In this paper, all graphs considered will be finite and without multiple loops or edges. Let $G = (V, E)$ be a graph without an isolated vertex. A set $D \subseteq V(G)$ is a dominating set of $G$ if every vertex in $V(G) - D$ is adjacent to at least one vertex in $D$. A set $D \subseteq V(G)$ is a paired dominating set of $G$ if it is dominating and the induced subgraph $\langle D \rangle$ has a perfect matching. The paired domination number $\gamma_p(G)$ is the cardinality of a smallest paired dominating set of $G$. This type of domination was introduced by Haynes and Slater in [5,4] and is studied, for example, in [6,7].

For two vertices $x$ and $y$, let $d_G(x, y)$ denote the distance between $x$ and $y$ in $G$. If $D$ is a set of vertices of $G$ and $x$ is a vertex of $G$, then the distance from $x$ to $D$, denoted by $d_G(x, D)$, is the minimum distance from $x$ to a vertex of $D$. A set $D \subseteq V(G)$ is a $k$-distance dominating set of $G$ if every vertex in $V(G) - D$ is within distance $k$ of at least one vertex in $D$. The $k$-distance domination number $\gamma^k(G)$ of $G$ equals the minimum cardinality among all $k$-distance dominating sets of $G$.

In this paper, we study a generalization of the paired domination number. We say that a set $D \subseteq V(G)$ is a $k$-distance paired dominating set of $G$ if $D$ is a $k$-distance dominating set of $G$ and the induced subgraph $\langle D \rangle$ has a perfect matching. The $k$-distance paired domination number $\gamma_{kp}^k(G)$ is the cardinality of a smallest $k$-distance paired dominating set of $G$.

Let $n(G)$ be the cardinality of the vertex set $V(G)$. The degree $d_G(x)$ of a vertex $x \in V(G)$ is the number of edges incident with $x$. The set $N^k_G[x] = \{y \in V(G) : d_G(x, y) \leq k\}$ is called the closed $k$-neighbourhood of $x$ in $G$. Define

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Theorem 3. Let $D$ be a dominating set of $G$ and $z \in D$. We say that $z$ is a private $k$-distance neighbour of $x$, with respect to a set $D$. Similarly, define $PN_G^k[x, D] = N_G^k[x] - N_G^k[D - \{x\}]$ be the private $k$-distance neighbourhood of a vertex $x$, with respect to a set $D$. If $z \in PN_G^k[x, D]$, we say that $z$ is a private $k$-distance neighbour of $x$, with respect to a set $D$. Similarly, define $PN_G^k[x, y, D] = (N_G^k[x] \cup N_G^k[y]) - N_G^k[D - \{x, y\}]$ to be the private $k$-distance neighbourhood of vertices $x$ and $y$, with respect to a set $D$. If $z \in PN_G^k[x, y, D]$, we say that $z$ is a private $k$-distance neighbour of $\{x, y\}$, with respect to a set $D$. The diameter $diam(G)$ of a connected graph $G$ is the number max$_{x,y \in V(G)} d_G(x,y)$. An edge $k$-subdivision in a nonempty graph $G$ is an operation of removal of an edge $xy \in E(G)$ and the addition of a new path $(z_1, \ldots, z_k)$ and edges $xz_1$ and $yz_k$. For a tree $T$, let $n_1(T)$ be the number of all leaves of $T$, that is the number of vertices with degree one in $T$. A vertex $x \in V(T)$ is called a support if $x$ is a neighbour of a leaf. For any unexplained terms and symbols see [2,3].

In this paper, we investigate properties of the $k$-distance paired domination number of a graph. We also give an upper bound and a lower bound on the $k$-distance paired domination number of a non-trivial tree $T$ in terms of the size of $T$ and the number of leaves in $T$ and we characterize the extremal trees.

2. Preliminary results

We begin with basic properties of $k$-distance paired dominating sets.

Observation 1. Every graph $G$ with no isolated vertex has a $k$-distance paired dominating set and the number $\gamma^k_p(G)$ is even.

Proposition 2. For any graph $G$ with no isolated vertex, $\gamma^k(G) \leq \gamma^k_p(G) \leq \gamma_p(G)$.

Proof. If $D$ is a minimum paired dominating set of $G$, then $D$ is a $k$-distance paired dominating set of $G$ and therefore, $\gamma^k_p(G) \leq |D| = \gamma_p(G)$. Similarly, if $D$ is a minimum $k$-distance paired dominating set of $G$, then $D$ is a $k$-distance dominating set of $G$ and thus $\gamma^k(G) \leq |D| = \gamma^k_p(G)$. □

Haynes and Slater [5] have proved what follows.

Theorem 3 (Haynes and Slater [5]). A graph $G$ with no isolated vertex has $\gamma_p(G) = n(G)$ if and only if each component of $G$ is $K_2$.

Here we prove a similar result for the $k$-distance paired domination number of a graph.

Proposition 4. If $G$ is a graph with no isolated vertex, then $\gamma^k_p(G) = n(G)$ if and only if each component of $G$ is $K_2$.

Proof. For $k = 1$ the result follows from Theorem 3. Thus we consider only $k \geq 2$. If each component of $G$ is $K_2$, then clearly $\gamma^k_p(G) = n(G)$. Assume now that $\gamma^k_p(G) = n(G)$. Then Proposition 2 implies that $\gamma_p(G) = n(G)$ and by Theorem 3, each component of $G$ is $K_2$. □

Proposition 5. For a path $P_n$ and a cycle $C_n$ on $n \geq 3$ vertices, $\gamma^k_p(P_n) = \gamma^k_p(C_n) = 2\lceil \frac{n}{2k+2} \rceil$.

Proof. Let $D = \{x_i, y_i : i = 1, 2, \ldots, q\}$ be a minimum $k$-distance paired dominating set of a path $P_n = (v_1, \ldots, v_n)$, where $\{x_i, y_i : i = 1, 2, \ldots, q\}$ is a perfect matching of $\{D\}$. Then each pair $\{x_i, y_i\}$ dominates at most $2k$ vertices of $V(P_n) - D$. Hence, $2q + 2k \cdot q \geq n$ and thus $q \geq \frac{n}{2k+2}$, which implies that $\gamma^k_p(P_n) \geq 2\lceil \frac{n}{2k+2} \rceil$.

On the other hand, it is immediate that for $n \leq k + 2$ the set $D = \{v_1, v_2\}$ is a minimum $k$-distance paired dominating set of cardinality $2 = 2\lceil \frac{n}{2k+2} \rceil$. Similarly, for $n \geq k + 3$ and $n = 1 \mod(2k + 2)$ it is possible to see that the set $D = \{v_{2(i+k+1)-1}, v_{2(i+k+1)-k} : i = 1, \ldots, \lceil \frac{n}{2k+2} \rceil - 1\} \cup \{v_{n-k}, v_{n-k+1}\}$ is a $k$-distance paired dominating set of cardinality $2\lceil \frac{n}{2k+2} \rceil$. Further, for all other $n$, the set $D = \{v_{2(i+k+1)-1}, v_{2(i+k+1)-k} : i = 1, \ldots, \lceil \frac{n}{2k+2} \rceil - 1\} \cup \{v_{n-k}, v_{n-k+1}\}$ is a $k$-distance paired dominating set of cardinality $2\lceil \frac{n}{2k+2} \rceil$, which completes the proof that $\gamma^k_p(P_n) = 2\lceil \frac{n}{2k+2} \rceil$. The proof for a cycle is similar and thus is omitted. □

Haynes and Slater [5] have proved that if a graph $G$ has no isolated vertex, then $\gamma_p(G) \geq n(G)/\Delta(G)$. We generalize this result for the $k$-distance paired domination number for the case $\Delta(G) \geq 3$. (Propositions 4 and 5 determine the $k$-distance paired domination number of a graph $G$ when $\Delta(G) = 1$ or $\Delta(G) = 2$.)
Fig. 1. A graph $G$ with $i_p^k(G) = \frac{n(G)(A-2)}{(A-1)^{k+1} - 1}$ for $k = 2$.

**Proposition 6.** If $G$ has no isolated vertex and $A = \Delta(G) \geq 3$, then

$$i_p^k(G) \geq \frac{n(G)(A-2)}{(A-1)^{k+1} - 1}.$$

**Proof.** Each vertex $v$ in a minimum $k$-distance paired dominating set $D$ of $G$ dominates at most $A - 1$ vertices of $V(G) - D$ at distance 1 from $v$, at most $(A - 1)^2$ vertices of $V(G) - D$ at distance 2 from $v$ and so on (Fig. 1). Thus $v$ dominates at most $(A - 1) + (A - 1)^2 + \cdots + (A - 1)^k = \frac{(A-1)^{k+1}-A+1}{A-2}$ vertices of $V(G) - D$. Hence,

$$n(G) \leq i_p^k(G) + \frac{n(G)(A-2)}{(A-1)^{k+1} - 1} - 1,$$

which gives

$$i_p^k(G) \geq \frac{n(G)(A-2)}{(A-1)^{k+1} - 1}.$$  \square

3. **Complexity issues for the $k$-distance paired dominating set decision problem**

In this section, we consider the decision problem of $k$-DISTANCE PAIRED DOMINATING SET as follows:

**$k$-DISTANCE PAIRED DOMINATING SET ($k$-PDS)**

**INSTANCE:** A connected graph $H = (V_H, E_H)$ and a positive integer $q$.

**QUESTION:** Does $G$ have a $k$-distance paired dominating set of size at most $q$?

Haynes and Slater [5] have proved that 1-PDS decision problem is NP-complete. Here we show that the decision problem $k$-PDS for $k > 1$ stays NP-complete even when restricted to bipartite graphs.

**Theorem 7.** $k$-DISTANCE PAIRED DOMINATING SET problem for $k > 1$ for bipartite graphs is NP-complete.

**Proof.** To prove the statement we use a polynomial reduction from DOMINATING SET (DS), which is known to be NP-complete problem (see [1]).

**DOMINATING SET (DS)**

**INSTANCE:** A graph $G = (V, E)$ and a positive integer $j$.

**QUESTION:** Does $G$ have a dominating set of size at most $j$?

It is possible to observe that $k$-PDS problem for bipartite graphs is in NP class of decision problems as it is easy to verify in polynomial time whether a given subset of vertices of a graph is a $k$-distance paired dominating set.

For a given instance of DS problem, which is a graph $G$ and a positive integer $j$, we construct a bipartite graph $H$ and an integer $q$ as follows:

$$(V, 1) = V(G) \times \{1\}, (V, 2) = V(G) \times \{2\}, \ldots, (V, 2k + 2) = V(G) \times \{2k + 2\},$$

$$V_H(H) = \bigcup_{i=1}^{2k+2} (V, i),$$
The obtained graph $H$ is bipartite, as every cycle in $H$ has an even length (see Fig. 2).

Assume first that $G$ has a dominating set of cardinality at most $j$, say $\{u_1, \ldots, u_i\}$ is a dominating set of $G$ and $i \leq j$. Then the set $\{(u_1, k + 1), (u_1, k + 2), (u_2, k + 1), (u_2, k + 2), \ldots, (u_i, k + 1), (u_i, k + 2)\}$ is clearly a $k$-distance paired dominating set of size $2i + j$ for $k > 1$.

Assume now that $H$ has a $k$-distance paired dominating set of size at most $q = 2j$. We shall show that $G$ has a dominating set of size at most $j$. Since $\text{diam}(H) = 2k + 1$, it is possible to observe that $H$ has a $k$-distance paired dominating set of size at most $q$, denoted $D_p$, which is a subset of $V(G) \times \{k + 1, k + 2\}$. Let $M$ be the perfect matching of $(D_p)$. Then $(u, k + 1) \in D_p$ if and only if $(u, k + 2) \in D_p$, and $(u, k + 1)(u, k + 2) \in M$. Hence, $D = \{u \in V(G) : (u, k + 1) \in D_p\}$ is a dominating set of $G$ of cardinality at most $j$.

It is obvious that the transformation used is polynomial as $H$ has $(2k + 2)n(G)$ vertices and $(2k + 1)n(G) + 4m(G)$ edges. □

4. Upper bound on the $k$-distance paired domination number of a tree

We begin this section with the following useful lemma.

Lemma 8. If $T$ is a tree with $\gamma_p^k(T) \geq 4$, then there exists an edge $e \in E(T)$ such that for the two connected components of $T - e$ denoted $T_1$ and $T_2$ is $\gamma_p^k(T_1) + \gamma_p^k(T_2) = \gamma_p^k(T)$. In addition, $n(T_1) \geq k + 1$ and $n(T_2) \geq k + 2$.

Proof. Let $S = (s_0, s_1, \ldots, s_l)$ be a longest path in $T$. Since $\gamma_p^k(T) \geq 4$, it follows that $l \geq 2k + 2$. It is easy to see that if $l = 2k + 2$, then the edge $e = s_{k}s_{k+1}$ has desired properties. Clearly $n(T_1) \geq k + 1$ and $n(T_2) \geq k + 2$, where $s_k \in T_1$ and $s_{k+1} \in T_2$. Thus we consider $l \geq 2k + 3$.

Among all minimum $k$-distance paired dominating sets of $T$ consider those which have a perfect matching containing $s_{k}s_{k+1}$ and $s_{l-k}s_{l-k-1}$. Denote the set of such minimum $k$-distance paired dominating sets of $T$ by $\mathcal{D}$. Let $D$ be an element of $\mathcal{D}$ such that:

$$\sum_{v \in D} d_T(v, S) = \min \left\{ \sum_{v \in D'} d_T(v, S) : D' \in \mathcal{D} \right\}. $$

Denote $P_1 = \{v \in V(T) : d_T(v, S) = d_T(v, s_i)\}$.

Since $S$ is a longest path in $T$, it is easy to observe that $s_k$ $k$-dominates each vertex of $P_0 \cup P_1 \cup \cdots \cup P_k$. Moreover, $D \cap (P_0 \cup \cdots \cup P_k) = \{s_0\}$, because $D$ is a minimum $k$-distance paired dominating set of $T$ and $s_k \in D$. If $D \cap P_{k+1} \neq \{s_{k+1}\}$, then $|P_{k+1}| \geq k + 2$, because otherwise $s_{k+1}$ would dominate all vertices of $P_{k+1}$. Let $s_ku \in E(T)$ be an edge such that $u \in P_{k+1} \cap D$ and let $v \in P_{k+1} \cap D$ be the vertex paired with $u$ in $(D)$. Let $T_1$ and $T_2$ be the two components of $T - s_k s_{k+1}$, where $s_k \in V(T_1)$ and $s_{k+1} \in V(T_2)$. Then $\gamma_p^k(T_1) = 2$ and $\gamma_p^k(T_2) = \gamma_p^k(T) - 2$, because

\begin{align*}
E_H(H) &= \{(u, i)(u, i + 1) : i = 1, 2, \ldots, 2k + 1, u \in V(G)\} \\
&\cup \{(u, k)(v, k + 1) : u, v \in V(G), uv \in E(G)\} \\
&\cup \{(u, k + 2)(v, k + 3) : u, v \in V(G), uv \in E(G)\},
q = 2j.
\end{align*}

Fig. 2. Reduction from DS to $k$-PDS for $G = P_3$.  

\[ E_H(H) = \{(u, i)(u, i + 1); i = 1, 2, \ldots, 2k + 1, u \in V(G)\} \]

\[ \cup \{(u, k)(v, k + 1); u, v \in V(G), uv \in E(G)\} \]

\[ \cup \{(u, k + 2)(v, k + 3); u, v \in V(G), uv \in E(G)\}, \]

\[ q = 2j. \]
the set $D' = D - \{s_k, v\}$ is a minimum $k$-distance paired dominating set of $T_2$. Note that $s_{k+1}$ and $u$ are paired in the subgraph induced by $D'$ in $T_2$. It is clear that $n(T_1) \geq k + 1$ and $n(T_1) = k + 1$ if and only if $T_1$ is a path on $k + 1$ vertices. Moreover, since $s_{i-k}, s_{i-k-1} \in V(T_2)$, we see that $n(T_2) \geq k + 2$.

From now on we consider the case $D \cap P_{k+1} = \{s_{k+1}\}$. Assume that there exists a vertex $x \in P_i$, where $i \geq k + 1$, such that $x$ is a private $k$-distance neighbour of the pair $\{s_k, s_{k+1}\}$ with respect to $D$. (Observe that $s_{k+1}$ may also be a private $k$-distance neighbour of $\{s_k, s_{k+1}\}$ with respect to $D$.) We claim that in this situation $D \cap (P_0 \cup \cdots \cup P_i) = \{s_k, s_{k+1}\}$. The result is clear when $x \in P_{k+1}$. Hence, let $i \geq k + 2$. We have already justified that $D \cap (P_0 \cup \cdots \cup P_k) = \{s_k\}$ and we have assumed that $D \cap P_{k+1} = \{s_{k+1}\}$. Thus, suppose that there exists a vertex $u$ which belongs to $D \cap (P_{k+2} \cup \cdots \cup P_i)$. Clearly, $u \notin V(S)$, because otherwise $x$ would not be a private $k$-distance neighbour of $\{s_k, s_{k+1}\}$ with respect to $D$. Hence, there exists a vertex $v \in D$ matched with $u$ in $D$ and $u, v \in P_j$, where $j \in \{k + 2, \ldots, i\}$. Moreover, $d_T(u, v) \leq j - k$, because $S$ is a longest path in $T$ and $\sum_{v \in D} d_T(v, S)$ is minimum. Consequently, $d_T(u, v) \leq j - k - 1$. Further, since $s_{k+1}$ $k$-dominates $x$, we see that

$$d_T(s_{k+1}, x) = d_T(s_{k+1}, s_i) + d_T(s_i, x) = i - k - 1 + d_T(s_i, x) \leq k,$$

which gives $d_T(s_i, x) \leq 2k - i + 1$. Since $u$ does not $k$-dominate $x$, we obtain

$$k < d_T(u, x) \leq d_T(u, s_j) + d_T(s_j, s_i) + d_T(s_i, x) \leq (j - k - 1) + (i - j) + (2k - i + 1) = k,$$

which is impossible. Hence, if there exists a vertex $x \in P_i$, where $i \geq k + 1$, such that $x$ is a private $k$-distance neighbour of $\{s_k, s_{k+1}\}$ with respect to $D$, then $D \cap (P_0 \cup \cdots \cup P_i) = \{s_k, s_{k+1}\}$.

Now we prove that if there exists a vertex $x \in P_i$, where $i \geq k + 1$, such that $x$ is a private $k$-distance neighbour of $\{s_k, s_{k+1}\}$ with respect to $D$, then $\{s_k, s_{k+1}\} k$-dominates each vertex of $P_0 \cup \cdots \cup P_i$. Of course, $s_k$ $k$-dominates each vertex belonging to $P_0 \cup \cdots \cup P_k$, so suppose that there exists $y \in P_j$, where $j \in \{k + 1, \ldots, i\}$ such that $y$ is $k$-dominated neither by $s_k$ nor by $s_{k+1}$. Then $y$ is $k$-dominated by a vertex $u \in D \cap P_h$ for some $h \in \{i + 1, \ldots, i\}$. Since $s_{k+1}$ $k$-dominates $x$, we obtain $d_T(s_i, x) \leq 2k - i + 1$ and since $s_{k+1}$ does not $k$-dominate $y$, we obtain $d_T(s_j, y) > 2k - j + 1$. As $u$ does not $k$-dominate $x$ we have

$$k < d_T(u, x) = d_T(u, s_h) + d_T(s_h, s_j) + d_T(s_j, x) \leq d_T(u, s_h) + (h - i) + (2k - i + 1).$$

Hence, $d_T(u, s_h) > 2i - k - h - 1$. Since $h > j$ and $u$ $k$-dominates $y$, we obtain

$$k > d_T(u, y) = d_T(u, s_h) + d_T(s_h, s_j) + d_T(s_j, y) > (2i - k - h - 1) + (h - j) + (2k - j + 1) = 2i - 2j + k,$$

which gives $i < j$, a contradiction. We have proved that if there exists a vertex $x \in P_i$, where $i \geq k + 1$, such that $x$ is a private $k$-distance neighbour of $\{s_k, s_{k+1}\}$ with respect to $D$, then $\{s_k, s_{k+1}\}$ $k$-dominates each vertex of $P_0 \cup \cdots \cup P_i$.

Let $i$ be the maximum integer such that $x \in P_i$, where $x$ is a private $k$-distance neighbour of $\{s_k, s_{k+1}\}$ with respect to $D$. We consider two cases depending on $i$.

**Case 1:** $i \in \{k + 1, \ldots, 2k + 1\}$. Let $T_1$ and $T_2$ be the two components of $T - s_is_{i+1}$, where $s_i \in V(T_1)$ and $s_{i+1} \in V(T_2)$. It is immediate that $\gamma_p^{k}(T_1) = 2$ and $\gamma_p^{k}(T_2) = \gamma_p^{k}(T) - 2$, because $\{s_k, s_{k+1}\}$ $k$-dominates each vertex of $V(T_1)$ and $\{s_k, s_{k+1}\}$ does not have a $k$-private neighbour with respect to $D$ in $V(T_2)$. It is clear that $n(T_1) \geq k + 2$. Moreover, since $s_{i-k}, s_{i-k-1} \in V(T_2)$, we see that $n(T_2) \geq k + 2$.

**Case 2:** $i \in \{0, \ldots, k\}$. Let $T_1$ and $T_2$ be the two components of $T - s_is_{i+1}$, where $s_k \in V(T_1)$ and $s_{i+1} \in V(T_2)$. It is immediate that $\gamma_p^{k}(T_1) = 2$ and $\gamma_p^{k}(T_2) = \gamma_p^{k}(T) - 2$, because the set $\{s_{k-1}, s_k\}$ is a minimum $k$-distance dominating set of $T_1$ and the pair $\{s_{k-1}, s_k\}$ does not have a $k$-private neighbour with respect to $D$ among vertices of $T_2$. It is clear that $n(T_1) \geq k + 1$ and $n(T_1) = k + 1$ if and only if $T_1$ is a path on $k + 1$ vertices. Moreover, $n(T_2) \geq k + 2$.

Hence, if $T$ is a tree with $\gamma_p^{k}(T) \geq 4$, then there exists an edge $e \in E(T)$ such that $\gamma_p^{k}(T_1) + \gamma_p^{k}(T_2) = \gamma_p^{k}(T), n(T_1) \geq k + 1$ and $n(T_2) \geq k + 2$, where $T_1$ and $T_2$ are the two components of $T - e$. □

**5. Upper bound on the $k$-distance paired domination number of a connected graph**

Haynes and Slater [5] have proved that if $G$ is a connected graph with $n(G) \geq 3$, then $\gamma_p(G) \leq n(G) - 1$ and they characterized the extremal graphs. In what follows we generalize their result for the $k$-distance paired domination number of a graph. We first present an upper bound for the $k$-distance paired domination number of a tree.
Theorem 9. If $T$ is a tree of size at least $k + 2$, then

\[ \gamma_p^k(T) \leq \frac{2^{n(T)} - 1}{k + 1}, \]

which is sharp as shown in Lemma 12.

Proof. We use induction on the $k$-distance paired domination number of a tree. Let $T$ be a tree with $n(T) \geq k + 2$. If $\text{diam}(T) \leq 2k + 1$, then $\gamma_p^k(T) = 2$ and the result clearly holds. Thus assume $\text{diam}(T) \geq 2k + 2$ which implies that $\gamma_p^k(T) \geq 4$. Assume also that the result holds for all trees $T'$ with $n(T') \geq k + 2$ and $\gamma_p^k(T') < \gamma_p^k(T)$. Applying Lemma 8 to tree $T$ we conclude that there exists an edge $e \in E(T)$ such that $\gamma_p^k(T_1) + \gamma_p^k(T_2) = \gamma_p^k(T)$, where $T_1$ and $T_2$ are the two components of $T - e$ such that $n(T_1) \geq k + 1$ and $n(T_2) \geq k + 2$. Now we consider two cases.

Case 1: $n(T_1) = k + 1$. Then by induction hypothesis applied to $T_1$ and $T_2$ we have $\gamma_p^k(T_1) \leq 2\frac{n(T_1) - 1}{k + 1}$ and $\gamma_p^k(T_2) \leq 2\frac{n(T_2) - 1}{k + 1}$. Hence,

\[ \gamma_p^k(T) = \gamma_p^k(T_1) + \gamma_p^k(T_2) \leq 2\frac{n(T_1) + n(T_2) - 2}{k + 1} \leq 2\frac{n(T) - 1}{k + 1}. \]

Case 2: $n(T_1) \geq k + 2$. Then by induction hypothesis applied to $T_2$ we see that $\gamma_p^k(T_2) \leq 2\frac{n(T_2) - 1}{k + 1}$. Hence,

\[ \gamma_p^k(T) = \gamma_p^k(T_1) + \gamma_p^k(T_2) \leq 2 + 2\frac{n(T_2) - 1}{k + 1} = 2\frac{n(T) - 1}{k + 1}. \]

Therefore, if $T$ is a tree with $n(T) \geq k + 2$, then $\gamma_p^k(T) \leq 2\frac{n(T) - 1}{k + 1}$. □

Proposition 10. Let $uv$ be any edge of a graph $G$. If $G - uv$ does not contain an isolated vertex, then $\gamma_p^k(G) \leq \gamma_p^k(G - uv)$.

Proof. If $D$ is a minimum $k$-distance paired dominating set of $G - uv$, then $D$ is a $k$-distance paired dominating set of $G$ and therefore, $\gamma_p^k(G) \leq |D| = \gamma_p^k(G - uv)$. □

Corollary 11. If $G$ is a connected graph of size at least $k + 2$, then $\gamma_p^k(G) \leq 2\frac{n(G) - 1}{k + 1}$.

Proof. It follows from Proposition 10 that $\gamma_p^k(G) \leq \gamma_p^k(T)$ for every spanning tree $T$ of $G$. Consequently, by Theorem 9, $\gamma_p^k(G) \leq \gamma_p^k(T) \leq 2\frac{n(G) - 1}{k + 1}$. □

For a positive integer $t$ define $K_{1,t}^k$ to be a $k$-subdivided star obtained from the star $K_{1,t}$ by $k$-subdividing each edge of the star. A vertex $x \in V(K_{1,t}^k)$ is a central vertex of $K_{1,t}^k$ if $\max\{d_{K_{1,t}^k}(x, v) : v \in V(K_{1,t}^k)\} = k + 1$. Now we characterize all trees $T$ for which $\gamma_p^k(T) = 2\frac{n(T) - 1}{k + 1}$. For this purpose, we define $\mathcal{A}$ to be the family of trees such that:

(i) each $k$-subdivided star $K_{1,t}^k$ belongs to $\mathcal{A}$;
(ii) each tree $T$ on $k + 2$ vertices belongs to $\mathcal{A}$;

Lemma 12. If $T$ is a tree belonging to the family $\mathcal{A}$, then $\gamma_p^k(T) = 2\frac{n(T) - 1}{k + 1}$.

Proof. If $T$ is a $k$-subdivided star $K_{1,t}^k$, then it is easy to check that $\gamma_p^k(T) = 2t$ and $n(T) = t(k + 1) + 1$. Thus the equality $\gamma_p^k(T) = 2\frac{n(T) - 1}{k + 1}$ clearly holds. If $T$ is a tree on $k + 2$ vertices, then $\text{diam}(T) \leq k + 1$. For this reason $\gamma_p^k(T) = 2$ and the equality $\gamma_p^k(T) = 2\frac{n(T) - 1}{k + 1}$ holds. □

Lemma 13. If $T$ is a tree with $\gamma_p^k(T) = 2\frac{n(T) - 1}{k + 1}$, then $T$ belongs to the family $\mathcal{A}$.
**Theorem 14.** If \( T \) is a tree with \( \gamma_p^k(T) = 2 \frac{n(T) - 1}{k + 1} \). Assume first that \( \gamma_p^k(T) = 2 \). This implies that \( n(T) = k + 2 \), so \( T \) belongs to the family \( \mathcal{P} \).

Assume now that \( \gamma_p^k(T) = 4 \). By assumption we have that \( n(T) = 2k + 3 \) and \( \text{diam}(T) \geq 2k + 2 \). We conclude that \( T \) is a path on \( 2k + 3 \) vertices and so \( T \) is the \( k \)-subdivided star \( K^k_{1,2} \). Therefore, \( T \) belongs to the family \( \mathcal{P} \).

Now let \( \gamma_p^k(T) \geq 6 \) and suppose that the result is true for all trees \( T' \) with \( \gamma_p^k(T') < \gamma_p^k(T) \). By Lemma 8, there exists an edge \( e \in \mathcal{E}(T) \) such that \( \gamma_p^k(T) = \gamma_p^k(T_1) + \gamma_p^k(T_2) \), where \( T_1 \) and \( T_2 \) are the two components of \( T - e \) such that \( n(T_1) \geq k + 1 \) and \( n(T_2) \geq k + 2 \). Let \( e = uv \) be the edge of \( T \) found in the same way as in the proof of Lemma 8 where \( u \in V(T_1) \) and \( v \in V(T_2) \). Consider two cases.

Case 1: \( n(T_1) \geq k + 2 \). Then by Theorem 9 we have \( \gamma_p^k(T_1) \leq 2 \frac{n(T_1) - 1}{k + 1} \) and \( \gamma_p^k(T_2) \leq 2 \frac{n(T_2) - 1}{k + 1} \). Hence,

\[
2 \frac{n(T) - 1}{k + 1} = \gamma_p^k(T) = \gamma_p^k(T_1) + \gamma_p^k(T_2) \leq 2 \frac{n(T_1) + n(T_2) - 2}{k + 1},
\]

a contradiction, because \( 2 \frac{n(T_1) + n(T_2) - 2}{k + 1} > 2 \frac{n(T_1) + n(T_2) - 2}{k + 1} \).

Case 2: \( n(T_1) = k + 1 \). Then \( T_1 \) is a path on \( k + 1 \) vertices in which \( u \) is a leaf in \( T_1 \) and \( \gamma_p^k(T_1) = 2 \). By Theorem 9 we see that \( \gamma_p^k(T_2) \leq 2 \frac{n(T_2) - 1}{k + 1} \). Hence,

\[
2 \frac{n(T_1) - 1}{k + 1} = \gamma_p^k(T) = \gamma_p^k(T_1) + \gamma_p^k(T_2) \leq 2 \frac{n(T_1) + n(T_2) - 2}{k + 1} = 2 \frac{n(T) - 1}{k + 1}.
\]

This implies that \( \gamma_p^k(T_2) = 2 \frac{n(T_2) - 1}{k + 1} \) and thus by induction hypothesis \( T_2 \in \mathcal{P} \). Since \( \gamma_p^k(T) \geq 6 \) and \( \gamma_p^k(T_1) = 2 \), we see that \( \gamma_p^k(T_2) \geq 4 \). Thus \( T_2 \) is a \( k \)-subdivided star \( K^k_k \), for some \( t \geq 2 \). Therefore, \( \gamma_p^k(T_2) = 2t \) and so \( \gamma_p^k(T) = 2t + 2 \). Suppose \( v \in V(T_2) \) is any vertex different from the central vertex of \( T_2 \). Then there exists a \( k \)-distance paired dominating set of \( T \) of cardinality \( 2t \) containing \( u \) and \( v \), a contradiction. Hence, \( v \in V(T_2) \) is the central vertex of \( T_2 \) and for this reason \( T \) is a \( k \)-subdivided star \( K^k_{1,t+1} \). Therefore, \( T \) belongs to the family \( \mathcal{P} \).

The proof is completed. \( \square \)

The following result follows immediately from Lemmas 12 and 13.

**Theorem 14.** If \( T \) is a tree, then

\[ \gamma_p^k(T) = 2 \frac{n(T) - 1}{k + 1} \]

if and only if \( T \) belongs to the family \( \mathcal{P} \).

Now we characterize all connected graphs \( G \) for which \( \gamma_p^k(G) = 2 \frac{n(G) - 1}{k + 1} \). For this purpose, we define \( \mathcal{P} \) to be the family of trees such that:

(i) each \( k \)-subdivided star \( K^k_{1,t} \) belongs to \( \mathcal{P} \);
(ii) each connected graph \( G \) on \( k + 2 \) vertices belongs to \( \mathcal{P} \);
(iii) the cycle \( C_{2k+3} \) belongs to \( \mathcal{P} \).

**Lemma 15.** If a graph \( G \) belongs to the family \( \mathcal{P} \), then \( \gamma_p^k(G) = 2 \frac{n(G) - 1}{k + 1} \).

**Proof.** If \( G \) is a \( k \)-subdivided star \( K^k_{1,t} \), then by Lemma 12, \( \gamma_p^k(G) = 2 \frac{n(G) - 1}{k + 1} \). Now if \( G \) is a connected graph on \( k + 2 \) vertices, then \( \gamma_p^k(G) = 2 \) and thus the equality \( \gamma_p^k(G) = 2 \frac{n(G) - 1}{k + 1} \) holds. Finally, if \( G \) is the cycle \( C_{2k+3} \), then \( \gamma_p^k(G) = 4 \) and \( n(G) = 2k + 3 \). Thus the equality \( \gamma_p^k(G) = 2 \frac{n(G) - 1}{k + 1} \) also holds. \( \square \)

**Lemma 16.** If \( G \) is a connected graph and \( \gamma_p^k(G) = 2 \frac{n(G) - 1}{k + 1} \), then \( G \) belongs to the family \( \mathcal{P} \).

**Proof.** For all trees the result follows from Lemma 13. Let \( G \) be a connected graph. By Theorem 9, for a tree \( T \) of order at least \( k + 2 \) we have that \( \gamma_p^k(T) \leq 2 \frac{n(T) - 1}{k + 1} \) and from Proposition 10 follows that \( \gamma_p^k(G) \leq \gamma_p^k(T) \) for every spanning
tree $T$ of a connected graph $G$. Consequently, if $\gamma^k_p(G) = 2 \frac{n(G) - 1}{k + 1}$, then $\gamma^k_p(T) = 2 \frac{n(T) - 1}{k + 1}$ for every spanning tree $T$ of $G$. Thus, if $\gamma^k_p(G) = 2 \frac{n(G) - 1}{k + 1}$, then each spanning tree $T$ of $G$ belongs to the family $\mathcal{R}^p$.

Let $G$ be a connected graph which is not a tree and with $\gamma^k_p(G) = 2 \frac{n(G) - 1}{k + 1}$. Assume first that $\gamma^k_p(G) = 2$. This implies that $n(G) = k + 2$, so $G$ belongs to the family $\mathcal{R}$.

Assume now that $\gamma^k_p(G) = 4$. This implies that $n(G) = 2k + 3$ and each spanning tree $T$ of $G$ must be the $k$-subdivided star $K^k_{1,2}$ which is the path $P_{2k+3}$. Denote by $u$ and $v$ the two leaves in $T = K^k_{1,2}$ and let $e \in E(T)$. Observe that $\gamma^k_p(T + e) = 4$ if $e = uv$ and it is easy to see that $\gamma^k_p(T + e) = 2$ if $e \neq uv$. We conclude that $G$ is the cycle $C_{2k+3}$ and in this way $G$ belongs to the family $\mathcal{R}$.

Now let $\gamma^k_p(G) \geq 6$. Then every spanning tree $T$ of $G$ is a $k$-subdivided star $K^k_{1,t}$ for some $t \geq 3$. Observe that $\gamma^k_p(K^k_{1,t} + e) < \gamma^k_p(K^k_{1,t})$ for each $e \in E(K^k_{1,t})$. Thus if $\gamma^k_p(G) = 2 \frac{n(G) - 1}{k + 1}$ and $\gamma^k_p(G) \geq 6$, then $G$ is a tree, a contradiction. Hence, if $G$ is a graph with $\gamma^k_p(G) = 2 \frac{n(G) - 1}{k + 1}$ and $\gamma^k_p(G) \geq 6$, then $G$ is a $k$-subdivided star $K^k_{1,t}$ for some positive integer $t$ and belongs to the family $\mathcal{R}$. $\square$

The following result is immediate from Lemmas 15 and 16.

**Theorem 17.** If $G$ is a connected graph, then

$$\gamma^k_p(G) = 2 \frac{n(G) - 1}{k + 1},$$

if and only if $G$ belongs to the family $\mathcal{R}$.

Since each graph $G$ belonging to the family $\mathcal{R}$ has $\text{diam}(G) \leq 2k + 2$, we have the following corollary.

**Corollary 18.** If $G$ is a connected graph and $\text{diam}(G) \geq 2k + 3$, then $\gamma^k_p(G) \leq 2 \frac{n(G) - k - 2}{k + 1}$.

Corollary 9 and Theorem 17 for $k = 1$ have been announced by Haynes and Slater in [5].

6. Lower bound on the $k$-distance paired domination number of a tree

In this section, we prove that for any tree of order $n(T) \geq 2$ and with $n_1(T)$ leaves we have

$$\gamma^k_p(T) \geq \frac{n(T) + 2k - k \cdot n_1(T)}{k + 1}.$$

We begin with the following lemma.

**Lemma 19.** If $T$ is a tree of order $n(T) \geq 2$ and $\gamma^k_p(T) = 2$, then $k \cdot n_1(T) \geq n(T) - 2$.

**Proof.** If a tree $T$ has $\gamma^k_p(T) = 2$, then obviously $\text{diam}(T) \leq 2k + 1$. We proceed by induction on the number of leaves of a tree. If $n_1(T) = 2$, then $T$ is a path. Since $\gamma^k_p(T) = 2$, the path has at most $2k + 2$ vertices. Thus

$$k \cdot n_1(T) = k \cdot 2 = (2k + 2) - 2 \geq n(T) - 2 = n(T) + 2k - (k + 1)\gamma^k_p(T).$$

Now let $T$ be a tree with $\gamma^k_p(T) = 2$ and $n_1(T) > 2$ and assume that the result is true for all trees $T'$ with $\gamma^k_p(T') = 2$ and $n_1(T') < n_1(T)$. Let $(x_0, \ldots, x_q)$ be a shortest path in $T$, where $d_T(x_0) = 1$ and $d_T(x_q) > 2$. Observe that $1 \leq q \leq k$, because $\text{diam}(T) \leq 2k + 1$ as $\gamma^k_p(T) = 2$ and $T$ is not a path. Denote $V_1 = \{x_0, x_1, \ldots, x_{q-1}\}$ and let $T'$ be the induced subgraph of $V(T) - V_1$. Obviously $|V_1| \leq k$. Then $n_1(T') = n_1(T) - 1$ and $n(T') \geq n(T) - k$. By induction,

$$k \cdot n_1(T') \geq n(T') - 2$$
and hence,
\[ k \cdot n_1(T) \geq n(T) - 2. \]

**Theorem 20.** If \( T \) is a tree of order \( n(T) \geq 2 \), then
\[
(k + 1) \gamma_{p}^{k}(T) \geq n(T) + 2k - k \cdot n_1(T).
\]

**Proof.** We proceed by induction on the \( k \)-distance paired domination number of a tree. If \( \gamma_{p}^{k}(T) = 2 \), then the result follows immediately from Lemma 19.

Let \( T \) be a tree with \( \gamma_{p}^{k}(T) \geq 4 \) and assume that the result is true for all trees \( T' \) with \( \gamma_{p}^{k}(T') < \gamma_{p}^{k}(T) \). Lemma 8 implies that there exists an edge \( e \) of \( T \) such that \( \gamma_{p}^{k}(T) = \gamma_{p}^{k}(T_1) + \gamma_{p}^{k}(T_2) \), where \( T_1 \) and \( T_2 \) are the two components of \( T - e \). Obviously \( n(T_1) \geq 2, n(T_2) \geq 2, \gamma_{p}^{k}(T_1) < \gamma_{p}^{k}(T) \) and \( \gamma_{p}^{k}(T_2) < \gamma_{p}^{k}(T) \). Thus, by induction hypothesis applied to \( T_1 \) and \( T_2 \), we have
\[
(k + 1) \gamma_{p}^{k}(T_1) \geq n(T_1) + 2k - k \cdot n_1(T_1)
\]
and
\[
(k + 1) \gamma_{p}^{k}(T_2) \geq n(T_2) + 2k - k \cdot n_1(T_2).
\]
From those inequalities we obtain
\[
(k + 1) \gamma_{p}^{k}(T) \geq n(T) + 4k - k \cdot (n_1(T_1) + n_1(T_2)).
\]
Observe that \( n_1(T_1) + n_1(T_2) \leq n_1(T) + 2 \), so
\[
(k + 1) \gamma_{p}^{k}(T) \geq n(T) + 2k - k \cdot n_1(T),
\]
which completes the proof of the bound. \( \square \)

A \textit{k-double star}, denoted \( S^k(q,r) \), where \( q, r \) and \( k \) are positive integers, for \( k > 1 \) is the tree obtained from two \((k - 1)\)-subdivided stars \( K_{1,q}^{k-1} \) and \( K_{1,r}^{k-1} \) by adding the edge incident with one central vertex of \( K_{1,q}^{k-1} \) and with one central vertex of \( K_{1,r}^{k-1} \). A \textit{1-double star} \( S^1(q,r) \), is the tree obtained from two stars \( K_{1,q} \) and \( K_{1,r} \) by adding the edge incident with one central vertex of \( K_{1,q} \) and with one central vertex of \( K_{1,r} \).

We are now in a position to provide a constructive characterization of all trees \( T \) for which \((k + 1) \gamma_{p}^{k}(T) = n(T) + 2k - k \cdot n_1(T) \). For this purpose, we introduce the following operation: if \( T_1 \) and \( T_2 \) are vertex disjoint trees and \( u \) and \( v \) are any two leaves belonging to \( T_1 \) and \( T_2 \), respectively, then by \( T_1 \oplus_{uv} T_2 \) we denote a tree obtained from \( T_1 \) and \( T_2 \) by adding the edge \( uv \).

Let \( \mathcal{R}_p \) denote the family of trees such that:

(i) every \( k \)-double star \( S^k(q,r) \) belongs to \( \mathcal{R}_p \);
(ii) \( T_1 \oplus_{uv} T_2 \) belongs to \( \mathcal{R}_p \) if only \( T_1 \) and \( T_2 \) belong to \( \mathcal{R}_p \).

**Observation 21.** If \( T \) is a tree belonging to the family \( \mathcal{R}_p \), then either \( T \) is a \( k \)-double star or \( T \) can be obtained from \( k \)-double stars \( S^k_1(q_1,r_1), \ldots, S^k_j(q_j,r_j) \), where \( j \geq 2 \), using operation \( \oplus_{uv} \).

**Lemma 22.** If \( T \) is a tree belonging to the family \( \mathcal{R}_p \), then
\[
(k + 1) \gamma_{p}^{k}(T) = n(T) + 2k - k \cdot n_1(T).
\]

**Proof.** If \( T \) is a \( k \)-double star, then \( \gamma_{p}^{k}(T) = 2, n(T) = k \cdot n_1(T) + 2 \) and certainly \( (k + 1) \gamma_{p}^{k}(T) = n(T) + 2k - k \cdot n_1(T) \). Otherwise, if \( T \) is a tree obtained from \( k \)-double stars \( S^k_1(q_1,r_1), \ldots, S^k_j(q_j,r_j) \) (\( j \geq 2 \)), then it is easily seen that
\[\gamma_p^k(T) = 2j\] (Fig. 3). Moreover,
\[n(T) = \sum_{i=1}^{j} n(S_i^k(q_i, r_i)) = \sum_{i=1}^{j} (k \cdot n_1(S_i^k(q_i, r_i)) + 2)\]
\[= 2j + k \cdot \sum_{i=1}^{j} n_1(S_i^k(q_i, r_i))\]
and
\[n_1(T) = \sum_{i=1}^{j} n_1(S_i^k(q_i, r_i)) - 2(j - 1)\].

It is easy to check that we also have \((k + 1)\gamma_p^k(T) = n(T) + 2k - k \cdot n_1(T)\).

**Lemma 23.** If \(T\) is a tree with \(\gamma_p^k(T) = 2\) and \(k \cdot n_1(T) = n(T) - 2\), then \(T\) belongs to the family \(\mathcal{R}_p\).

**Proof.** We proceed by induction on the number of leaves of a tree. If \(T\) is a tree with \(n_1(T) = 2\), then \(T\) is a path. As \(\gamma_p^k(T) = 2\) and \(k \cdot n_1(T) = n(T) - 2\), the path has exactly \(2k + 2\) vertices. Thus \(T\) is a \(k\)-double star \(S^k(1, 1)\).

Now let \(T\) be a tree with \(\gamma_p^k(T) = 2\), \(n_1(T) > 2\) and such that \(k \cdot n_1(T) = n(T) - 2\). Assume that the result is true for all trees \(T'\) with \(\gamma_p^k(T') = 2\) and \(n_1(T') < n_1(T)\). Let \((x_0, \ldots, x_q)\) be a shortest path in \(T\), where \(d_T(x_0) = 1\) and \(d_T(x_q) > 2\). Observe that \(1 \leq q \leq k\), since \(\text{diam}(T) \leq 2k + 1\) and \(T\) is not a path. Denote \(V_1 = \{x_0, x_1, \ldots, x_{q-1}\}\) and let \(T'\) be the induced subgraph of \(V(T) - V_1\). Obviously \(|V_1| \leq k\). Then \(n_1(T') = n_1(T) - 1\), \(\gamma_p^k(T') = 2\) and \(n(T') = n(T) - p\). By Lemma 19 applied to \(T'\) we have
\[k \cdot n_1(T') \geq n(T') - 2.\]
Hence,
\[k \cdot n_1(T) \geq n(T) - p + k - 2.\]
Since \(k \cdot n_1(T) = n(T) - 2\), we conclude that \(p = k\) and thus \(k \cdot n_1(T') = n(T') - 2\). By induction, \(T'\) is a \(k\)-double star. This also implies that \(T\) is a \(k\)-double star and for this reason \(T\) belongs to the family \(\mathcal{R}_p\).

**Lemma 24.** If \(T\) is a tree with \((k + 1)\gamma_p^k(T) = n(T) + 2k - k \cdot n_1(T)\), then \(T\) belongs to the family \(\mathcal{R}_p\).

**Proof.** We proceed by induction on the \(k\)-distance paired domination number of a tree. If \(T\) is a tree with \(\gamma_p^k(T) = 2\), then the result follows immediately from Lemma 23.

Let \(T\) be a tree with \(\gamma_p^k(T) \geq 4\) and such that \((k + 1)\gamma_p^k(T) = n(T) + 2k - k \cdot n_1(T)\). Assume that the result is true for all trees \(T'\) with \(\gamma_p^k(T') < \gamma_p^k(T)\). Lemma 8 applied to \(T\) implies that there exists an edge \(e \in E(T)\) such that \(\gamma_p^k(T) = \gamma_p^k(T_1) + \gamma_p^k(T_2)\), where \(T_1\) and \(T_2\) are the two components of \(T - e\). Of course \(n(T_1) \geq 2\), \(n(T_2) \geq 2\),
\( \gamma_p^k(T_1) < \gamma_p^k(T) \) and \( \gamma_p^k(T_2) < \gamma_p^k(T) \). It is easy to observe that \( n_1(T) \geq n_1(T_1) + n_1(T_2) - 2 \). Thus, by Theorem 20, \((k + 1)\gamma_p^k(T_1) \geq n(T_1) + 2k - k \cdot n_1(T_1)\) and \((k + 1)\gamma_p^k(T_2) \geq n(T_2) + 2k - k \cdot n_1(T_2)\). Therefore,
\[
(k + 1)\gamma_p^k(T) \geq n(T) + 2k - k(n_1(T_1) + n_1(T_2) - 2) \geq n(T) + 2k - k \cdot n_1(T).
\]
Since \((k + 1)\gamma_p^k(T) = n(T) + 2k - k \cdot n_1(T)\), we conclude that
\[
(k + 1)\gamma_p^k(T) = n(T) + 2k - k(n_1(T_1) + n_1(T_2) - 2) = n(T) + 2k - k \cdot n_1(T)
\]
and hence,
\[
n_1(T) = n_1(T_1) + n_1(T_2) - 2,
\]
\[
(k + 1)\gamma_p^k(T_1) = n(T_1) + 2k - n_1(T_1),
\]
\[
(k + 1)\gamma_p^k(T_2) = n(T_2) + 2k - n_1(T_2).
\]
Thus, by induction, \( T_1 \) and \( T_2 \) belong to the family \( \mathcal{R}_p \) and, if \( e = uv \), then \( d_{T_1}(u) = d_{T_2}(v) = 1 \), that is \( u \) and \( v \) are leaves in \( T_1 \) and \( T_2 \), respectively. Therefore, \( T \) is obtained from \( T_1 \) and \( T_2 \) by adding an edge incident with a leaf of \( T_1 \) and with a leaf of \( T_2 \), and we conclude that \( T \in \mathcal{R}_p \). \( \square \)

The following result is obvious from Lemmas 22 and 24.

**Theorem 25.** If \( T \) is a tree on \( n(T) \geq 2 \) vertices, then
\[
(k + 1)\gamma_p^k(T) = n(T) + 2k - k \cdot n_1(T)
\]
if and only if \( T \) belongs to the family \( \mathcal{R}_p \).

**References**


