On some properties of distance measures

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Abstract

Based on the axiom definition of distance measure, some new formulas of fuzzy entropy induced by distance measure and some new properties of distance measure are given. A characterization of a σ-distance measure is a metric defined on all fuzzy sets is obtained. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Distance measure is a term that describes the difference between fuzzy sets. Liu [5] gave the axiom definition of distance measure and discussed the relationships between distance measure and fuzzy entropy, similarity measure. Distance measure can be considered as a dual concept of similarity measure. Many researchers, such as Yager [10], Kosko [8] and Kaufmann [7] had used distance measure to define fuzzy entropy. Liu [5] extended Yager’s formula to give a general relationship between distance (or similarity measure) and fuzzy entropy, obtained some important conclusions. Using the concept of σ-distance measure, as defined by Liu, we have studied some properties of σ-distance measure and obtained a result about fuzzy entropy induced by σ-distance measure [9]. This result is a generalization of Kosko’s formula [8]. In this paper, some new properties of distance measure are given, and some new formulas of fuzzy entropy induced by distance measure are shown, among which Kaufmann’s formula is extended to some extent.

In the study of distance measure, one interesting problem is the relation between distance measure and metric defined on all fuzzy sets. We shed some light on this problem in this paper, to obtain a characterization of a σ-distance measure as a metric defined on all fuzzy sets.

2. Fuzzy entropy and distance measure

Throughout this paper, $R^+ = [0, +\infty)$, $F(X)$ expresses the set of all fuzzy sets on universal set $X$, $P(X)$ expresses the set of all crisp sets on universal set $X$. $\mu_A(x)$ is the membership function of $A \in F(X)$, $[a]$ is the fuzzy set of $X$ for which $\mu_{[a]}(x) = a, \forall x \in X (a \in [0, 1])$. For fuzzy set $A$, we use $A^c$ to express the complement of $A$, i.e., $\mu_{A^c}(x) = 1 - \mu_A(x), \forall x \in X$. For two fuzzy sets $A$ and $B$, $A \cup B$ the union of $A$ and $B$ is defined as $\mu_{A \cup B}(x) =$
max(\(\mu_A(x), \mu_B(x)\)), \(A \cap B\) the intersection of \(A\) and \(B\) is defined as \(\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))\). The fuzzy set \(A^*\) is called the sharpness of \(A\), if \(\mu_{A^*}(x) = \frac{1}{2}\) and \(\mu_{A^*}(x) \leq \mu_A(x)\) when \(\mu_A(x) \leq \frac{1}{2}\). For fuzzy set \(A\), the crisp sets \(A_{\text{near}}\), \(A_{\text{far}} \in \mathcal{P}(X)\) are defined as

\[
\mu_{A_{\text{near}}}(x) = \begin{cases} 
1, & \mu_A(x) \geq \frac{1}{2}, \\
0, & \mu_A(x) < \frac{1}{2}, 
\end{cases}
\]

\[
\mu_{A_{\text{far}}}(x) = \begin{cases} 
0, & \mu_A(x) \geq \frac{1}{2}, \\
1, & \mu_A(x) < \frac{1}{2}. 
\end{cases}
\]

To make the statements more clear, the following definitions will be based on \(\mathcal{F}(X)\), and not on a subclass of \(\mathcal{F}(X)\) as done by Xuechang [5].

**Definition 2.1** (Liu Xuechang [5]). A real function \(e : \mathcal{F}(X) \rightarrow [0, 1]\) is called an entropy on \(\mathcal{F}(X)\), if \(e\) has the following properties:

- (EP1) \(e(D) = 0\), \(\forall D \in \mathcal{P}(X)\);
- (EP2) \(e([\frac{1}{2}, 1]) = \max_{A \in \mathcal{F}(X)} e(A)\);
- (EP3) \(e(A^*) = e(A)\), \(\forall A \in \mathcal{F}(X)\);
- (EP4) \(e(A) = e(A^c), \forall A \in \mathcal{F}(X)\).

**Definition 2.2** (Liu Xuechang [5]). A real function \(d : \mathcal{F}(X) \times \mathcal{F}(X) \rightarrow [0, 1]\) is called a distance measure, if \(d\) has the following properties:

- (DP1) \(d(A, B) = d(B, A), \forall A, B \in \mathcal{F}(X)\);
- (DP2) \(d(A, A) = 0, \forall A \in \mathcal{F}(X)\);
- (DP3) \(d(D, D^c) = \max_{A, B \in \mathcal{F}(X)} d(A, B), \forall D \in \mathcal{P}(X)\);
- (DP4) \(\text{if } A \subset B \subset \mathcal{F}(X)\), then \(d(A, C) \geq d(A, B)\) and \(d(A, C) \geq d(B, C)\).

If we normalize \(e\) and \(d\), we can make \(0 \leq e(A) \leq 1\) for \(A \in \mathcal{F}(X)\), \(0 \leq d(A, B) \leq 1\) for \(A, B \in \mathcal{F}(X)\). In this paper, we only consider the entropy and distance measure that are normalizations.

**Proposition 2.1** (Liu Xuechang [5]). If \(d\) is a distance measure on \(\mathcal{F}(X)\), define

\[
e(A) = 1 - d(A, A^c), \forall A \in \mathcal{F}(X).
\]

Then \(e\) is an entropy on \(\mathcal{F}(X)\).

We call \(e\) mentioned in Proposition 2.1 the entropy generated by distance measure \(d\) and denote it by \(e(d)\). To describe the relation between the fuzziness of \(A\), \(A \cap D\) and \(A \cup D (D, D^c \in \mathcal{P}(X))\), Xuechang [5] introduced the concept of \(\sigma\)-entropy. Similar method is also used for distance measure.

**Definition 2.3** (Liu Xuechang [5]). A fuzzy entropy \(e\) is called a \(\sigma\)-entropy on \(\mathcal{F}(X)\), if \(e\) satisfies

\[
e(A) = e(A \cap D) + e(A \cup D^c), \quad \forall D \in \mathcal{P}(X).
\]

**Definition 2.4** (Liu Xuechang [5]). A distance measure \(d\) is called a \(\sigma\)-distance measure on \(\mathcal{F}(X)\), if for any \(A, B \in \mathcal{F}(X)\) and \(D \in \mathcal{P}(X)\),

\[
d(A, B) = d(A \cap D, B \cap D) + d(A \cap D^c, B \cap D^c)
\]

holds.

**Theorem 2.1.** If \(d\) is a \(\sigma\)-distance measure on \(\mathcal{F}(X)\), then \(e(d)\) is a \(\sigma\)-entropy on \(\mathcal{F}(X)\).

**Proof.** The theorem can be proved by Theorems 8.1 and 7.2 in [5].

In [5], it was proved that \(e\) is a \(\sigma\)-entropy if \(e(A) + e(B) = e(A \cap B) + e(A \cup B)\) for any \(A, B \in \mathcal{F}(X)\). Next we give another expression of \(\sigma\)-entropy.

**Theorem 2.2.** Let \(e\) be an entropy on \(\mathcal{F}(X)\); then \(e\) is a \(\sigma\)-entropy iff for any \(A \in \mathcal{F}(X)\) and \(D \in \mathcal{P}(X)\),

\[
e(A) = e(A \cap D) + e(A \cap D^c), \quad \forall D \in \mathcal{P}(X)
\]

holds.

**Proof.** We only prove that (2.1) implies (2.3). The proof of the conclusion of (2.3) implies that (2.1) is similar. From (2.1), we have

\[
e(A \cup D) = e((A \cup D) \cap D) + e((A \cup D) \cap D^c)
\]

\[
= e(D) + e(A \cap D^c) = e(A \cap D^c).
\]

Similarly,

\[
e(A \cup D^c) = e(A \cap D).
\]

Thus

\[
e(A \cup D) + e(A \cup D^c) = e(A \cap D) + e(A \cap D^c) = e(A)
\]

and the conclusion follows.
Theorem 2.3. If $e$ is an entropy on $\mathbf{F}(X)$, then $e' = e/(2-e)$ is also an entropy on $\mathbf{F}(X)$.

Proof. It is clear that $0 \leq e'(A) \leq 1$ for any $A \in \mathbf{F}(X)$ and $e'$ satisfies (EP1), (EP2) and (EP4). For (EP3), from $e(A^*) \leq e(A)$, we have $2-e(A) \leq 2-e(A^*)$. Thus

$$e'(A^*) = \frac{e(A^*)}{2-e(A^*)} \leq \frac{e(A)}{2-e(A)} = e'(A).$$

3. Fuzzy entropy induced by distance measure

One commonly used $\sigma$-distance measure between fuzzy sets is the Hamming distance $d_{H}$ [5]. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set, then $d_{H}$ is defined as

$$d_{H}(A, B) = \frac{1}{n} \sum_{i=1}^{n} |\mu_{A}(x_i) - \mu_{B}(x_i)|.$$

Kaufman [7] defined an entropy generated by distance measure which is different from $e(d)$:

$$e_{K}(A) = \frac{2}{n^p} \sum_{i=1}^{n} |\mu_{A}(x_i) - \mu_{A_{near}}(x_i)|^p$$

$$\quad p \in [1, +\infty)$$

For $p = 1$, $e_{K}(A) = 2/n \sum_{i=1}^{n} |\mu_{A}(x_i) - \mu_{A_{near}}(x_i)| = 2d_{H}(A, A_{near})$. Using $d_{H}$, Kosko [8] also defined an entropy which is different from the entropy $e(d)$: $e_{K}(A) = d_{H}(A, A_{near})/d_{H}(A, A_{far})$. Using Kosko’s idea, we have given a general conclusion about the relation between entropy and distance measure in [9] (see Theorem 3.1). In this section, we will give another relation between entropy and distance measure which extends Kaufmann’s formula to some extent. First we state some properties about $\sigma$-distance measure; the proofs are contained in [9].

Proposition 3.1 (Fan and Xie [9]). Let $d$ be a $\sigma$-distance measure on $\mathbf{F}(X)$; then

1. $d(A, A_{near}) \geq d(A^*, A_{near})$;
2. $d(A, A_{far}) \leq d(A^*, A_{far})$.

For crisp set $D \subset X$, use $\frac{1}{2}D$ to express that the fuzzy set satisfies

$$\mu_{(1/2)D}(x) = \begin{cases} \frac{1}{2}, & x \in D, \\ 0, & x \notin D. \end{cases}$$

Proposition 3.2 (Fan and Xie [9]). Let $d$ be a $\sigma$-distance measure on $\mathbf{F}(X)$; if $d$ satisfies $d(\frac{1}{2}D, [0]) = d(\frac{1}{2}D, D)$ for any crisp set $D$, then $d$ has the following properties:

1. $d(A, A_{near}) \leq d(A, A^*_{far})$;
2. $d(A, A_{far}) = d(A, A^*_{near}) = d(A, A^*_{far})$.

Theorem 3.1 (Fan and Xie [9]). Let $d$ be a $\sigma$-distance measure on $\mathbf{F}(X)$; if $d$ satisfies the following properties,

1. $d(\frac{1}{2}D, [0]) = d(\frac{1}{2}D, D), \forall D \in \mathbf{P}(X)$;
2. $d(A^*, B^*) = d(A, B), \forall A, B \in \mathbf{F}(X)$,
then $e_{K}(A) = d(A, A_{near})/d(A, A_{far})$ is a fuzzy entropy.

Proof. From Proposition 3.2, we have $0 \leq e_{K}(A) \leq 1$.

(i) $e_{K}$ satisfies (EP1): $\forall D \subset \mathbf{P}(X), D_{near} = D, D_{far} = D^c$, so $e_{K}(D) = 0$.

(ii) $e_{K}$ satisfies (EP2): because $[\frac{1}{2}]_{near} = [1], [\frac{1}{2}]_{far} = [0]$, thus

$$e_{K}([\frac{1}{2}]) = d([\frac{1}{2}], [1]) + 1 - d([\frac{1}{2}], [0]) = 1.$$

(iii) $e_{K}$ satisfies (EP3): Let $A^*$ is the sharpened version of $A$; then $A^*_{near} = A_{near}, A^*_{far} = A_{far}$. From Proposition 3.1, we have

$$e_{K}(A^*) = d(A^*, A_{near}) + 1 - d(A^*, A_{far})$$

$$\leq d(A, A_{near}) + 1 - d(A, A_{far}) = e_{K}(A).$$

(iv) $e_{K}$ satisfies (EP4): From $d(A, B) = d(A^c, B^c)$, obtain

$$d(A, A_{near}) = d(A^c, (A_{near})^c) = d(A^c, A_{far}),$$

$$d(A, A_{far}) = d(A^c, (A_{far})^c) = d(A^c, A_{near}).$$

From Proposition 3.2, we have

$$e_{K}(A) = d(A, A_{near}) + 1 - d(A, A_{far})$$

$$= d(A^c, A_{far}) + 1 - d(A^c, A_{near})$$

$$= d(A^c, A^*_{near}) + 1 - d(A^c, A^*_far) = e_{K}(A^c).$$
Note: If \(d\) also satisfies \(d(A, A_{\text{near}}) = 1 - d(A, A_{\text{far}})\), then \(e_F(A) = 2d(A, A_{\text{near}})\) is an entropy. For Hamming distance \(d_H\), we have \(d_H(A, A_{\text{near}}) = 1 - d_H(A, A_{\text{far}})\). Therefore, \(e_F(A) = 2d_H(A, A_{\text{near}})\) is an entropy. So Theorem 3.2 can be considered as a generalization of Kaufmann’s formula to some extent.

**Proposition 3.3.** For Hamming distance \(d_H\) and entropy \(e_F(A) = 2d_H(A, A_{\text{near}})\), we have \(e'_F = e_K\), where \(e'_F\) is defined in Theorem 2.3.

**Proof.** For any \(A \in F(X)\), we have
\[
e'_F(A) = \frac{e_F(A)}{2 - e_F(A)} = \frac{2d_H(A, A_{\text{near}})}{2 - 2d_H(A, A_{\text{near}})} = \frac{d_H(A, A_{\text{near}})}{1 - d_H(A, A_{\text{far}})} = e_K(A).
\]

In the following, we will give some formulas of fuzzy entropy induced by distance measure which are different from the above.

**Theorem 3.3.** If \(d\) is a distance measure on \(F(X)\), define
\[
e_1(A) = \frac{d(A \cup A^c, X)}{d(A \cap A^c, X)}.
\]
Then \(e_1\) is a fuzzy entropy.

**Proof.** \(e_1\) obviously satisfies \(0 \leq e_1(A) \leq 1\), (EP1), (EP2) and (EP4). For (EP3), from
\[
0 \subseteq A^c \cap A^c \subseteq A \cap A^c \subseteq A \cup A^c \subseteq A^c \cup A^c \subseteq X,
\]
we have \(d(A^c \cup A^c, X) \leq d(A \cup A^c, X), d(A^c \cap A^c, X) \geq d(A \cap A^c, X)\). Thus
\[
e_1(A^c) = \frac{d(A^c \cup A^c, X)}{d(A^c \cap A^c, X)} \leq \frac{d(A \cup A^c, X)}{d(A \cap A^c, X)} = e_1(A).
\]
Similarly, we have the following theorem.

**Theorem 3.4.** If \(d\) is a distance measure on \(F(X)\), define
\[
e_2(A) = \frac{d(A \cap A^c, 0)}{d(A \cup A^c, 0)}.
\]
Then \(e_2\) is a fuzzy entropy.

**Proposition 3.4.** If \(d\) is a distance measure and satisfies \(d(A, B) = d(A^c, B^c)\), then \(e_1 = e_2\).

We will now state two induced relations; the proofs are similar to the proof of Theorem 4.1 and are omitted.

**Theorem 3.4.** If \(d\) is a distance measure on \(F(X)\) and satisfies \(d([\frac{1}{2}], X) = d([\frac{1}{2}], 0)\), define
\[
e_3(A) = \frac{d(A \cup A^c, X)}{d(A \cap A^c, 0)}.
\]
Then \(e_3\) is a fuzzy entropy.

**Theorem 3.5.** If \(d\) is a distance measure on \(F(X)\) and satisfies \(d([\frac{1}{2}], X) = d([\frac{1}{2}], 0)\), define
\[
e_4(A) = \frac{d(A \cap A^c, 0)}{d(A \cup A^c, X)}.
\]
Then \(e_4\) is a fuzzy entropy.

**Proposition 3.5.** If \(d\) is a distance measure and satisfies \(d(A, B) = d(A^c, B^c)\), then \(e_3 = e_4\).

4. Some properties of \(\sigma\)-distance measure

**Theorem 4.1.** Let \(d\) be a distance measure on \(F(X)\); then \(d\) is a \(\sigma\)-distance measure iff for any \(A, B \in F(X)\) and \(D \in P(X)\),
\[
d(A, B) = d(A \cup D, B \cup D) + d(A^c \cap D, B^c \cap D) \tag{4.1}
\]
holds.

**Proof.** We only prove that (2.2) implies (4.1). The proof of the conclusion of (4.1) implies (2.2) is similar.

From (2.2), we have
\[
d(A \cup D, B \cup D) = d(A \cup D) \cap D, (B \cup D) \cap D
\]
\[+ d((A \cup D) \cap D^c, (B \cup D) \cap D^c)
\]
\[= d(D, D) + d(A \cap D^c, B \cap D^c)
\]
\[= d(A \cap D^c, B \cap D^c).
\]
Similarly \(d(A \cup D^c, B \cup D^c) = d(A \cap D, B \cap D)\).
Thus

\[ d(A \cup D, B \cup D) + d(A \cup D^c, B \cup D^c) \]

\[ = d(A \cap D, B \cap D) + d(A \cap D^c, B \cap D^c) \]

\[ = d(A, B) \]

and the conclusion follows.

**Proposition 4.1.** If \( d \) is a \( \sigma \)-distance measure on \( \mathbf{F}(X) \), then we have

\[ d(A, A \cup B) = d(B, A \cap B), \]  \hspace{1cm} (4.2)

\[ d(A, A \cap B) = d(B, A \cup B). \]  \hspace{1cm} (4.3)

**Proof.** We only prove (4.2). The proof of the conclusion of (4.3) is similar.

Let \( D = \{ x \in X \mid \mu_A(x) \leq \mu_B(x) \} \); from (2.2) we have

\[ d(A, A \cup B) = d(A \cap D, (A \cup B) \cap D) \]

\[ + d(A \cap D^c, (A \cup B) \cap D^c) \]

\[ = d(A \cap D, B \cap D) + d(A \cap D^c, A \cap D^c) \]

\[ = d(A \cap D, B \cap D), \]

\[ d(B, A \cap B) = d(B \cap D, A \cap B \cap D) \]

\[ + d(B \cap D^c, A \cap B \cap D^c) \]

\[ = d(B \cap D, A \cap D) + d(B \cap D^c, B \cap D^c) \]

\[ = d(B \cap D, A \cap D). \]

Therefore \( d(A, A \cup B) = d(B, A \cap B). \)

**Proposition 4.2.** If \( d \) is a \( \sigma \)-distance measure on \( \mathbf{F}(X) \), then for \( A, B \in \mathbf{F}(X) \) we have

\[ d(A, B) = d(A \cap B, A \cup B) \]

\[ = d(A, A \cap B) + d(A, A \cup B) \]

\[ = d(A, A \cap B) + d(A, A \cup B) \]

\[ = d(B, A \cap B) + d(B, A \cup B). \]

**Proof.** Let \( D = \{ x \in X \mid \mu_A(x) \leq \mu_B(x) \} \); we have proved in Proposition 4.1 that \( d(A, A \cup B) = d(B, A \cap B) = d(A \cap D, B \cap D) \). Similarly, we can prove that \( d(A, A \cap B) = d(B, A \cup B) = d(A \cap D^c, B \cap D^c) \). From (2.2), we have

\[ d(A, B) = d(A \cap D, B \cap D) + d(A \cap D^c, B \cap D^c) \]

\[ = d(A, A \cap B) + d(A, A \cup B) \]

\[ = d(B, A \cap B) + d(B, A \cup B). \]

In the proof of Theorem 8.2 in [5] (step (6)), it was proved that \( s(A, B) = s(A \cap B, A \cup B) \) for a normal \( \sigma \)-similarity measure \( s \). Because \( d = 1 - s \) we have \( d(A, B) = d(A \cap B, A \cup B) \).

**Theorem 4.2.** If \( d \) is a \( \sigma \)-distance measure on \( \mathbf{F}(X) \), \( A, B, C \in \mathbf{F}(X) \), then we have

\[ d(A, B) + d(A, C) = d(A, B \cap C) + d(A, B \cup C). \]  \hspace{1cm} (4.4)

**Proof.** Let \( D = \{ x \in X \mid \mu_A(x) \leq \mu_B(x) \} \); from (2.2) we have

\[ d(A, B \cap C) \]

\[ = d(A \cap D, B \cap C \cap D) + d(A \cap D^c, B \cap C \cap D^c) \]

\[ = d(A \cap D, B \cap D) + d(A \cap D^c, C \cap D^c), \]

\[ d(A, B \cup C) = d(A \cap D, (B \cup C) \cap D) \]

\[ + d(A \cap D^c, (B \cup C) \cap D^c) \]

\[ = d(A \cap D, C \cap D) + d(A \cap D^c, B \cap D^c). \]

Thus,

\[ d(A, B \cap C) + d(A, B \cup C) \]

\[ = d(A \cap D, B \cap D) + d(A \cap D^c, C \cap D^c) \]

\[ + d(A \cap D, C \cap D) + d(A \cap D^c, B \cap D^c) \]

\[ = d(A, B) + d(A, C). \]

**Corollary.** If \( d \) is a \( \sigma \)-distance measure on \( \mathbf{F}(X) \), then for any \( A, B, C \in \mathbf{F}(X) \) we have

(1) \( d(A, B \cup C) \leq d(A, B) + d(A, C) \);

(2) \( d(A, B \cap C) \leq d(A, B) + d(A, C) \).
5. A relation between distance measure and metric

Metric is an important concept in mathematics; we call \( \rho \) a metric on set \( U \), if \( \rho \) satisfies [4]

1. \( \rho(x, y) \geq 0 \) and \( \rho(x, y) = 0 \) iff \( x = y \);
2. \( \rho(x, y) = \rho(y, x) \) for all \( x, y \in U \);
3. \( \rho(x, z) \leq \rho(x, y) + \rho(y, z) \) for all \( x, y, z \in U \).

Now we give a condition for which a distance measure is a metric on \( F(X) \). First we give a lemma.

**Lemma 5.1** (Liu Xuechang [5]). If \( D_1, D_2, \ldots, D_n \in P(X) \), \( \bigcup_{i=1}^{n} D_i = X \), \( D_i \cap D_j = 0 \) for \( i \neq j \) and \( d \) is a \( \sigma \)-distance measure, then \( d(A, B) = \sum_{i=1}^{n} d(A \cap D_i, B \cap D_i) \). Especially \( d(A, B) = \sum_{i=1}^{n} d(A(x_i), B(x_i)) \) when \( X = \{x_1, x_2, \ldots, x_n\} \) is a finite set.

**Theorem 5.1.** Let \( d \) be a \( \sigma \)-distance measure on \( F(X) \). If \( d \) satisfies

1. \( d(A, B) = 0 \) iff \( A = B \);
2. \( d(A, B) + d(B, C) = d(A, C) \) for \( A \subset B \subset C \).

Then \( d \) is a metric on \( F(X) \).

**Proof.** It is clear that \( d \) satisfies (1) and (2) of the definition of metric. Now we prove \( d \) satisfies (3) of the definition of metric. For fuzzy sets \( A, B, C \in F(X) \), from (4.4), we have

\[
\begin{align*}
&d(A, B) + d(B, C) = d(B, A \cap C) + d(B, A \cup C).
\end{align*}
\]

Thus we can suppose that \( A \subset C \). Let

\[
\begin{align*}
D_1 &= \{x \in X \mid \mu_B(x) \leq \mu_A(x) \leq \mu_C(x)\}, \\
D_2 &= \{x \in X \mid \mu_B(x) < \mu_A(x) \leq \mu_C(x)\}, \\
D_3 &= \{x \in X \mid \mu_A(x) < \mu_B(x) \leq \mu_C(x)\}.
\end{align*}
\]

From Lemma 5.1, we have

\[
\begin{align*}
d(A, B) &= d(A \cap D_1, B \cap D_1) + d(A \cap D_2, B \cap D_2) + d(A \cap D_3, B \cap D_3), \\
d(B, C) &= d(B \cap D_1, C \cap D_1) + d(B \cap D_2, C \cap D_2) + d(B \cap D_3, C \cap D_3), \\
d(A, C) &= d(A \cap D_1, C \cap D_1) + d(A \cap D_2, C \cap D_2) + d(A \cap D_3, C \cap D_3).
\end{align*}
\]

From \( B \cap D_1 \subset A \cap D_1 \subset C \cap D_1 \), we have

\[
\begin{align*}
d(A \cap D_1, C \cap D_1) &\leq d(B \cap D_1, C \cap D_1) + d(A \cap D_1, B \cap D_1).
\end{align*}
\]

From \( A \cap D_2 \subset B \cap D_2 \subset C \cap D_2 \), we have

\[
\begin{align*}
d(A \cap D_2, C \cap D_2) &\leq d(B \cap D_2, C \cap D_2) + d(A \cap D_2, B \cap D_2).
\end{align*}
\]

Summing up the above results, we have

\[
d(A, C) \leq d(A, B) + d(B, C).
\]

**Example 5.1.** Let \( X = \{x_1, x_2, \ldots, x_n\} \) be a finite set, and \( c_i \) be a family of constant numbers which satisfy the following condition: \( \sum_{i=1}^{n} c_i = 1 \). Then the following distance measure

\[
d_F(A, B) = \sum_{i=1}^{n} c_i |A(x_i) - B(x_i)|
\]

is a metric on \( F(X) \). Especially when \( c_i = 1/n \), we obtain the Hamming distance.

**Lemma 5.2.** Let \( \rho \) be a metric on \( [0, +\infty) \); then

\[
\rho(\lambda, \mu) = c|\lambda - \mu| \text{ iff } \rho \text{ satisfies the following conditions:}
\]

\[
\begin{align*}
&\rho(\lambda, \mu) = \rho(\lambda, \mu), \\
&\rho(\lambda + \mu) = \rho(0, \mu).
\end{align*}
\]

**Proof.** If \( \rho(\lambda, \mu) = c|\lambda - \mu| \), then obviously \( \rho \) satisfies (5.1) and (5.2). In the following, we shall prove that if \( \rho \) satisfies (5.1) and (5.2), then \( \rho(\lambda, \mu) = c|\lambda - \mu| \).

Let \( f(\lambda) = \rho(0, \lambda) \); then

\[
\begin{align*}
f(\lambda + \mu) &= \rho(0, \lambda + \mu) = \rho(0, \lambda) + \rho(\lambda, \lambda + \mu) \\
&= \rho(0, \lambda) + \rho(0, \mu) = f(\lambda) + f(\mu).
\end{align*}
\]

By [1, p. 34], we have \( f(\lambda) = c\lambda \).
Because \( \rho(\lambda, \mu) = \rho(\min(\lambda, \mu), \min(\lambda, \mu) + |\lambda - \mu|) = \rho(0, |\lambda - \mu|), \) thus \( \rho(\lambda, \mu) = c|\lambda - \mu| \).

**Note 1.** If we only consider metric \( \rho \) defined on \([0, 1]\) and valued in \([0, 1]\), the above conclusion is also correct.

**Note 2.** The two conditions on Lemma 5.2 have no relation.

(1) does not necessarily imply (2): define \( \rho_1(\lambda, \mu) = |\lambda^2 - \mu^2| \), then \( \rho_1 \) is a metric on \([0, +\infty)\) and satisfies \( \rho(\lambda, \gamma) = \rho(\lambda, \mu) + \rho(\mu, \gamma) \) for \( \lambda \leq \mu \leq \gamma \). From \( \rho_1(0, \mu) = \mu^2 \) and \( \rho_1(\lambda, \lambda + \mu) = |\lambda^2 - (\lambda + \mu)^2| = \mu^2 (2\lambda + \mu) \), we know that generally \( \rho_1(\lambda, \lambda + \mu) \neq \rho_1(0, \mu) \).

(2) does not necessarily imply (1): define \( \rho_2(\lambda, \mu) = |\lambda - \mu|/(1 + |\lambda - \mu|) \), then \( \rho_2(\lambda, \lambda + \mu) = \rho_2(0, \mu) \), but \( \rho_2(\lambda, \gamma) \neq \rho_2(\lambda, \mu) + \rho_2(\mu, \gamma) \) for \( \lambda \leq \mu \leq \gamma \). For example,

\[
\rho_2\left(\frac{1}{2}, \frac{1}{2}\right) + \rho_2\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{1 + \frac{1}{4}} + \frac{1}{1 + \frac{1}{4}} = 2 \cdot \frac{1}{1 + \frac{1}{4}} = \frac{2}{3} \\
\rho_2\left(0, \frac{1}{2}\right) = \frac{1}{1 + \frac{1}{2}} = \frac{1}{3}
\]

We use \( x_2 \) to express the fuzzy point, i.e.

\[
\begin{align*}
\lambda, & \quad y = x \\
0, & \quad y \neq x
\end{align*}
\]

(0 \leq \lambda \leq 1).

**Theorem 5.2.** Let \( X = \{x_1, x_2, \ldots, x_n\} \) be a finite set, and \( d \) be a \( \sigma \)-distance measure on \( F(X) \); then \( d = d_F \) iff \( d \) satisfies the following conditions:

1. \( d(x_i, x_j) = 0 \) iff \( \lambda = \mu \);
2. \( d(x_i, x_j) + d(x_j, x_k) = d(x_j, x_k) \) for \( 0 \leq \lambda \leq \mu \leq \gamma \leq 1 \) and \( x \in X \);
3. \( d(x_i, x_j) = d(x_0, x_{\lambda - \mu}) \), \( \forall x \in X \) and \( 0 \leq \lambda \leq 1, 0 \leq \mu \leq 1 \).

**Proof.** If \( d = d_F \), then \( d \) obviously satisfies conditions (1)–(3) of Theorem 5.2. In the following, we will prove that if \( d \) satisfies conditions (1)–(3) of Theorem 5.2, then \( d = d_F \).

Define \( \rho_3(\lambda, \mu) = d(x_0, x_{\lambda - \mu}) \) \( \forall x \in X \); then \( \rho_3 \) is a metric on \([0, 1]\). By Lemma 5.2, we know \( \rho_3(\lambda, \mu) = c\lambda |\lambda - \mu| \). From Lemma 5.1, we know \( d(A, B) = \sum_{i=1}^{n} d(A(x_i), B(x_i)) \)

\[
= \sum_{i=1}^{n} \rho(x_i, A(x_i), B(x_i)) \\
= \sum_{i=1}^{n} c\lambda |A(x_i) - B(x_i)|,
\]

where \( c\lambda \) is a suitable constant positive number. We can select \( c\lambda \) to satisfy \( 0 \leq d(A, B) \leq 1 \), i.e. \( \sum_{i=1}^{n} c\lambda = 1 \).

6. **For further reading**

The following references are also of interest to the reader: [2,3,6,11].

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**References**