Consensus of High Order Linear Multi-agent Systems Using Output Error Feedback

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Abstract—In this paper, we study the consensus problem of multi-agent systems in which each agent adopts the same linear model that can be of any order. We consider the case where only the relative output error between the neighboring agents can be measured. In order to solve the consensus problem, two kinds of decentralized control laws are designed. We first show that a static output error feedback control can solve the consensus problem if some further constraints on the system model is imposed. Then we use an observer based approach to design a dynamic output error feedback consensus control. We note that with the observer based approach, some information exchange between the agents is needed unless the associated adjacent graph is completely connected.

I. INTRODUCTION

In the past few years the study of multi-agent systems has attracted a considerable attention from various research communities. Multi-agent systems appear in various areas, such as cooperative control of unmanned aerial vehicles [1], consensus problem of communication networks [2], [3], [4], formation control of mobile robots [5], [6], etc.

In collective behaviors of multiple agents, consensus is one of the most important behaviors, which means that all agents will reach a common state eventually. There is already a large amount of literature concerning it, e.g., [2], [3], [4], [7], [8] and the references therein. So far the consensus problem that has been studied widely concerns agents with first or second order dynamics. For example, a pioneer work is the famous Vicsek model [9], in which a consensus scheme was proposed based on a simple discrete-time model for the headings of $n$ autonomous agents moving in a plane. Then some theoretical explanations for the consensus behavior of the Vicsek model were given in [2], [4], [10], etc. [3] solved the average-consensus problem of first order multi-agent system with strongly connected and balanced digraph. In [7], [8], [11], [12], to name a few, the consensus of second order multi-agent system is discussed. Various connectivity conditions are assumed to assure the consensus. A survey on consensus problem was given in [13], [14].

Recently, high order model or more general linear models are studied in for example [5], [15], [16], [17], [18]. However, in [15], [16], [17], they all assumed that the relative states between neighbors are available to the local input. In [5], Fax et al. gave a necessary and sufficient condition for the solvability of the consensus problem based on local dynamic output feedback control with fixed connection topology. But how to design the local dynamic output feedback is a very difficult problem and it seems that at present no answer is available for a general linear model. In [18], Ma et al. studied the consensusability of linear time-invariant multi-agent systems, where the admissible consensus protocol is based on a static output feedback.

Following our previous work [17], in this paper we consider a more general linear model, where the dynamics of each agent can be of any order. Different from [15], [16], [17], we consider the case that only the output error with the neighbors can be measured. We will provide two kinds of decentralized control law: one is based on static output error feedback, and the other is based on observers, thus dynamic output error feedback. Our contribution can be regarded to be two-fold: (i) under some mild conditions, a local static output error feedback control (rather than the full state error) is proven to be convergent; (ii) an observer-based dynamic output error feedback control is constructed for the general case and we also point out when this control scheme would require additional information exchange among the neighboring agents.

The rest of the paper is organized as follows. Section II provides some necessary preliminaries. Section III is our problem formulation. Section IV considers the consensus problem with static output error feedback. In Section V, we deals with the consensus problem by constructing decentralized observers. In Section VI, some illustrative examples are given. Section VII is the conclusion.

II. PRELIMINARIES

Graph theory is widely used in coordination problems of multiple agents. In this section, we recall some basic concepts and results on graph theory. More details can be found in for example [5], [19].

An undirected graph $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ of order $N$ consists of a vertex set $\mathcal{V} = \{1, 2, \cdots, N\}$, an edge set $\mathcal{E} = \{(i, j) : i, j \in \mathcal{V}\}$, and a weighted adjacency matrix $\mathcal{A} = [\alpha_{ij}] \in \mathbb{R}^{N \times N}$, where $\alpha_{ii} = 0$ and $\alpha_{ij} = \alpha_{ji} \geq 0$, $\alpha_{ij} > 0$ if and only if there is an edge between vertex $i$ and vertex $j$ and the two vertices are called adjacent (or they are mutual
neighbors). The set of neighbors of vertex \( i \) is denoted by \( \mathcal{N}_i = \{ j \in \mathcal{V} : (i, j) \in \mathcal{E}, \ j \neq i \} \).

A path of length \( l \) from vertex \( i \) to \( j \) is a sequence of \( l + 1 \) distinct vertices starting from \( i \) and ending at \( j \) such that the consecutive vertices are adjacent. For an undirected graph \( \mathcal{G} \), if there is a path between any two vertices, then \( \mathcal{G} \) is called connected, otherwise, disconnected. If there is an edge between any two vertices, then the graph is called complete graph.

Throughout this paper, we assume the graph is undirected and for simplicity, assume it is unweighted, that is, \( \mathcal{A} \) is a \( 0 \times 1 \) matrix. Define the Laplacian \( L \) with respect to the graph \( \mathcal{G} \) as

\[
L = [l_{ij}]_{N \times N}, \quad \text{where} \quad l_{ij} = \begin{cases} |\mathcal{N}_i|, & i = j, \\ -1, & j \in \mathcal{N}_i, \\ 0, & \text{otherwise}, \end{cases}
\]

where \( |\mathcal{N}_i| \) is the cardinality of the set \( \mathcal{N}_i \).

By the definition, every row sum of \( L \) is zero. 0 is an eigenvalue of \( L \) and \( \mathbf{1}_N \) is the associated eigenvector, that is, \( L \mathbf{1}_N = 0 \), where \( \mathbf{1}_N = [1, 1, \ldots, 1]^T \in \mathbb{R}^N \). If \( \mathcal{G} \) is connected, then 0 is the algebraically simple eigenvalue of \( L \) and all the other eigenvalues are positive.

At last, we list some properties of Kronecker product of matrices with appropriate dimension.

- \( \alpha (X \otimes Y) = (\alpha X) \otimes Y = X \otimes (\alpha Y) \), \( \alpha \in \mathbb{R} \);
- \( (X + Y) \otimes Z = (X \otimes Z) + (Y \otimes Z) \);
- \( Z \otimes (X + Y) = (Z \otimes X) + (Z \otimes Y) \);
- \( (X \otimes Y) \otimes Z = X \otimes (Y \otimes Z) \);
- \( (X \otimes Y)(Z \otimes W) = XZ \otimes YW \).

### III. PROBLEM FORMULATION

In this paper, we consider a multi-agent system with \( N \) agents. The dynamics of each agent is

\[
\begin{align*}
\dot{x}^i &= A x^i + B u^i \\
y^i &= C x^i, \quad i = 1, \ldots, N
\end{align*}
\]

where \( x^i = (x_1^i, x_2^i, \ldots, x_n^i)^T \in \mathbb{R}^n \), \( u^i \in \mathbb{R}^r \), \( y^i \in \mathbb{R}^m \) denote the state, control input, measurement output of the \( i \)-th agent, \( i = 1, \ldots, N \), respectively. \( A, B, C \) are matrices with appropriate dimension. We assume \( \text{rank}(B) = r \) and \( \text{rank}(C) = m \).

The topology of relationship of the \( N \) agents can be denoted by an undirected graph \( \mathcal{G} \).

**Definition 3.1:** For system (2), if for any initial value \( x^i(0) \),

\[
\lim_{t \to \infty} \|x^i - x^j\| = 0, \quad i, j = 1, \ldots, N,
\]

then the consensus is said to be achieved asymptotically.

In our previous work [17], we focused on the consensus problem of system (2), where the full state error is available. By constructing the local state error feedback control

\[
u^i = K \sum_{j \in \mathcal{N}_i(t)} (x^j - x^i), \quad i = 1, \ldots, N
\]

where \( \mathcal{N}_i(t) \) denotes the set of neighbors of agent \( i \) at time \( t \), we obtained the following result:

**Proposition 3.2:** ([17]) Consider system (2) and assume that it is controllable. If the adjacent graph is switching with positive dwell time and frequently connected, then the consensus can be achieved asymptotically via local state error feedback control (4) with suitably chosen coefficients \( K \).

Particularly, if the adjacent graph is fixed and connected, the conclusion is obviously true.

However, in many cases the full state error is not available and in this paper we will focus on such cases.

### IV. CONSENSUS WITH STATIC OUTPUT ERROR FEEDBACK

To begin with, we consider static output error feedback control for consensus.

Define

\[
z^i = \sum_{j \in \mathcal{N}_i} (y^j - y^i), \quad i = 1, \ldots, N.
\]

\( z^i \) is considered as the local output error, which is available for agent \( i \).

Based on the local information available, we propose the following local static output error feedback control for each agent \( i \)

\[
u^i = K z^i = K \sum_{j \in \mathcal{N}_i} (y^j - y^i), \quad i = 1, \ldots, N.
\]

The following observations are basically from [5] with some trivial modification.

Denote by \( x \) and \( z \) the stacks of vectors \( \{x^1, \ldots, x^N\} \) and \( \{z^1, \ldots, z^N\} \), respectively. From (5) and (2), we have

\[
z^i = \sum_{j \in \mathcal{N}_i} (y^j - y^i) = C \sum_{j \in \mathcal{N}_i} (x^j - x^i), \quad i = 1, \ldots, N.
\]

So

\[
z = (I_N \otimes C)(L \otimes I_n)x = (L \otimes C)x,
\]

where \( L \) is the Laplacian defined in (1).

Then the closed-loop system of (2) with the control law (6) becomes

\[
\dot{x} = [I_N \otimes A + (I_N \otimes BK)(L \otimes C)]x
\]

\[
= [I_N \otimes A + L \otimes BKC]x.
\]

Since \( L \) is symmetric, there is an orthogonal matrix \( T \) such that

\[
TLT^T = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)
\]

is diagonal, where \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N \) are the eigenvalues of \( L \). The first row \( T_1 \) of \( T \) is \( T_1^T = \frac{1}{\sqrt{N}} 1_N \), which is the eigenvector associated with the eigenvalue \( \lambda_1 = 0 \) of \( L \).

Now by the orthogonal transformation

\[
\tilde{x} = (T \otimes I_n)x,
\]

system (8) becomes

\[
\dot{\tilde{x}} = [I_N \otimes A + D \otimes BKC] \tilde{x},
\]

that is,

\[
\dot{x}^i = [A + \lambda_i BKC] x^i, \quad i = 1, \ldots, N.
\]
Note that (12) is a special case of equation (13) of [5].

Obviously, (also by Theorem 1 of [5]), if the adjacent graph is connected, the consensus can be achieved if and only if there is a gain matrix $K$ such that (12) can be stabilized simultaneously for $i = 2, \cdots, N$.

Next, for simplicity, we only discuss the single input single output case, i.e., $r = m = 1$. We assume that

A1. The transmission zeros of system $(A, B, C)$ are stable.  
A2. The relative degree of system $(A, B, C)$ is 1.

First, we list a Lemma from [20], which is useful to decide the stability of a polynomial.

Lemma 4.1: ([20]) Let $n$ be a positive integer and let $P(s)$ be a stable polynomial of degree $n - 1$:

$$P(s) = p_0 + p_1 s + \cdots + p_{n-1} s^{n-1}$$

with all $p_i > 0$.

Then there exists an $\alpha > 0$ such that

$$Q(s) = P(s) + p_n s^n$$

is stable if and only if $p_n \in [0, \alpha)$.

Based on Lemma 4.1, we can derive the following result.

Lemma 4.2: Consider a finite set of linear systems

$$\begin{cases}
\dot{x}_i = Ax_i + \lambda_i Bu_i \\
y_i = Cx_i,
\end{cases}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $y \in \mathbb{R}$, $(A, B, C)$ is completely controllable and observable, and all $\lambda_i > 0$, $i = 1, \cdots, l$.

Then under Assumption A1 and A2 there exists a common $K$ such that the output feedback

$$u_i = Ky_i, \quad i = 1, \cdots, l$$

stabilizes the $l$ systems in (13) simultaneously.

Proof. Without loss of generality, assume the pair $(A, B, C)$ is in Brunovsky canonical form. Let

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix},$$

$$B = [0, \cdots, 0, 1]^T, \quad C = [\beta_0, \beta_1, \cdots, \beta_{n-1}].$$

Then the characteristic polynomials for $A + \lambda_i BKC$ are

$$Q_i(s) = s^n - \lambda_i K \beta_{n-1} s^{n-1} - \cdots - \lambda_i K \beta_1 s - \lambda_i K \beta_0 + p_a(s),$$

$$= s^n - \lambda_i K P(s) + p_a(s), \quad i = 1, \cdots, l,$$

where $P(s) = \beta_{n-1} s^{n-1} + \cdots + \beta_1 s + \beta_0$, $p_a(s) = a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$. Note that for SISO system, $K \in \mathbb{R}$ is scalar.

By Assumption A1 and A2, $P(s)$ is Hurwitz and $\beta_{n-1} \neq 0$. We can assume, without loss of generality, all $\beta_i > 0$, $i = 0, \cdots, n - 1$.

Let $\lambda^*$ be the minimum $\lambda_i$. Using Lemma 4.1, there exists an $\alpha > 0$ such that when $K < -\frac{1}{\lambda^*_a}$, all the polynomials $s^n - \lambda_i K P(s)$, $i = 1, \cdots, l$ are also Hurwitz.

It is easy to see, for example, by singular perturbation analysis that when $K$ is chosen sufficiently negative, the effect of $p_a(s)$ on the roots of $Q_i(s)$ is negligible. The conclusion thus follows.

Now we are ready to consider the consensus problem. By Lemma 4.2 and the above analysis, we can obtain the following result immediately.

Theorem 4.3: Consider SISO system (2). Assume the adjacent graph is connected. If $(A, B, C)$ is controllable and observable and Assumption A1 and A2 hold, then the consensus can be achieved asymptotically via local output error feedback control (6).

Proof. Since the adjacent graph is connected, $\lambda_1 = 0$ is the single eigenvalue of $L$ and all the other eigenvalues are positive. By Lemma 4.2, under assumption A1 and A2, there is a common $K$ such that system (12) is stabilized for $i = 2, \cdots, N$, that is,

$$\hat{x}^i \to 0, \quad t \to \infty, \quad i = 2, \cdots, N.$$ 

Then it is easy to see by (10) and the construction of $T$,

$$x^i \to \frac{1}{\sqrt{N}} \hat{x}^i, \quad t \to \infty, \quad i = 1, \cdots, N.$$ 

So the consensus is achieved.

Remark 4.4: For multi-input multi-output case, if we assume system (2) is square, that is, $r = m$, then a similar result can be obtained if we modify Assumption A2 accordingly.

Remark 4.5: [18] also considered the consensus problem of system (2), in which, to assure the consensus, a rank condition is assumed, that is, assume

$$\text{rank}(C) = \text{rank} \begin{bmatrix} C \\ B^T P \end{bmatrix},$$

(15)

where $P$ is the solution of Riccati equation

$$A^T P + PA - PBB^T P + I_n = 0.$$ 

In fact, this rank condition (15) is quite strict since it holds only for some special matrices $C$. For example, for a simple second order system with $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, the solution to Riccati equation is $P = \begin{bmatrix} \sqrt{3} & 1 \\ 1 & \sqrt{3} \end{bmatrix}$, only when $C = \alpha [1, \sqrt{3}]$ with $\alpha \neq 0$, the rank condition is satisfied. In comparison, any $C = [\beta_0, \beta_1]$ with $\beta_0 \beta_1 > 0$ satisfies the Assumption A1 and A2 in Theorem 4.3.

Remark 4.6: It is quite obvious that, although the static local output error feedback is relatively easy to design, it can not be applied to any controllable and observable multi-agent systems. So in the next section, we consider dynamic output error feedback control.

V. CONSENSUS WITH DYNAMIC OUTPUT ERROR FEEDBACK

Recently, Fax et al. [5] gave a necessary and sufficient condition for the solvability of consensus problem based on local dynamic output feedback control. However, the
existence of the local dynamic output feedback and how to
design it if it exists is still an open problem.
In this section, we consider this problem from another
point of view. Since the local state error cannot be measured
directly, we try to estimate the local state error by the
observer.
Consider system (2) and the local output error (5). Let \( \hat{x}^i -
\hat{x}^j \) be the estimate of \( x^i - x^j \) by agent \( i \), \( i = 1, \cdots , N \), \( j \in \mathcal{N}_i \). Assume each \( \hat{x}^i \) satisfies
\[
\dot{\hat{x}}^i = A\hat{x}^i + G[C \sum_{j \in \mathcal{N}_i} (\hat{x}^i - \hat{x}^j) - z^j] + Bu^i, \quad i = 1, \cdots , N,
\]
where \( \mathcal{N}_i \) is the neighbors set of agent \( i \), the matrix \( G \) is
designable.

Based on the estimated state error \( \hat{x}^i - \hat{x}^j \), we propose the
following feedback control
\[
u^i = K \sum_{j \in \mathcal{N}_i} (\hat{x}^i - \hat{x}^j), \quad i = 1, \cdots , N.
\]

Then we have

**Theorem 5.1:** Consider system (2) with controllable and
observable \( (A, B, C) \). Assume the adjacent graph is
connected. Then the consensus is achieved via the observer (16)
and the local estimated state error feedback control (17).
Moreover, each estimated value \( \hat{x}^i - \hat{x}^j \) will converge to the
actual value \( x^i - x^j \), \( j \in \mathcal{N}_i \), \( i = 1, \cdots , N \).

To prove Theorem 5.1, we need the following Lemma
obtained in [17].

**Lemma 5.2:** ([17]) Consider a finite set of linear systems
\[
\dot{x}^i = Ax^i + \lambda_i Bu^i, \quad i = 1, \cdots , k,
\]
where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^r \), \( (A, B) \) is completely controllable,
rank(\( B \)) = \( r \), and \( \lambda_i > 0 \), \( i = 1, \cdots , k \). Then there exists a
\( K \) which simultaneously assigns the poles of \( k \) systems as
negative as possible. Precisely, for any \( M > 0 \) there exist
\[
u^i = Kx^i, \quad i = 1, \cdots , k,
\]
such that
\[
\text{Re}(\sigma(A + \lambda_i BK)) < -M, \quad i = 1, \cdots , k.
\]

**Proof of Theorem 5.1.** Let \( e^i = \hat{x}^i - x^i, \ i = 1, \cdots , N \). Then
according to system (2) and (16), \( e^i \) satisfies
\[
\dot{e}^i = Ae^i + GC\left[ \sum_{j \in \mathcal{N}_i} (\hat{x}^i - \hat{x}^j) - \sum_{j \in \mathcal{N}_i} (x^i - x^j) \right]
= Ae^i + GC \sum_{j \in \mathcal{N}_i} (e^i - e^j), \quad i = 1, \cdots , N.
\]

With the control (17), the closed loop of system (2) is
\[
\dot{x}^i = Ax^i + BK \sum_{j \in \mathcal{N}_i} (\hat{x}^i - \hat{x}^j), \quad i = 1, \cdots , N.
\]

Denote the concatenation of the vectors \( \{x^i\}, \{\hat{x}^i\}, \{e^i\}, \{u^i\} \) by \( x, \hat{x}, e \) and \( u \), respectively. Then with the control
\[
u = (I_N \otimes K)(L \otimes I_n)\hat{x} = (L \otimes K)\hat{x},
\]
the closed loop becomes
\[
\dot{x} = (I_N \otimes A)x + (I_N \otimes B)(L \otimes K)\hat{x}
= (I_N \otimes A)x + (L \otimes BK)\hat{x},
\]
\[
\dot{e} = (I_N \otimes A + L \otimes GC)e.
\]

Or it can be rewritten as
\[
\begin{bmatrix}
\dot{x} \\
\dot{e}
\end{bmatrix}
= \begin{bmatrix}
I_N \otimes A + L \otimes BK & L \otimes BK \\
0 & I_N \otimes A + L \otimes GC
\end{bmatrix}
\begin{bmatrix}
x \\
e
\end{bmatrix}
\]

Similar to Section IV, under the orthogonal transformation
\( \hat{x} = (T \otimes I_n)x \) and \( \hat{e} = (T \otimes I_n)e \), where \( T \) is defined in (9), the closed loop is changed into
\[
\begin{bmatrix}
\dot{\hat{x}} \\
\dot{\hat{e}}
\end{bmatrix}
= \begin{bmatrix}
A + \lambda_i BK & \lambda_i BK \\
0 & A + \lambda_i GC
\end{bmatrix}
\begin{bmatrix}
\hat{x} \\
\hat{e}
\end{bmatrix}, \quad i = 1, \cdots , N.
\]

The consensus can be achieved if and only if system (27)
is stabilized for \( i = 2, \cdots , N \). By Lemma 5.2 and the dual
principle, if \( (A, B, C) \) is controllable and observable, there
exist common gain matrices \( K \) and \( G \), such that system
(27) can be stabilized for \( i = 2, \cdots , N \), which solves the
consensus problem.

Furthermore, since \( \hat{e}^i \) converges to zero for \( i = 2, \cdots , N \),
by the definition of \( T \), we have \( e^i - \hat{e}^i \to 0 \), \( i = 2, \cdots , N \),
which implies that
\[
(x^i - \hat{x}^i) - (x^i - x^j) = (\hat{x}^i - x^i) - (\hat{x}^j - x^j)
= e^i - \hat{e}^i \to 0,
\]
\( \forall \ j \in \mathcal{N}_i \), \( i = 1, \cdots , N \).

The conclusion is obtained. \( \square \)

**Remark 5.3:** From the control (6) and (17), it is easy to
see that, if the consensus is achieved, all the agents will
coverge to their center \( x_c = \frac{1}{N} \sum_{i=1}^{N} x_i \). Under the control
(6) or (17), the center \( x_c \) satisfies the free drift equation
\[
\dot{x}_c = Ax_c
\]

**Remark 5.4:** In the observer (16) for agent \( i \), the information
of \( \hat{x}^j \), \( j \in \mathcal{N}_i \), is needed. For a general connected
graph, this exchange of information seems to be inevitable
unless the adjacent graph has some special structure as is
discussed next.

Finally, we discuss the complete graph case, in which
the observer for each agent can be designed without any
information exchange with the other agents.

When the graph is completely connected, the local state
error of agent \( i \) becomes
\[
\sum_{j \in \mathcal{N}_i} (x^i - x^j) = N(x^i - x_c).
\]
Let $\delta^i = x^i - x_c$, then $\delta^i$ satisfies
\[
\dot{\delta}^i = A\delta^i + Bu^i, \quad i = 1, \ldots, N.
\] (29)
Now, we need to estimate $\delta^i$. Denote the estimated value of $\delta^i$ by $\hat{\delta}^i$, which satisfies
\[
\dot{\hat{\delta}}^i = A\hat{\delta}^i + Bu^i + G(z^i - NC\hat{\delta}^i), \quad i = 1, \ldots, N.
\] (30)
where $z^i$ is defined in (5). For the complete graph case, $z^i = NC(x^i - x_c) = NC\delta^i$.
Let $e^i = \hat{\delta}^i - \delta^i$ be the difference between the estimated and actual state error, then
\[
\dot{e}^i = (A - NGC)e^i, \quad i = 1, \ldots, N.
\] (31)

Then we have the following result:

**Theorem 5.5:** Consider system (2). Assume the adjacent graph is complete. If $(A, B, C)$ is controllable and observable, then the consensus is achieved via the observer (30) and the local estimated state error feedback
\[
u^i = NK\hat{\delta}^i, \quad i = 1, \ldots, N.
\] (32)

**Proof.** By the control (32), the closed-loop system of (29) becomes
\[
\dot{\hat{\delta}}^i = A\hat{\delta}^i + NBK\hat{\delta}^i = (A + NBK)\hat{\delta}^i + NBKe^i, \quad i = 1, \ldots, N.
\] (33)

Associated with (31), we have
\[
\begin{bmatrix}
\dot{\hat{\delta}}^i \\
\dot{e}^i
\end{bmatrix} =
\begin{bmatrix}
A + NBK & NBK \\
0 & A - NGC
\end{bmatrix}
\begin{bmatrix}
\hat{\delta}^i \\
e^i
\end{bmatrix}, \quad i = 1, \ldots, N.
\] (34)

Since $(A, B, C)$ is controllable and observable, we can find matrices $K$ and $G$ such that $A + NBK$ and $A - NGC$ are stable, that is,
\[
\delta^i \to 0, \quad e^i \to 0, \quad \text{as } t \to \infty, \quad i = 1, \ldots, N,
\]
which show that $\hat{\delta}^i$ is the observer of $\delta^i$ and
\[
x^i - x_c \to 0, \quad i = 1, \ldots, N.
\]
So the consensus is achieved asymptotically and each agent converges to the trajectory of their center $x_c$. \qed

**Remark 5.6:** In fact, the results of this paper remain true if the adjacent graph is weighted and/or direct. For the weighted graph case, we only need to define the local output error
\[
z^i = \sum_{j \in N_i} \alpha_{ij} (y^i - y^j), \quad i = 1, \ldots, N.
\]
For the digraph case, when the graph has a spanning tree, the conclusion is also true.

**VI. ILLUSTRATIVE EXAMPLES**

In this section, we present two examples to validate the theoretical results in this paper.

**Example 6.1:** Consider a third order multi-agent system with 4 agents, satisfying
\[
\begin{align*}
\dot{x}^i &= Ax^i + bu^i \\
y^i &= cx^i \\
x^i &\in \mathbb{R}^3, \quad i = 1, 2, 3, 4,
\end{align*}
\] (35)
where
\[
A = \begin{bmatrix}
0 & 1 & 1 \\
-1 & 0 & 1 \\
-1 & -1 & 0
\end{bmatrix}, \quad b = \begin{bmatrix}
0 \\
1 \\
1
\end{bmatrix}, \quad c = [2, 1, 3].
\]

This example is found in [17], in which we used the full state error feedback. Now, we assume the full state error cannot be obtained directly and use the output error feedback. It is easy to see that $(A, b, c)$ is controllable and observable, and the Assumption A1 and A2 hold.

Assume the adjacent graph is given by Fig. 1. The Laplacian of the graph is
\[
L = \begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 0 & -1 \\
-1 & 0 & 2 & -1 \\
0 & -1 & -1 & 2
\end{bmatrix}
\]
The eigenvalues of $L$ are $\lambda_1 = 0$, $\lambda_2 = 0.5858$, $\lambda_3 = 2$, $\lambda_4 = 3.4142$.

Take $K = -2$, then by the static output error feedback (6), the consensus can be achieved under any initial values. Refer to Fig. 2 for the simulation result with the initial values
\[
x^1(0) = [1, 5, -4]^T, \quad x^2(0) = [-4, 6, 2]^T,
\]
\[
x^3(0) = [-3, 1, 7]^T, \quad x^4(0) = [8, -7, 0]^T.
\]
All four trajectories converge to a common circle which is the trajectory of their center $x_c$. \qed

\[\text{Fig. 1. The adjacent graph}\]

\[\text{Fig. 2. Consensus with static output error feedback}\]

**Example 6.2:** We also consider system (35) with the same $A$ and $b$ as in Example 6.1, but $c = [1, 0, 0]$. In this case, we can not use the static output error feedback since Assumption A1 is not satisfied.

Assume the topology graph is still given in Fig. 1. Now, we try to solve the consensus problem by constructing the observer to estimate the error $x^i - x^j, \quad j \in N_i$.\[\text{3689}\]
Choose
\[ K = [-20, -8, -4], \quad G = [-10, -15, -6]^T, \]
which assure \( A + \lambda_i BK \) and \( A + \lambda_i GC, \quad i = 2, 3, 4 \) are all stable. By Theorem 5.1, the consensus can be achieved by the control (17).

Fig. 3 shows the consensus with the initial values
\[
x^1(0) = [3, 1, -6]^T, \quad x^2(0) = [-3, 6, 1]^T, \\
x^3(0) = [-5, -1, 7]^T, \quad x^4(0) = [5, -9, 8]^T,
\]
The trajectories of four agents will converge to a common circle which is the trajectory of their center \( x_c \).

Fig. 4 is the error of \( e^i - e^3, \quad i = 2, 3, 4 \). All \( e^i - e^1 \to 0, \quad i = 2, 3, 4 \), which shows that the estimate \( \hat{x}^i - \hat{x}^3 \to x^i - x^3 \).

![Fig. 3. Consensus with observer](image1)

![Fig. 4. The error of \( e^i - e^3, \quad i = 2, 3, 4 \)](image2)

VII. CONCLUDING REMARKS

In this paper, the consensus problem was considered for multi-agent systems, in which all agents are modeled by an identical controllable and observable linear system. We considered the case that the full state error is not available. The contribution of this paper includes that (i) under some mild conditions, a local static output error feedback control (rather than the full state error) was provided; (ii) decentralized observers were constructed to estimate the relative state error, based on which decentralized controllers were designed. With both controllers, the consensus problem can be achieved asymptotically.

Further research will focus on the existence of the dynamic output error feedback proposed in [5] and the way to design it, which seems a highly nontrivial task. From this paper, however, one point is clear that the dynamic output error feedback as defined in [5] can not be obtained by the observer method in general.

REFERENCES


