Reconstruction of the deterministic dynamics of stochastic systems

M. Siefert, J. Peinke
FB 8 Physics, Carl-von-Ossietzky Universität Oldenburg, D–26123 Oldenburg, Germany,
phone: +49-(0)441-798-3516
mail: peinke@uni-oldenburg.de
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We show that based on the mathematics of Markov processes and in particular based on the
definition of Kramers-Moyal coefficients it is possible to estimate the deterministic part of the
dynamics for a broad class of nonlinear noisy systems. In particular we show that for different kinds
of noise perturbations, including non Langevin force with finite correlation time and independent
measurement noise, the deterministic part can be reconstructed.

INTRODUCTION

Scientists are frequently confronted with complex
dynamical systems, which can be found for example
in physics, medical science, environmental science, and
economics. Often the describing equations are not
known. Nevertheless, an experimenter can get access to
a detailed quantitative examination of the system due
to sophisticated methods of pure data analysis. In the
past decades several methods have been established to
grasp properties of distinct dynamical systems from
experimental data. On the one hand methods for pure
deterministic systems, on the other hand methods
for dynamical stochastic systems or a combination
of these systems exist [Schreiber & Kantz, 1997,
Soofi & Cao, 2002, Hilborn & Tufillaro, 1997,
Borland & Haken, 1992, Borland & Haken, 1992b,
Timmer, 2000, Klimonotovich, 1993, Stark et al., 1997,
Broomhead et al., 1998]. In [Friedrich & Peinke, 1997,
Siegert et al., 1998, Friedrich et al., 2000, Renner et al., 2001] it has been
proposed to use the mathematics of diffusion processes
to analyze data of systems showing complex behavior,
combining deterministic dynamics and stochastic dy-
namics. Given the measured state space variables as a
time series of a system, this approach enables to extract
from these data a stochastic differential equation. As a
result a general Langevin equation or a Fokker-Planck
equation is obtained.[1] With these equations a clear
separation between the deterministic and the stochastic
part of the time evolution is given. Before applying
this approach, neither preknowledge about the system
under consideration nor requirements on the form of
the differential equation have to be taken into account.
It can even be checked, whether the system can be
described by the Langevin equation or not (i.e. which
type of noise is present, this includes the verification
of the embedding dimension [Siefert et al., 2001]). Up
to now this method has been applied to dynamical
systems whose time evolution obeys the condition of a
Markov process, i.e. the noise-force of the corresponding
Langevin equation must be $\delta$–correlated. Definitely
for real experimental data this requirement can not be
fulfilled in an ideal way. Noise with finite correlation
time is the more realistic assumption (for example, the
diffusion of molecules, a well known diffusion process,
has a finite correlation time too, given by the path
length between two impacts). If the correlation time of
such cases are much smaller than the typical time of the
dynamics considered, it has been shown that the system
can be well approximated and described by a Langevin
equation [Siegert et al., 1998, Friedrich et al., 2000,
Renner et al., 2001]. The paper’s primary target is to
show that the method presented in the following does
not only work with a Langevin force but also with
different kinds of noise and correlated forces. To this end
we present the successful calculation of the deterministic
part from data of an experimental system affected by
different kind of forces. The paper’s structure is as
follows: First we describe the mathematics we are using
for the reconstruction of the dynamical equations from
given data sets. Next follows the presentation of the
experimental system. Finally, based on the resulting
data we give evidence that the drift coefficient, i.e. the
deterministic part, can be calculated independently of
the nature of the driving noisy forces.

DIFFUSION PROCESSES

In a first step we focus on nonlinear dynamical systems
with dynamical white noise, also known as diffusion pro-
cesses. These diffusion processes can be described by a
Langevin equation (in the Itô representation),

$$\frac{d}{dt} X_i(t) = D^{(1)}_i(X, t) + \sum_{j=1}^{n} \sqrt{D^{(2)}(X, t)}_{ij} \Gamma_j(t), \quad i = 1, \ldots, n$$

(1)
where $X(t)$ denotes the $n$-dimensional stochastic state vector. The drift coefficients, $D^{(1)}_i$, represent the deterministic part of the dynamics, and the diffusion coefficients, $D^{(2)}_{ij}$, determine the strength of the noise, including the general case of multiplicative noise when the coefficients $D^{(2)}_{ij}$ depend on $X$. The symbol $[,]_{ij}$ indicates that first the square root of the diagonalized matrix $D^{(2)}$ has to be taken; successively a back transformation gives the $ij$-coefficients. For the Langevin process the drift terms $D^{(1)}_i$ and the diffusion coefficients $D^{(2)}_{ij}$ are given as the limit of conditional moments
\[
D^{(1)}_i = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \langle X_i(t + \Delta t) - x_i(t) \rangle |_{X(t) = x}, \quad (2)
\]
\[
D^{(2)}_{ij} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \langle (X_i(t + \Delta t) - x_i(t))(X_j(t + \Delta t) - x_j(t)) \rangle |_{X(t) = x} . \quad (3)
\]
For numerical estimation the condition of the moments must be approximated by $X(t) \in U(x)$, for a sufficient small neighborhood $U$ of a fixed value $x$. This conditional moments can be estimated directly from given data sets in a parameter free way. The estimation of the diffusion coefficient in case of white noise was described in [Siegert et al., 1998, Friedrich et al., 2000] for example.

**EXPERIMENTAL SETUP**

Using Eq. (2) we analyze experimental data of a chaotic electronic oscillator. As circuitry we chose the Shinriki oscillator [Shinriki & Yamamoto, 1981] as shown in Fig. 1 with a basic frequency of 851 Hz. It is driven by a negative resistor which can be understood as an ohmic resistor with $R_{N,IC} < 0$. In Fig. 2 an exemplary phase space representation of the attractor for the experimental data without noisy perturbations is shown. The corresponding ODEs of the Shinriki oscillator are given by
\[
\dot{X}_1 = -\frac{X_1 - \Gamma_1(t)}{R_NC_1} - \frac{X_1}{R_1C_1} - \frac{f(X_1 - X_2)}{C_1} = D^{(1)}_1(X_1, X_2) + g_1F_1(t) \quad (4)
\]
\[
\dot{X}_2 = \frac{f(X_1 - X_2)}{C_2} - \frac{1}{R_3C_2}X_3 = D^{(1)}_2(X_1, X_2, X_3) + g_2F_2(t) \quad (5)
\]
\[
\dot{X}_3 = -\frac{R_3}{L}(X_2 - X_3) = D^{(1)}_3(X_2, X_3) + g_3F_3(t). \quad (6)
\]
where $X_i$ describe the voltage drops over different electronic elements as shown in Fig. 1. $f(\cdot)$ includes the strong nonlinearity of the Zener diodes. The functions $g_iF_i(t)$ represent the perturbing noisy forces. In principle, four different types of perturbations $g_iF_i(t)$ can be connected to the circuitry. They represent typical situation which can arise in a system with disturbing noise.

1. **Dynamical noise or Langevin force.** This kind of force is usually chosen to model continuous stochastic dynamics. It covers a very broad class of systems and provide a well developed mathematical theory. For an experimental realisation, a $\delta$-correlated noise source with a standard deviation $\sigma = 770$mV in series connection to the negative resistor. This results in a Langevin force $g_1F_1(t) = (\equiv \sum_{j=1}^{3} \sqrt{D^{(2)}_{ij}(X(t), j)}\Gamma_j(t), compare Eq. (4) with Eq. (1)) with the strength of $D^{(2)}_{ij} = 17.7 V^2/s$ ($g_2 = 0, g_3 = 0$). The attractor reconstructed from these data is shown in Fig. 3a.

2. **Measurement noise.** Very often, observed noise comes from the measuring process itself and doesn’t influence the system’s dynamic. To obtain such a signal we add $\delta$-correlated measurement noise (about 1% of $X_1$) to the noise free trajectory: $X_1 \rightarrow X_1$+measurement noise ($g_1 = 0, g_2 = 0, g_3 = 0$). The attractor is shown in Fig. 4a.

3. **Dynamical noise, no Langevin force.** A new type of dynamics can be achieved by connecting the $\delta$-correlated noise directly to one state variable, here parallel to $X_3$ with an amplitude of 10%. This noise also occurs in the voltage $X_2$ and acts there as a Langevin force ($g_1 = 0, g_2 = (R_I+R_V)^{-1}, g_3 = R_3/(R_3+R_V)^{-1}, F_3(t) = 0.0001V(t)$, where $R_I$ is the inner resistance of the noise source and $R_V$ is the multiplier of the noise source). Note that $g_3F_3(t)$ with $F_3(t) = 0.0001V(t)$ is not any more a Langevin force. The attractor is shown in Fig. 5a.

4. **Periodic force.** In realistic situations, noise has always finite correlation time. As a force with infinite correlation time we use a fast oscillating signal which is in series connection with the negative resistor (The amplitude of 400 mV is of the same order as the Langevin force.). Its frequency is 27 kHz ($g_1 = (R_{N,IC} + R_I)^{-1}, g_2 = 0, g_3 = 0, F_1(t) = X_0 \sin(\omega t)$ with $(2\pi)^{-1}\omega = 27$ kHz). The sinusoidal force has no phase locking and no frequency locking to the dynamics of the Shinriki oscillator. The resulting attractor is shown in Fig. 6a.

**EXPERIMENTAL RESULTS**

After preparing the four different systems, we measure the three state variables, which means the three voltages $X_1, X_2$ and $X_3$. In figure parts a of Figs. 3-6, the
phase state representation are shown, respectively. To give evidence of the practicability of Eq. (2) we show the reconstructed deterministic part of Eq. (4), which we obtained from these experimental data. Here an exemplary cut through \( \{D^{(1)}, X\} \) has been chosen in such a way that the nonlinearity becomes obvious. By measuring the electronic elements \((R_N, C_1, R_1 \text{ and } f(\cdot))\) we can directly compare the characteristics gained from Eq. (4) with the reconstructed one (part b of Figs. 3-6). Except for small deviations, the drift term is well reproduced for all presented kind of noise. For the Langevin force case, see Fig. 3, the used noise contains frequencies up to 37 MHz. The discrepancy between the reconstructed characteristic and the measured one is due to frequency dependent impedances. For a verification, we analogously analyzed a numerically calculated time series and found that this discrepancy vanishes. Due to the fractal geometry of the attractor, the state space is not covered uniformly thus we find in Fig. 5b and 6b gaps in the reconstructed curves.

CONCLUSION

We have presented a chaotic oscillator with four different driving forces. Three different \( \delta \)-correlated noises and one fast oscillating sinusoidal force which represents the case of a correlated noise. We have shown that the drift coefficient \( D^{(1)} \) can successfully be reconstructed from the data for all presented cases. This result motivates to apply the present approach to systems which does not fulfill the Markov properties in a stringent sense. At last we note that in our analysis here we investigated only cases of additive noise. From previously results (see [Friedrich & Peinke, 1997, Renner et al., 2002, Renner et al., 2001, Renner, 2001]) we do not expect principal difficulties to extend our work to multiplicative noise, even though it is more difficult to realize it experimentally.


More precisely speaking, we reconstruct the flow field of the dynamics, from which a functional form can be estimated.
FIG. 1: Circuitry of the Shinriki oscillator. Here the noise source is series to the negative resistor.
FIG. 2: Trajectory for the Shinriki oscillator in the phase space without noise.
FIG. 3: a) Trajectory for the Shinriki oscillator in the phase space with dynamical noise. b) The corresponding trajectory in the phase space.
FIG. 4: a) Trajectory for the Shinriki oscillator in the phase space with measurement noise. b) The corresponding trajectory in the phase space.
FIG. 5: a) Trajectory for the Shinriki oscillator in the phase space with dynamical noise, which is not a Langevin force. b) The corresponding trajectory in the phase space.
FIG. 6: a) Trajectory for the Shinriki oscillator in the phase space with a sinusoidal force. b) The corresponding trajectory in the phase space.