Real root classification of parametric spline functions

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A R T I C L E   I N F O

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A B S T R A C T

The real root classification of a given parametric spline function is a collection of possible cases of its real root distribution on every interval, together with the conditions of its coefficients must be satisfied for each case. This paper presents an algorithm to deal with the real root classification of a given parametric spline function. Two examples are provided to illustrate the proposed algorithm is flexible.

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1. Introduction

The counting and classifying of the real/imaginary roots of a given polynomial have been the subject of many investigations. The classical Sturm Theorem and Tarski’s Theorem are efficient methods for determining the numbers of real roots of polynomials with constant coefficients, but inconvenient for those with symbolic coefficients (see [1] for details). Fortunately, there are several different methods to determine the number of the distinct real roots of polynomials with symbolic coefficients. Yang et al. [2,3] established the complete discrimination system of a real parametric polynomial, which is sufficient for determining the numbers and multiplicities of the real/imaginary roots, namely, determining the complete root classification. In parallel, Gonzalez-Vega et al. [4] proposed the use of Sturm–Habicht sequences to solve the real roots of univariate polynomials. In 2009, Liang and Jeffrey [5] proposed automatic computation of the complete root classification for a real parametric polynomial. More importantly, the algorithm offered improved efficiency and a new test for non-realizable conditions.


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algorithm to isolate the real solutions of two piecewise algebraic curves based on the Krawczyk–Moore iterative algorithm. At the same time, Wu [14] presented an algorithm for computing the real intersection points of piecewise algebraic curves which is primarily based on the interval zeros of the univariate interval polynomial in Bernstein form. A question will arise: Can we count or classify the real roots of a given spline function?

Given a set of spline knots $\Xi: -\infty < x_0 < x_1 < \cdots < x_n < x_{n+1} < \infty$. The univariate spline space $\mathcal{S}_n[x_0, x_1, \ldots, x_{n+1}]$ is defined as follows:

$$\mathcal{S}_n[x_0, x_1, \ldots, x_{n+1}] = \{ S(x) \in C^{n-1}|S_i(x) \in P_n[x], \quad i = 0, 1, \ldots, N \},$$

where $S_i(x)$ denotes the restriction of $S(x)$ over the interval $[x_i, x_{i+1}]$, $i = 0, 1, \ldots, N$, and $P_n[x]$ denotes the set of univariate polynomials with degree $\leq n$ in variable $x$. The function $S(x)$ is said to be of class $C^{n-1}$ if and only if the derivatives $S'(x), S'(x), \ldots, S^{(n-1)}(x)$ exist and are continuous. Certainly, $\mathcal{S}_n[x_0, x_1, \ldots, x_{n+1}]$ is a linear space and its dimension is $n + N + 1$.

It is well-known that an arbitrary univariate spline $S(x) \in \mathcal{S}_n[x_0, x_1, \ldots, x_{n+1}]$ has the following unified representation [15]

$$S(x) = S_0(x) + \sum_{j=1}^{N} c_j(x - x_j)^n,$$

where truncation function means $x_\ast = x$ for $x \geq 0$ and $x_\ast = 0$ for $x < 0$, and $S_0(x)$ is a univariate polynomial with degree $\leq n$ on the initial interval $[x_0, x_1]$.

Firstly, we have the coarse bound of the number of real roots of a given univariate spline.

**Theorem 1.1.** If $S(x) \in \mathcal{S}_n[x_0, x_1, \ldots, x_{n+1}]$, then the number of the roots of $S(x)$ is not greater than $n + N$.

**Proof.** Obviously, the number of real roots of $S(x)$ on $[x_0, x_{n+1}]$ is equal to the summation of the number of real roots of $S_i(x)$ on $[x_i, x_{i+1}]$, $i = 0, 1, \ldots, N$. Suppose the number of the real roots of $S(x)$ on $[x_0, x_{n+1}]$ is $M$, then $S'(x)$ has at least $M - 1$ real roots. Inductively, $S^{(n-1)}(x)$ has at least $M - (n - 1)$ real roots. However, it is obvious that piecewise linear spline $S^{(n-1)}(x)$ has at most $n + N$ real roots on $[x_0, x_n]$. Therefore, $M - (n - 1) \leq N + 1$, i.e., $M \leq n + N$.

It is pointed out that the “parametric” spline function means it contains symbolic coefficients and also allows to have some certain constant coefficients.

Secondly, we make the following two conventions for a given parametric spline.

- $S(x)$ is assumed to be “regular”. It means that none of the spline knots is the real roots of $S(x)$, i.e.

$$S(x_i) \neq 0, \quad i = 0, 1, \ldots, N + 1.$$  (3)

- $S(x)$ is assumed to be non-degenerate. That’s to say,

$$\text{deg}(S_i(x)) = n, \quad i = 0, 1, \ldots, N,$$

where $S_i(x) = S_0(x) + \sum_{j=1}^{n} c_j(x - x_j)^n$. For example, if we give a parametric spline $S(x) = 2x^4 + bx^2 + c + 2(x - 1)^4 + d(x - 2)^4 \in \mathcal{S}_4[0, 1, 2, 3]$, then $S(x)$ has at most 6 real roots (counted with multiplicities) on $(0, 3)$ from Theorem 1.1 until now. In fact, we inevitably encounter another important problem that we want to know all the possible cases of the number of real roots of $S(x)$ and its distribution, as well as the conditions on these symbolic coefficients. □

Undoubtedly, one can deal with real root classification of piecewise polynomial on every interval individually. However, the spline function satisfies certain continuity on adjacent knots, we naturally ask a question: Can we tackle the real root classification of parametric splines on the whole? As the authors knowledge, there is no existing result on the real root classification of a given parametric spline till now. In this article, we mainly generalize the methods in papers [3,5] to solve the real root classification of a given univariate parametric spline. From now on, the parametric spline (2) is assumed to be regular and non-degenerate if not specified.

The rest of this paper is organized as follows. In Section 2, we recall several basic definitions and results on determining the number of real roots of a given parametric polynomial. In Section 3, we give the algorithm to tackle the real root classification of a given parametric spline, which is the main part of this paper. Finally, two illustrated examples are provided to show the algorithm is flexible in Section 4. Also, we conclude this paper in Section 5.

2. Preliminary

In this section, we shall mostly review the existing work with respect to the algorithm proposed by Yang et al. (see [3] for details and references therein). Let $f(x) \in \mathbb{R}[x]$ and write

$$f(x) = a_0x^a + a_1x^{a-1} + \cdots + a_{n-1}x + a_n.$$
Definition 2.1. The following \((2n + 1) \times (2n + 1)\) matrix in term of the coefficients,

\[
\begin{pmatrix}
    a_0 & a_1 & a_2 & \cdots & a_n \\
    0 & na_0 & (n-1)a_1 & \cdots & a_{n-1} \\
    a_0 & a_1 & \cdots & a_{n-1} & a_n \\
    0 & na_0 & \cdots & 2a_{n-2} & a_{n-1} \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & a_0 & a_1 & \cdots & a_n \\
    0 & 0 & a_1 & \cdots & a_n \\
\end{pmatrix}
\]

is called the discriminant matrix of \(f(x)\) and denoted by \(\text{Discr}(f)\).

Definition 2.2. Let \(d_k\) be the \(k\)th principal minor of \(\text{Discr}(f)\), and let \(D_k = d_{2k}\). The \(n\)-tuple

\[
\{D_1, D_2, \ldots, D_n\}
\]

is called the discriminant sequence of polynomial \(f(x)\).

Definition 2.3. We call the list

\[
\{\text{sign}(D_1), \text{sign}(D_2), \ldots, \text{sign}(D_n)\}
\]

the sign list of a given sequence \(\{D_1, D_2, \ldots, D_n\}\), where \(\text{sign}(x)\) is the signum function.

Definition 2.4. The revised sign list \([e_1, e_2, \ldots, e_n]\) of \(f(x)\) (rsf\((f)\) for short) is constructed from the sign list \(s = [s_1, s_2, \ldots, s_n]\) of \(f(x)\) as follows.

- If \([s_i, s_{i+1}, \ldots, s_{i+j}]\) is a section of \(s\), where
  \[
  s_i \neq 0: \quad s_{i+1} = s_{i+2} = \cdots = s_{i+j-1} = 0; \quad s_{i+j} \neq 0,
  \]
  then we replace the subsection \([s_{i+1}, s_{i+2}, \ldots, s_{i+j-1}]\) with \([-s_i, -s_i, s_i, -s_i, -s_i, \ldots]\) such that \(s_{i+r} = (-1)^{(r+1)/2} s_i, r = 1, 2, \ldots, j - 1\).
- Otherwise, let \(e_k = s_k\), i.e., no changes for other terms.

Theorem 1.1. Let \(\text{rsf}(f)\) be the revised sign list of \(f(x)\). If the number of non-vanishing elements in \(\text{rsf}(f)\) is \(l\), and the number of sign changes in \(\text{rsf}(f)\) is \(v\), then \(f(x)\) has \(v\) pairs of distinct complex conjugate roots and \(l - 2v\) distinct real roots.

The following definition is no more than subresultant polynomial sequence of \(f(x)\) and \(f'(x)\).

Definition 2.5. Let \(M = \text{Discr}(f)\), the discriminant matrix of \(f(x)\). By \(M_{ki}\) denote the submatrix formed by the first \(2k\) row of \(M\), the first \(2k - 1\) columns and the \(2k + i\) column of \(M\). Then, construct polynomials

\[
P_k(f) = \sum_{i=0}^{k} \text{det}(M_{n-k+i}) x^{k-i}
\]

for \(k = 0, 1, \ldots, n - 1\). We call the \(n\)-tuple

\[
\{P_0(f), P_1(f), \ldots, P_{n-1}(f)\}
\]

the multiple factor sequence of \(f(x)\).

Proposition 2.1. If the number of the 0’s in the \(\text{rsf}(f)\) is \(k\), then \(P_k(f) = \gcd(f(x), f'(x))\).

Definition 2.6. Let \(\Delta^1(f) = \gcd(f(x), f'(x))\), and let \(\Delta^0(f) = f(x), \Delta^j(f) = \Delta(\Delta^{j-1}(f)), j = 1, 2, \ldots\). Then \(\{\Delta^0(f), \Delta^1(f), \Delta^2(f), \ldots\}\) is called the \(\Delta\)-sequence of \(f(x)\).

In fact, the discriminant sequences of those polynomials in the \(\Delta\)-sequence is enough to determining the multiplicities of real roots of \(f(x)\).

Proposition 2.2. If \(\Delta^j(f)\) has \(k\) real roots with multiplicities \(n_1, n_2, \ldots, n_k\) and \(\Delta^{j-1}(f)\) has \(m\) distinct real roots, then \(m \geq k\), and the multiplicities of these \(m\) distinct roots are \(n_1 + 1, n_2 + 1, \ldots, n_k + 1, 1, \ldots, 1\) respectively.
It tells us that we can do in this way for \( \lambda \)-sequence until for some \( j \) that the revised sign list of the discriminant sequence of \( f \) does not contains 0 and compute the numbers of the distinct real roots of \( \Delta(f) \) (using Theorem 2.1), then compute the numbers with multiplicities of the real roots for \( \Delta^{-1}(f) \) (using Proposition 2.2), and then for \( \Delta^{-2}(f), \ldots \), until we obtain the real root classification for \( f(x) \) at last.

We consider how to compute the number of real roots for \( f(x) \) on an interval \((a, b)\). Let 
\[
\Phi(x) = (1-x)^n f \left( \frac{b - ax}{1 - x} \right),
\]
\[
h(x) = \Phi(-x^2).
\]

It is easy to see that the number of the roots of a polynomial \( f(x) \) in the interval \((a, b)\), with \( f(a) \neq 0 \) and with \( f(b) \neq 0 \) is equal to the number of negative roots of \( \Phi(x) \). The number of negative roots of \( \Phi(x) \) equals the half number of the roots of \( h(x) \). Yang et al. [3] presented a better method to construct the discriminant sequence of \( h(x) \) directly from the discriminant matrix of \( f(x) \) directly.

**Definition 2.7.** By \( \{d_1, d_2, \ldots, d_{2n+1}\} \) denote the principal minor sequence of \( \text{Discr}(f) \), we call \( 2n \)-tuple \( \{d_1d_2d_3, \ldots, d_{2n}d_{2n+1}\} \)
the negative root discriminant sequence, and denoted by n.r.d. \( (f) \).

**Proposition 2.3.** Let \( \{d_1, d_2, \ldots, d_{2n+1}\} \) be the principal minor sequence of \( \text{Discr}(f) \) and let \( h(x) = f(-x^2) \). Assume \( a_0 \neq 0 \). Then, the discriminant sequence of \( h(x) \)
\[
D = \{D_1(h), D_2(h), \ldots, D_{2n}(h)\}
\]
is equal to n.r.d. \( (f) \), i.e., \( D_n(h) = d_n(f)d_{n+1}(f) \), up to a factor of the same sign as \( a_0 \).

Liang et al. [5] gave an improved algorithm to tackle the complete root classification of a parametric polynomial directly using sign lists.

**Definition 2.8** [1]. Let \( s = [s_n, \ldots, s_0] \) be a finite list of elements in \( \mathbb{R} \) such that \( s_n \neq 0 \). Let \( m < n \) be such that \( s_{n-1} = \cdots = s_{m+1} = 0 \), and \( s_m \neq 0 \), and \( s' = [s_m, \ldots, s_0] \). If there is no such \( m \), then \( s' \) is the empty list. We define inductively \( PmV(s) = \{0, \text{PmV}(s') + \epsilon_{n-m}\text{sign}(s_{n}s_m), \text{PmV}(s'), \text{if } n - m \text{ odd; } \text{if } n - m \text{ even.} \}
\]
where \( \epsilon_{n-m} = (-1)^{(n-m)(n-m-1)/2} \).

\( PmV(s) \) is called “generalized permeance minus variation”. When all the elements of \( s \) are non-zero, \( PmV(s) \) is the difference between the number of sign permeance and the number of sign variations.

The following theorem give the number of distinct real roots in term of sign lists directly.

**Theorem 2.2** [5]. Let \( D = [D_1, \ldots, D_n] \) be the discriminant sequence of polynomial \( f(x) \) with degree \( n \). If \( PmV(D) = r \), then \( f(x) \) has \( r + 1 \) distinct real roots.

The next theorem can be used to detect the non-realizable sign lists in the input conditions.

**Theorem 2.3** [5]. Let \( s = [s_1, \ldots, s_n], \) and \( e = [e_1, \ldots, e_n] \) be the sign list and the revised sign list of \( f(x) \), respectively. Then \( PmV(s) = PmV(e) \)

### 3. Real root classification of parametric splines

Let \( S(x) \in \mathbb{S}_n[x_0, x_1, \ldots, x_{N+1}] \) be a univariate parametric spline. The real root classification (RRC) of \( S(x) \) on \([x_0, x_{N+1}]\) is denoted by
\[
\{n_1^{[0]}, n_2^{[0]}, \ldots, n_1^{[1]}, n_2^{[1]}, \ldots, n_1^{[N]}, n_2^{[N]}, \ldots\},
\]
where \( n_i^{[k]} \) denotes the multiplicities of the distinct real roots of \( S(x) \) on the interval \([x_i, x_{i+1}]\). The purpose of RRC of a given parametric spline \( S(x) \) is to derive a collection of all possible cases of its real roots on \([x_0, x_{N+1}]\), together with the conditions of its coefficients must be satisfied for each case.

Let
\[
\Phi_i(x) = (1-x)^{n_i} S_i \left( \frac{x_{i+1} - x_i x}{1 - x} \right),
\]
\[
h_i^+(x) = \Phi_i(-x^2), \quad i = 0, 1, \ldots, N.
\]
With the above discussion, we can easily have

**Proposition 3.1.**

\[
\text{Zero}|S_i(x)|_{[x_i, x_{i+1}]} = \text{Zero}|\Phi^{\delta}(x)|_{[-\infty, 0]} = \frac{1}{2}\text{Zero}|h^{\delta}(x)|_{[-\infty, +\infty]},
\]

where, \text{Zero}|p(x)|_{(a, b)} denotes the number of real roots of \(p(x)\) on \((a, b)\) (counted with multiplicities).

As to the number of distinct real roots of \(S(x)\), we have the following conclusion.

**Proposition 3.2.** Given a parametric spline \(S(x)\) and let \(\Phi^{\delta}(x)\) and \(h^{\delta}(x)\) be defined as (6). By \(\{d_1^{\delta}, d_2^{\delta}, \ldots, d_{2n}^{\delta}\}\) denote the principal minor sequence of \(D^{\delta}\). Set

\[
D^{[i]} = \{d_1^{[i]} d_2^{[i]} d_3^{[i]} \ldots d_{2n}^{[i]} d_{2n+1}^{[i]}\}, \quad i = 0, 1, \ldots, N.
\]

If \(\text{PMV}(D^{[i]}) = r_i\), then the number of the distinct real roots of \(S(x)\) is \(\frac{1}{2}\sum_{i=0}^{N} (r_i + 1)\).

**Proof.** It is obvious that the number of distinct real roots of \(S(x)\) is the summation of the number of distinct real roots of \(S_i(x)\), \(i = 0, 1, \ldots, N\). From Proposition 2.3, we know \(D^{[i]}\) is the discriminant sequence of \(h^{\delta}(x)\). Then, we can easily arrive the result from Proposition 3.1 and Theorem 2.2.

**Remark 3.1.** Though we construct the discriminant sequence of \(h^{\delta}(x)\) from the discriminant matrix of \(\Phi^{\delta}(x)\) directly, we delete the non-zero positive factor in every term of \(D^{\delta}\). It greatly reduces the computational cost compared to algorithm in [5].

**Remark 3.2.** Since the spline function satisfies continuity \(S_{i+1}(x) - S_i(x) = c_{i+1}(x - x_i)^n\), the only difference between \(D^{\delta}\) and \(D^{[i+1]}\) is that the latter has a more parameter \(c_{i+1}\). That’s to say, if the \(\text{sign}(D^{[i]})\) is determined, then the possibility of \(\text{sign}(D^{[i+1]})\) becomes very small. This phenomenon can be seen from our numerical experiments clearly.

Now, we introduce some notations used in the subsequent algorithms.

- **PRRC**\((n, N)\): the possible real root classification of \(S(x) \in \mathbb{S}_{2}[x_0, \ldots, x_{N+1}]\) on every interval \((x_i, x_{i+1})\), \(i = 0, 1, \ldots, N\). It is the union of the set of partition of \(0, 1, \ldots, n + N\) (using Theorem 1.1). For example,

\[
\text{PRRC}(2, 1) = \{[0, 0], [0, 1, 1], [0, 1, 0, 0], [1, 1, 1], [1, 0, 0, 0], [1, 1, 0, 0], [1, 0, 0, 0]\}.
\]

It tells us that it possibly exists twelve different cases for real root classification of a given parametric spline \(S(x) \in \mathbb{S}_{2}[x_0, x_1, x_2]\) on \((x_0, x_2)\). Every element in **PRRC**\((n, N)\) has the same meaning as defined in (5).

- **rrc**: the element in **PRRC**\((n, N)\) and it has the form \(\text{rrc} = \text{rrc}_0 \land \text{rrc}_1 \land \cdots \land \text{rrc}_n\), where \(\text{rrc}_i = \{n_1^{[i]}, n_2^{[i]}, \ldots\}\) reveals the number and multiplicity of real roots of \(S(x)\) on \((x_i, x_{i+1})\). Take \(\text{rrc} = \{2\} \land \{1, 1\}\) for example, it means that \(S(x)\) has one real root with multiplicity 2 on \((x_0, x_1)\) and two simple real roots on \((x_1, x_2)\).

- **L(rrc)**: we define \(L(\text{rrc}) = \{L_0\} \land \{L_1\} \land \cdots \land \{L_n\}\), where \(L_i = \Phi(\text{rrc}_i)\) denotes the number of elements in \(\text{rrc}_i\), i.e., the number of distinct real roots of \(S(x)\) on interval \((x_i, x_{i+1})\). Take \(\text{rrc} = \{2\} \land \{1, 1\}\) for example, we have \(L(\text{rrc}) = \{1, 1\} \land \{2\}\) and it shows that \(S(x)\) has one distinct real root on \((x_0, x_1)\) and two distinct real roots on \((x_1, x_2)\).

- **MS**: the set of possible multi-sign lists from the discriminant sequence of \(D = D^{[0]} \land D^{[1]} \land \cdots \land D^{[N]}\) and denoted by \(\text{MS} = \{ms | ms = ms^{[0]} \land ms^{[1]} \land \cdots \land ms^{[N]}\}\), where \(S^{[i]}\) denotes the set of all possible sign lists from \(D^{[i]}\), \(i = 0, 1, \ldots, N\).

- **Classify**\((S^{[i]}, r_i)\): the subset of \(S^{[i]}\) of which the maximal index of non-vanishing members is \(r_i\). For example, if the multi-sign list \(S^{[i]} = \{[1, -1, -1, 1], [1, -1, 1, 1], [1, -1, 0, 0]\}\), then

\[
\text{Classify}(S^{[i]}, 4) = \{[1, -1, -1, 1], [1, -1, 1, 1]\},
\]

\[
\text{Classify}(S^{[i]}, 2) = \{[1, -1, 0, 0]\}.
\]

- **MinusOne**\((\text{rrc})\): \(\text{MinusOne}(\text{rrc}) = \text{rrc}_0 - \{1\} \land \text{rrc}_1 - \{1\} \land \cdots \land \text{rrc}_n - \{1\}\). Take \(\text{rrc} = \{3\} \land \{2, 1\}\) for example, we have

\[
\text{MinusOne}(\text{rrc}) = \{2\} \land \{1, 0\}\]

We first compute all possible multi-sign list MS for \(S(x)\) for a given \(\text{rrc} \in \text{PRRC}(n, N)\).
Algorithm 3.1. SplineMS(S(x), rrc)

Input A given parametric spline S(x) ∈ Sn[x0, x1, ..., xN+1] and a given rrc
Output The set MS of S(x) w.r.t. rrc.

Step 1. For every i ∈ {0, 1, ..., N}. Set Φ[i](x) = (1 - x)nS_i(\frac{\sum_{j=0}^{i-1} x_j}{x_i - x_0}) and h[i](x) = Φ[i](−x^2), we compute the discriminant sequence D[i] of each polynomial h[i](x) (By using Proposition 3.2).

Step 2. Compute L(rrc) = {L0} \land {L1} \land ... \land {L_N} and the MS0 (all possible multi-sign lists from the discriminant sequence D = D[0] \land D[1] \land ... \land D[N]).

Step 3. Compute MS = {ms ∈ MS0 | PmV(ms[i]) = PmV(rsl(ms[i])) = 2L_i - 1, i = 0, ..., N}.

Step 4. Return MS.

From Proposition 3.2, we know that SplineMS(S(x), rrc) is a set of all possible multi-sign lists for S(x) having Li distinct real roots on every interval [x_i, x_{i+1}], i = 0, ..., N. The algorithm also gives the necessary and sufficient conditions for S(x) ∈ Sn[x0, x1, ..., xN+1] having \sum_{i=0}^{N} Li total distinct real roots on [x0, xN+1]. Obviously, if Li = 0 or Li = n, then we do not require any further computation for S(x) on [x_i, x_{i+1}].

Take PRRC(3,1) for example, we perform the algorithm SplineMS(S(x), rrc) and will obtain the necessary and sufficient conditions for S(x) having rrc as possible cases of its real root distribution directly when rrc ∈ {{1, 1} \land {0}, {0} \land {0}, {0} \land {1, 1, 1}}.

It is noted that the number of real roots h[i](x) is even. That’s to say, the number of non-vanishing members of the revised sign list of D[i] is even.

Since S[i] denotes the set of all possible sign lists from discriminant sequence D[i] of polynomial h[i](x) with degree 2n and we set G_k = Classify(S[i], k), Then S[i] = \bigcup_{k=1}^{2n} G_{2k} because the number of real roots of h[i](x) is always a multiple of 2. From the Proposition 2.1, we know \Delta_i'(h[i]) = P_{2n-2k}(h[i]). It is a polynomial of degree 2n - 2k and it is used to determine the multiplicity of the distinct real root of S_i(x) on interval (x_i, x_{i+1}). By Proposition 2.2, the distinct real roots of \Delta_i'(h[i]) is MinusOne(rrc). And, the above computation can be repeated for \Delta_i'(h[i]) and MinusOne(rrc) until the last entry of the sign list is non-zero.

The algorithm to compute the multiplicity of the distinct real root according to the mixed lists is outlined as follows.

Algorithm 3.2. MultiRoot(Si(x), rrc)

Input Si(x) and rrc.
Output A sequence of mixed lists (the conditions for Si(x) having rrc as a possible case of its real root distribution).

Step 1. Set Φ[i](x) = (1 - x)nS_i(\frac{\sum_{j=0}^{i-1} x_j}{x_i - x_0}) and h[i](x) = Φ[i](−x^2).

Step 2. Compute h[i](x) = \Delta_i'(h[i](x)) (Using Proposition 2.1).

Step 3. Compute m_i = \deg h[i](x), rrc := MinusOne(rrc), and L_i = \Theta(rrc).

Step 4. Compute S[i] = {ms[i] | PmV(ms[i]) = PmV(rsl(ms[i])) = 2L_i - 1}.

Step 5. If S[i] = ∅, then return NULL.

else if L_i = 0 or L_i = m_i or Classify(S[i], m_i) = S[i], then return S[i].

else for j from 2L_i to m by 2 do

G_j = Classify(S[i], j)
if G_j ≠ ∅ then

if i = m then return G_j

else go to Step 2.

else return NULL.

It is from the discussion (see [5] for details) that output can be viewed as a couple of tree whose nodes consist mix lists. The necessary and sufficient conditions for a polynomial Si(x) on [x_i, x_{i+1}] having rrc as a possible case of its real root distribution is a single chain of mixed lists connected using conjunction (\land). Now, suppose that the leaves are labeled by k. Let Ξ_i be the chain of mixed lists obtained by connecting the nodes from the kth leaf to its root using conjunction. Then, the output conditions can be expressed as the disjunction of these chains.

Finally, we outline the algorithm to derive the real root classification of a given parametric spline, together with the conditions of its coefficients must be satisfied.
Algorithm 3.3. RRC(S(x))

**Input** Given a parametric spline $S(x) \in S_n[x_0, x_1, \ldots, x_{N+1}]$.

**Output** The real root classification of $S(x)$, as well as the necessary and sufficient conditions of its coefficients must be satisfied.

**Step 1.** Compute PRRC($n, N$) and $L(\text{rrc}) = \{L_0\} \land \{L_1\} \land \cdots \land \{L_N\}$. Then, MS is the multi-sign lists of $S(x)$ having $\text{rrc} = \text{rrc}_0 \land \text{rrc}_1 \land \cdots \land \text{rrc}_N$ as possible cases of its real root distribution. The algorithm also gives the necessary and sufficient conditions of its coefficients must be satisfied for each case.

**Step 2.** If $L_0 = 0$ or $L_1 = n$ for all $l \in \{0, 1, \ldots, N\}$, then we run SplineMS($S(x)$, $\text{rrc}$) and obtain MS. Else, we perform MultiRoot($S(x)$, $\text{rrc}$) and obtain $S^5$. Then, MS = $\{m^0, m^1, \ldots, m^N, m^5\} \in S^5$.

**Step 3.** Output MS and the conditions on its necessary and sufficient coefficients according to the multi-sign lists MS.

It is obvious that the set MS is the multi-sign lists of $S(x)$ having $\text{rrc} = \text{rrc}_0 \land \text{rrc}_1 \land \cdots \land \text{rrc}_N$ as possible cases of its real root distribution. The algorithm also gives the necessary and sufficient conditions of its coefficients must be satisfied for each case.

**Remark 3.3.** Although the Step 3 in Algorithm 3.1 is used to filter the non-realizable sign lists to a certain extent, it is not guaranteed that all non-realizable sign lists are detected. That's to say, the output contains excessive non-realizable conditions. It is an important and complex problem on semi-algebraic set and it is difficult to study. In this article, we use the tofind command in DISCOVERER package developed by Xia [16] to determine whether these conditions can be realized or not.

4. Illustrated examples

In this section, two simple examples are provided to show the proposed algorithm is flexible.

**Example 4.1.** Find the real root classification of a given parametric spline $S(x) = x^2 + bx + c + dx^4$ on $(-\infty, +\infty)$.

This problem is equivalent to find the number of negative real roots of $S_0(x) = x^2 + bx + c$, and positive real roots of $S_1(x) = (1 + d)x^4 + bx + c$, respectively.

We assume the conditions $c = 0$ and $1 + d = 0$ hold since $S(x)$ is regular and non-degenerate.

Set $\Phi^0(x) = S_0(x)$, $\Phi^1(x) = S_1(-x)$ and compute the negative root discriminant sequences of Discr($\Phi^0(x)$) and Discr($\Phi^1(x)$):

$$D^0 = \left\{ 2, 2b, b(b^2 - 4c), c(b^2 - 4c)^2 \right\} = \left\{ 1, \Delta_1, \Delta_1\Delta_3, \Delta_2\Delta_3^2 \right\}.$$

$$D^1 = \left\{ (1 + d)^3, -2b(1 + d)^4(4c - b^2 - 4c(1 + d)), c(1 + d)^4(b^2 - 4c(1 + d))^2 \right\} = \left\{ \Delta_4, -\Delta_1, -\Delta_1\Delta_3, \Delta_2\Delta_3^2 \right\},$$

where,

$$\Delta_1 = b, \quad \Delta_2 = c, \quad \Delta_3 = b^2 - 4c, \quad \Delta_4 = 1 + d, \quad \Delta_5 = b^2 - 4c(1 + d)$$

and $x \equiv y$ denotes that $x$ equals $y$ by discarding a positive factor.

The real root classification of $S(x)$ is:

1. $\{1,1\} \land \{0\} \iff [\Delta_1 > 0 \land \Delta_2 > 0 \land \Delta_3 > 0 \land \Delta_4 > 0]$
2. $\{1,1\} \land \{1\} \iff [\Delta_1 > 0 \land \Delta_2 > 0 \land \Delta_3 > 0 \land \Delta_4 < 0]$
3. $\{2\} \land \{0\} \iff [\Delta_1 > 0 \land \Delta_2 = 0 \land \Delta_3 > 0]$
4. $\{2\} \land \{1\} \iff [\Delta_1 > 0 \land \Delta_2 = 0 \land \Delta_4 < 0]$
5. $\{1\} \land \{0\} \iff [\Delta_1 = 0 \land \Delta_2 < 0 \land \Delta_3 < 0 \land \Delta_4 < 0 \land \Delta_5 > 0]$
6. $\{1\} \land \{1\} \iff [\Delta_1 = 0 \land \Delta_2 < 0 \land \Delta_3 < 0 \land \Delta_4 < 0 \land \Delta_5 > 0]$
7. $\{1\} \land \{1,1\} \iff [\Delta_1 = 0 \land \Delta_2 < 0 \land \Delta_3 > 0 \land \Delta_4 < 0 \land \Delta_5 > 0]$
8. $\{1\} \land \{2\} \iff [\Delta_1 = 0 \land \Delta_2 > 0 \land \Delta_3 < 0 \land \Delta_4 < 0 \land \Delta_5 = 0]$
9. $\{0\} \land \{0\} \iff [\Delta_1 = 0 \land \Delta_2 > 0 \land \Delta_3 < 0 \land \Delta_4 < 0 \land \Delta_5 > 0]$
10. $\{0\} \land \{1\} \iff [\Delta_1 = 0 \land \Delta_2 > 0 \land \Delta_3 < 0 \land \Delta_4 < 0 \land \Delta_5 < 0]$
11. $\{0\} \land \{2\} \iff [\Delta_1 = 0 \land \Delta_2 > 0 \land \Delta_3 < 0 \land \Delta_4 < 0 \land \Delta_5 = 0]$
12. $\{0\} \land \{1,1\} \iff [\Delta_1 = 0 \land \Delta_2 > 0 \land \Delta_3 < 0 \land \Delta_4 < 0 \land \Delta_5 < 0]$. 

Let us explain several cases of the RRC of $S(x)$ on $(\infty, \infty)$. For example, the algorithm `SplineMS($S(x), \{1, 1\} \land \{0\}$)` is designed to generate the conditions for $S(x)$ having two distinct real roots on $(\infty, 0)$ and no real root on $(0, \infty)$. It turns out that the output is MS = $\{1, 1, 1\} \land \{1, -1, -1, -1, 1\}$. We run the algorithm `SplineMS($S(x), \{2\} \land \{0\}$)` and the output is MS = $\{1, 1, 0, 0\} \land \{1, -1, 1, 1\} \land \{1, 1, 1, 1\} \land \{1, -1, 0, 0\}$. The termination conditions are certainly satisfied and MS is just want. These conditions can be expressed as $\Delta_1 > 0 \land \Delta_2 > 0 \land \Delta_3 > 0 \land \Delta_4 > 0$.

We consider the element $\{2\} \land \{0\}$ of PRRC and explain the case (3). We run the algorithm `SplineMS($S(x), \{2\} \land \{0\}$)` and the output is MS = $\{1, 1, 1, 1\} \land \{1, -1, -1\} \land \{1, 0, 0\} \land \{1, -1, 1, 1\} \land \{1, -1, 0, 0\}$. We also have to determine the possible sign lists of the polynomials in $\Lambda$ sequence of $h^0(x)$ and $\Lambda^0(h^0(x)) = 8bx^2 - 16c$. We do the algorithm `MultiRoot($S_0(x), \{2\}$)` and found the sign list of $\Lambda^1(h^0(x))$ is always $\{1, 1, 1, 1\}$. The termination condition holds. Thus, case (3) holds if and only if MS = $\{1, 1, 0, 0\} \land \{1, -1, -1\} \land \{1, -1, 1\} \land \{1, -1, 1\} \land \{1, 1, -1\} \land \{1, 1, -1\}$). These conditions can be expressed as $\Delta_1 > 0 \land \Delta_2 > 0 \land \Delta_3 > 0$. Other cases can be explained similarly.

Now we consider case (9). It describes that $S(x) > 0$, $\forall x \in [\infty, \infty]$. It equals to the output of algorithm `SplineMS($S(x), \{0\} \land \{0\}$`). The outcome is MS = $\{1, 0, 0, 1\} \land \{1, 0, 0, 1\} \land \{1, 1, -1\} \land \{1, 1, -1\} \land \{1, 1, -1\} \land \{1, 1, -1\}$. Thus, we can establish the necessary and sufficient conditions for $S(x) > 0$, $\forall x \in [\infty, \infty]$ as defined in case (9).

From the RRC of $S(x)$, we can obtain the conditions on $b, c, d$ such that $S(x) > 0$, $\forall x \in (-\infty, \infty)$, i.e., case (3), case (9) or case (11) holds. Meanwhile, we also found the maximum of the number of the real roots of $S(x)$ is 3 (counted with multiplicity) if and only if one of cases (2), (4), (7) and (8) holds.

The RRC of a given parametric spline is useful and we can know the distribution of real roots conveniently by solving it. For example, given a spline function with constant coefficients $S(x) = x^3 + \sqrt{2}x - \sin 1 + \ln 2 \cdot x^2$. It is easy to see $\Delta_3 < 0 \land \Delta_4 > 0$ and it corresponds to the case (6). Therefore, $S(x)$ has one single real root on $(-\infty, 0)$ and one single real root on $(0, \infty)$.

**Example 4.2.** Solve the real root classification of a given cubic parametric spline $S(x) \in S_3[\infty, 0, 1]$ and $S(x) = x^3 + bx + c + dx^3$.

This problem is equivalent to finding the number of negative real roots of $S_0(x) = x^3 + bx + c$ and real roots of $S_1(x) = (1 + d)x^3 + bx + c$. We respectively consider the conditions $1 + d \neq 0$.

Also, the conditions $1 + d \neq 0$, $c \neq 0$ and $b + c + d \neq 0$ hold since the spline is regular and non-degenerate.

Set $\Phi^0(x) = S_0(x)$, $\Phi^1(x) = (1 - x)^3S_0(\frac{1}{x}) = 1 + b + x + c - (2b + 3c)x(x + b + 3c)x^2 - cx^3$, and compute the negative root discriminant sequence of $\text{Discr}(\Phi^0(x))$ and $\text{Discr}(\Phi^1(x))$, respectively.

$$D^0 = \left\{3, 0, 0, 24b^3, 4b^2 \left(4b^3 + 27c^2\right), c \left(4b^3 + 27c^2\right)^2 \right\} \Rightarrow \{1, 0, 0, \Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_6, \Delta_7, \Delta_8\}$$

$$D^1 = \left\{-3c^3, 3c^3(2b + 3c), 2b^2c^2(2b + 3c), -2b^2c^2 \left(4b^3 + 27c^2\right), c^4(1 + d) \left(2b^3 + 3bc + 2b^2c + 9c^2 + 3bcd + 9c^2d\right),
\right.$$  

$$c^4(1 + d) \left(2b^3 + 3bc + 2b^2c + 9c^2 + 3bcd + 9c^2d\right) \left(4b^3 + 27c^2 + 27c^2d\right),$$

$$\Rightarrow \{\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5, \Delta_6, \Delta_7, \Delta_8\},$$

where

$$\Delta_1 = b, \quad \Delta_2 = c$$

$$\Delta_3 = 4b^2 + 27c^2, \quad \Delta_4 = b + 3c$$

$$\Delta_5 = 1 + d$$

$$\Delta_6 = 2b^3 + 3bc + 2b^2c + 9c^2 + 3bcd + 9c^2d, \quad \Delta_7 = 4b^2 + 27c^2 + 27c^2d,$$

$$\Delta_8 = 1 + b + c + d.$$

By a thorough and detailed analysis on the sign list of $D^0$ and $D^1$, the real root classification of $S(x)$ is:

1. $\{2\} \land \{1\} \iff \Delta_1 < 0 \land \Delta_2 < 0 \land \Delta_3 = 0 \land \Delta_5 > 0 \land \Delta_7 < 0$.
2. $\{2\} \land \{0\} \iff \Delta_1 < 0 \land \Delta_2 < 0 \land \Delta_3 = 0 \land \Delta_5 < 0 \land \Delta_7 < 0 \land \Delta_8 < 0 \land \Delta_5 = 0 \land \Delta_7 > 0 \land \Delta_8 > 0$.
3. $\{1, 1\} \land \{1\} \iff \Delta_2 < 0 \land \Delta_3 < 0 \land \Delta_5 > 0 \land \Delta_7 > 0$.
4. $\{1, 1\} \land \{0\} \iff \Delta_2 < 0 \land \Delta_3 < 0 \land \Delta_7 < 0 \land \Delta_8 < 0 \land \Delta_5 > 0$.
5. $\{1\} \land \{2\} \iff \Delta_2 > 0 \land \Delta_8 < 0 \land \Delta_7 = 0$.
6. $\{1\} \land \{1, 1\} \iff \Delta_2 > 0 \land \Delta_8 < 0 \land \Delta_7 < 0$.
7. $\{1\} \land \{1\} \iff \Delta_2 > 0 \land \Delta_8 < 0$.
Table 1
Running times of the two examples.

<table>
<thead>
<tr>
<th>Example</th>
<th>Example 4.1</th>
<th>Example 4.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Running times (s)</td>
<td>45.3</td>
<td>97.6</td>
</tr>
</tbody>
</table>

Let us explain several cases of the RRC of $S(x)$ on $(-\infty, 1)$. For example, For the case $\{1,1,1\} \land \{0\} \in \text{PRRC}(3,1)$ and the output of SplineMS $\emptyset, \{1,1,1\} \land \{0\}$ is the empty sequence NULL, they are not realizable. So it is not realizable that $S(x)$ has three distinct real roots on $(-\infty, 0)$ and no real root on $(0,1)$. Similarly, the elements in $\{1,1,1\} \land \{1,2,1\} \land \{0\}, \{2,1\} \land \{1\}, \{3\} \land \{0\}, \{3\} \land \{1\}, \{0\} \land \{1,1,1\}, \{1\} \land \{1,1,1\}, \{0\} \land \{1,2\}, \{1\} \land \{1,2\}, \{0\} \land \{3\}, \{1\} \land \{3\}, \{1\} \land \{1,1,1\}, \{1\} \land \{2\}, \{2\} \land \{1,1\}, \{2\} \land \{2\}$ all not realizable.

From the RRC of $S(x)$, we can obtain the conditions on $b$, $c$, $d$ such that $s(x) \geq 0$. For $x \in (-\infty, 1)$, i.e., case (2), case (9) or case (12) holds. Meanwhile, we found the maximum of the number of the real roots of $S(x)$ on $(-\infty, 1)$ is 3 if and only if one of the following cases (1), (3), (5) and (6) holds.

The discussion of computational complexity of the algorithm, at least on the computational efficiency is vital. We list the running times for the above two examples. Obviously, the proposed algorithm is relatively complex and time-consuming (see Table 1).

**Remark 4.1.** Since the real root classification of spline functions is relatively complex especially when the splines possess higher degree and more parameters, so we only give two relatively simple examples to verify the flexibility of the proposed method. Indeed, we implement the above two examples by using Maple and DISCOVERER even though they may be solved by hand. We can implement other examples and the algorithm is relatively time-consuming even though we add manual intervention.

5. Conclusion

In this paper, we propose an algorithm for the real root classification of a given spline function. It is primarily based on the existing work on complete root classification of a parametric polynomial. However, the proposed algorithm is not guaranteed that all non-realizable sign lists are detected and deleted. Moreover, the automatic computation of real root classification of a given spline function deserves further consideration. We will continue this issues in future work.

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References