Dynamics of a predator-prey system with pulses

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\textbf{Abstract}

In this paper, we investigate the dynamic behaviors of a Holling II two-prey one-predator system with impulsive effect concerning biological control and chemical control strategy-periodic releasing natural enemies and spraying pesticide at different fixed moment. By using the Floquet theory of linear periodic impulsive equation and small-amplitude perturbation method, we show that there exists a globally asymptotically stable two-prey eradication periodic solution when the impulsive period is less than some critical value. Further, we prove that the system is permanent if the impulsive period is larger than some critical value, and meanwhile the conditions for the extinction of one of the two-prey and permanence of the remaining two species are given. Finally, we give numerical simulation, with increasing of predation rate for the super competitor and impulsive period, the system displays complicated behaviors including a sequence of direct and inverse cascades of periodic-doubling, periodic-halving, chaos and symmetry breaking bifurcation. Our results suggest a new approach in the pest control.

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1. Introduction

It is well known, many insects are beneficial to human, but some insects are harmful to human, only these harmful insects can cause economic damage as their population reaching economic injury level. Controlling harmful insects and other arthropods has become an important issue in recent years, for instance, minimizing losses due to insect pests and insect vectors remains an essential component of the programmes in the Office of Agricultural Entomology in China. Overusing a single control tactic is discouraged to minimize damage to non-target organisms, and to preserve the quality of the environment. Overusing a single control tactic also can lead pest to produce resistance to chemical control (for example, pesticide), it will be more difficult to control pest later. Then biological and chemical control were introduced.

Biological control \cite{1–7} is the purposeful introduction one or more natural enemies of an exotic pest, specifically for the purpose of suppressing the abundance of the pest in a new target region to a level at which it no longer causes economic damage. Virtually all insect and mite pests have some natural enemies. Natural enemies are able to play a more active role in suppressing insect pests. Usually, predators feed on not only insect pests but also other insects. There may be more than one pest species – for example, the two species of aphids predominant in small grains: the English grain aphid and the oatbird cherry aphid. Aphids’ high reproductive rate enables their populations to quickly build up to levels that can cause an economic loss. However, aphids are usually kept in check by biological control agents, such as lady beetles, parasitic wasps, and syrphid fly maggots which are often abundant in small grains. One approach to biological control is augmentation, which is manipulation of existing natural enemies to increase their effectiveness. This can be achieved by mass production and periodic releasing natural enemies, and by genetic enhancement of the enemies to increase their effectiveness at control.

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Wherever possible, different pest control techniques should work together rather than against each other. In this paper, according to periodic biological and chemical control, and based on two-prey one-predator system with Holling II functional response, we suggest a simple mathematical model with pulses to describe the process of periodic releasing natural enemies and spraying pesticide (or harvesting pests) at different fixed moments. System with impulsive effects describing evolution processes are characterized by the fact that at certain moments of time they abruptly experience a change of state. Processes of such character are studied in almost every domain of applied sciences. Numerous examples are given in Bainov’s and his collaborator’s books [8,9]. Some impulsive equations have been recently introduced into population dynamics in relation to: vaccination [10,11], population ecology [12,13], and impulsive birth [14,15], chemotherapeutic [16,17].

The paper is arranged like this. A Holling II two-prey one-predator system concerning biological and chemical control is given in Section 2. In Section 3, we give some notations and lemmas. In Section 4, we consider the local stability and global asymptotic stability of the two-predator periodic solution by using Floquet theory for the impulsive equation, small-amplitude perturbation skills and techniques of comparison, and in Section 5 we show that the system is permanent if the impulsive period is larger than some critical value. Moreover, we give the sufficient conditions for one of two-prey extinction and the remaining two species permanence. A brief discussion and further numerical simulation are given in the last section.

2. Model formulation

Based on many experiments, Holling [18] suggested three different kinds of functional response for different kinds of species to model the phenomenon of predation, which made the standard Lotka–Volterra systems more realistic. Liu and Chen [13] investigated complex dynamics of Holling type II Lotka–Volterra predator-prey system with impulsive perturbations on the predator. Zhang and Chen [19] studied a Holling II functional response food chain model with impulsive perturbations. [13] investigated complex dynamics of Holling type II Lotka–Volterra predator-prey system with impulsive perturbations on pesticide (or harvesting pests) at different fixed moments. That is, we consider the following impulsive differential equations:

where \( x_i(t) \) (\( i = 1, 2 \)) is the population size of prey (pest) species and \( z(t) \) is the population size of predator (natural enemies) species, \( b_i > 0 \) (\( i = 1, 2, 3 \)) are intrinsic rates of increase or decrease, \( \alpha > 0 \) and \( \beta > 0 \) are parameters representing competitive effects between two-prey, \( \eta > 0 \) and \( \mu > 0 \), \( \frac{q_i(t)}{1+q_i(t)} \) and \( \frac{d_i(t)}{1+d_i(t)} \) are the Holling type II functional responses, \( d > 0 \) is the rate of converting prey into predator.

Model (2.1) with constant periodic releasing predator and spraying pesticide (or harvesting pests) was studied by Song and Li [20]:

where \( \Delta x_i(t) = x_i(t^+) - x_i(t) \), \( \Delta z(t) = z(t^+) - z(t) \). \( T \) is the period of the impulsive effect, \( p_i > 0 (i = 1, 2) \) is the proportionality constant which represents the rate of mortality due to applying pesticide. \( q > 0 \) is the number of predator released each time.

Now we will develop systems (2.1) and (2.2) by introducing a constant periodic releasing natural enemies and spraying pesticide (or harvesting pests) at different fixed moment. That is, we consider the following impulsive differential equations:

where \( \Delta x_i(t) = x_i(t^+) - x_i(t) \), \( \Delta z(t) = z(t^+) - z(t) \). \( T \) is the period of the impulsive effect, \( p_i > 0 (i = 1, 2) \) is the proportionality constant which represents the rate of mortality due to applying pesticide. \( q > 0 \) is the number of predator released each time.
where $0 \leq l \leq 1$, $\Delta x(t) = x(t^+) - x(t)$, $\Delta z(t) = z(t^+) - z(t)$, $0 \leq p_i < 1$, $0 \leq p < 1$, $i = 1, 2$, represents the fraction of pests and predator which die due to the pesticide at $t = (n + l - 1)T$, $q > 0$ is the number of predator released at $t = nT$, $n \in \mathbb{Z}_+$ and $\mathbb{Z}_+ = \{1, 2, \ldots\}$, $T$ is the period of the impulsive effect. We will use a combination of biological and chemical tactics to eradicate the pest or keep the pest population below the damage level.

3. Notations and definitions

Let $R_+ = [0, \infty)$, $R^+_1 = \{x \in R^+_1 : x > 0\}$. $\Omega = \text{int}R^+_1, Z_+$ be the set of all non-negative integers. Denote $f = (f_1, f_2, f_3)$, the map defined by the right-hand side of the first three equations of system (2.3). Let $V_0 = \{V : R_+ \times R^+_1 \rightarrow R_+\}$, then $V$ is said to belong to class $V_0$ if

(i) $V$ is continuous in $((n - 1)T, (n + l - 1)T] \times R^+_1$ and $((n + l - 1)T, nT] \times R^+_1$, for each $x \in R^+_1$, $n \in Z_+$, \[ \lim_{t \to (n + l - 1)T^+} V(t, y) = V((n + l - 1)T^+, x) \] and \[ \lim_{t \to nT^+} V(t, y) = V(nT^+, x) \] exist.

(ii) $V$ is locally Lipschitzian in $x$.

**Definition 3.1.** If $V \in V_0$, then for $(t, x) \in ((n - 1)T, (n + l - 1)T] \times R^+_1$ and $((n + l - 1)T, nT] \times R^+_1$, the upper right derivative of $V(t, x)$ with respect to the impulsive differential system (2.3) is defined as $D^+V(t, x) = \lim_{h \to 0^+} \sup |V(t + h, x + hf(t, x)) - V(t, x)|$.

The solution of the system (2.3) is a piecewise continuous function $x : R_+ \rightarrow R^+_1$, $x(t)$ is continuous in $((n - 1)T, (n + l - 1)T]$ and $((n + l - 1)T, nT]$, $n \in Z_+, 0 \leq l \leq 1$. Obviously, the smoothness properties of $f$ guarantee the global existence and uniqueness of solution of system (2.3), for details see [8,9].

**Definition 3.2.** The species $x_i$, $i = 1, 2$ of (2.3) is said to be permanent if there exist positive constants $m, M$ and $T_0$ such that each positive solution $(x_1(t), x_2(t), z(t))$ of the system (2.3) satisfies $m \leq x_i(t) \leq M$, $i = 1, 2$ ($m \leq z(t) \leq M$) for all $t > T_0$. If all species of the system are permanent, then the system is called permanent.

The following lemma is obvious.

**Lemma 3.1.** Let $x(t)$ is a solution of system (2.3) with $x(0^+) \geq 0$, then $x(t) \geq 0$ for all $t \geq 0$. And $x(t) > 0$ for $t \geq 0$ if $x(0^+) > 0$.

We will use a basic comparison result from Theorem 3.1 in [8]. For convenience, we state it in our notations.

**Lemma 3.2.** Let $V : R_+ \times R^+_1 \rightarrow R_+$ and $V \in V_0$. Assume that

$$
\begin{aligned}
D^+V(t, x) &\leq g(t, V(t, x)), \\
V(t, x(t^+)) &\leq \phi_n(V(t, x(t))), \\
V(t, x(t^+)) &\leq \psi_n(V(t, x(t))),
\end{aligned}
$$

where $g : R_+ \times R_+ \rightarrow R$ is continuous in $((n - 1)T, (n + l - 1)T]$ and $((n + l - 1)T, nT]$, for $x \in R_+, n \in Z_+$, such that \[ g(t, y) = g((n + l - 1)T^+, x), \lim_{n \to (n + l - 1)T^+} g(t, y) = g(nT^+, x) \] exist, functions $\phi_n, \psi_n : R_+ \rightarrow R_+$ are non-decreasing. Let $r(t)$ be the maximal solution of the scalar impulsive differential equation

$$
\begin{aligned}
\dot{u}(t) &= g(t, u(t)), \\
u(t^+) &= \phi_n(u(t)), \\
u(t^+) &= \psi_n(u(t)), \\
u(0^+) &= u_0,
\end{aligned}
$$

existing on $[0, \infty)$. Then $V(0^+, u_0) \leq u_0$ implies that $V(t, x(t)) \leq r(t)$, $t \geq 0$, where $x(t)$ is any solution of (2.3).

Similar result can be obtained when all the directions of the inequalities in the lemma are reversed and $\phi_n, \psi_n$ are non-increasing. Note that if we have some smoothness conditions of $g(t)$ to guarantee the existence and uniqueness of solutions for (3.2), then $r(t)$ is exactly the unique solution of (3.2).

For convenience, we give some basic properties of the following system:

$$
\begin{aligned}
\dot{z} &= -b_3z(t), \\
\Delta z(t) &= -p_2z(t), \\
\Delta z(t) &= q, \\
z(0^+) &= z_0 \geq 0.
\end{aligned}
$$

**Lemma 3.3.** System (3.3) has a positive periodic solution $\tilde{z}(t)$ and for every solution $z(t)$ of (3.3) with initial value $z(0^+) = z_0 \geq 0$, we have $z(t) \rightarrow \tilde{z}(t)$ as $t \rightarrow \infty$. 

\[ \]
Proof. Clearly
\[
\tilde{z}(t) = \begin{cases}
\frac{q \exp(-b_2(t-(n-1)T))}{(1-p) \exp(-b_2 T)} - (n-1)T < t \leq (n+l-1)T, \\
\frac{q(1-p) \exp(-b_2 T)}{1-(1-p) \exp(-b_2 T)} - (n+l-1)T < t \leq nT,
\end{cases}
\]
is a positive periodic solution of (3.3). The solution of (3.3) with initial value \(z(0^+) = z_0 \geq 0\) is
\[
z(t) = \begin{cases}
(1-p)^{n-1} (z_0 - \frac{q}{1-(1-p) \exp(-b_2 T)}) \exp(-b_2 t) + \tilde{z}(t), & (n-1)T < t \leq (n+l-1)T, \\
(1-p)^n (z_0 - \frac{q}{1-(1-p) \exp(-b_2 T)}) \exp(-b_2 t) + \tilde{z}(t), & (n+l-1)T < t \leq nT,
\end{cases}
\]
n \in Z_. Hence, \(|z(t) - \tilde{z}(t)| \to 0\) as \(t \to \infty\). The proof is complete. □

Therefore, the system (2.3) has a two-pest eradication periodic solution
\[
(0, 0, \tilde{z}(t)) = \begin{cases}
(0, 0, \frac{q \exp(-b_2(t-(n-1)T))}{1-(1-p) \exp(-b_2 T)}), & (n-1)T < t \leq (n+l-1)T, \\
(0, 0, \frac{q(1-p) \exp(-b_2 T)}{1-(1-p) \exp(-b_2 T)}), & (n+l-1)T < t \leq nT.
\end{cases}
\]

4. Extinction

In this section, we study the stability of the two-pest eradication periodic solution of the full system (2.3).

Theorem 4.1. Let \((x_1(t), x_2(t), z(t))\) be any solution of (2.3). Then \((0, 0, \tilde{z}(t))\) is globally asymptotically stable provided that
\[
\begin{align*}
f_1(T) &= b_1 T - \frac{q_1}{b_1 (1-p) \exp(-b_2 T)} - \ln \left( \frac{1}{1-p} \right) \leq 0, \\
f_2(T) &= b_2 T - \frac{q_2}{b_2 (1-p) \exp(-b_1 T)} - \ln \left( \frac{1}{1-p_2} \right) \leq 0.
\end{align*}
\]

Proof. Similar to Theorem 3.1 of Liu et al. [21], we can prove that the two-pest eradication periodic solution \((0, 0, \tilde{z}(t))\) is locally asymptotically stable, we omit it here.

In the following, we prove the global attractivity. Noting that
\[
\begin{cases}
\dot{x}_1 \leq x_1(b_1 - x_1), & t \neq (n+l-1)T, \\
\Delta x_1 = -p_1 x_1, & t = (n+l-1)T.
\end{cases}
\]

Consider the following impulsive differential equation:
\[
\left\{ \begin{array}{ll}
\dot{w}(t) = w(t)(b_1 - w(t)), & t \neq (n+l-1)T, \\
\Delta w(t) = -p_1 w(t), & t = (n+l-1)T, \\
w(0^+) = x_1(0^+) \geq 0.
\end{array} \right.
\]

If \(t \neq (n+l-1)T\), we have \(x_1(t) \leq w(t)\) and \(w(t) \to b_1\) as \(t \to \infty\); if \(t = (n+l-1)T\), we have \(x_1(t^+) = (1-p_1)x_1(t) < x_1(t)\). Hence, there exists a \(\varepsilon_1 > 0\) such that \(x_1(t) < b_1 + \varepsilon_1\) for \(t > 0\). Similarly, we can assume that \(x_2(t) < b_2 + \varepsilon_2\) for \(t > 0\). Choose a \(\varepsilon > 0\) such that \(\delta = (1-p_1) \exp \left( \int_0^T (b_1 - \frac{q_1}{b_1 (1-p) \exp(-b_1 T)}) \, dt \right) < 1\). Note that \(\tilde{z}(t) \geq -b_2 z(t)\), considering the comparison system:
\[
\begin{cases}
\dot{y}(t) = -b_3 y(t), & t \neq (n+l-1)T, \ t \neq nT, \\
\Delta y(t) = -p y(t), & t = (n+l-1)T, \\
y(t^+) = q, & t = nT, \\
y(0^+) = z_0 \geq 0.
\end{cases}
\]

From Lemmas 3.2 and 3.3, we have \(z(t) \geq y(t)\) and \(y(t) \to \tilde{z}(t)\) as \(t \to \infty\). Then
\[
z(t) \geq y(t) \to \tilde{z}(t) - \varepsilon
\]
holds for all \(t\) large enough. For simplification, we may assume that (4.3) holds for all \(t \geq 0\). From (2.3) we get
\[
\begin{cases}
\dot{x}_1(t) \leq x_1(t) \left( b_1 - \frac{q_1 x_1(t)}{b_1 (1-p) \exp(-b_1 T)} \right), & t \neq (n+l-1)T, \\
\Delta x_1(t) = -p_1 x_1(t), & t = (n+l-1)T.
\end{cases}
\]
Integrate (4.4) on \((n + l - 1)T, (n + l)T\), which yields
\[ x_1((n + l)T) \leq (1 - p_1)x_1((n + l - 1)T) \exp \left( \int_{(n + l - 1)T}^{(n + l)T} \left( b_1 - \frac{\eta(1 - p)}{1 + \omega t(b_1 + \varepsilon_1)} \right) dt \right) = x_1((n + l - 1)T) \delta. \]

Thus \( x_1((n + l)T) \leq x_1(nT) \delta^l \) and \( x_1((n + l)T) \to 0 \) as \( n \to \infty \). Therefore, \( x_1(t) \to 0 \) as \( n \to \infty \), since \( 0 < x_1(t) \leq (1 - p_1)x_1((n + l - 1)T) \exp(b_1 T) \) for \( (n + l - 1)T < t < (n + l)T \). By the same method we can prove \( x_2(t) \to 0 \) as \( n \to \infty \), so we omit it.

Next, we prove that \( z(t) \to \hat{z}(t) \) as \( t \to \infty \) if \( \lim_{n \to \infty} x_1(t) = 0 \) and \( \lim_{n \to \infty} x_2(t) = 0 \). For \( 0 < \varepsilon < \frac{b_1}{\eta(1 - p)} \), there exists a \( T_1 > 0 \) such that \( 0 < x_1(t) < \varepsilon \) and \( 0 < x_2(t) < \varepsilon \) hold for all \( t > T_1 \). Without loss of generality, we may assume that \( 0 < x_1(t) < \varepsilon \) and \( 0 < x_2(t) < \varepsilon \) hold for all \( t > 0 \). Then we have \(-b_2 z(t) < \hat{z}(t) < z(t)(-b_3 + \eta \varepsilon + \Delta \mu)\). By Lemmas 3.2 and 3.3, we obtain \( y(t) \leq z(t) \leq y_1(t) \) and \( y_1(t) \to \hat{z}(t) \) as \( t \to \infty \), where \( y(t) \) and \( y_1(t) \) are solutions of Eq. (4.2) and the following impulsive differential equation:

\[
\begin{align*}
\dot{y}_1(t) &= y_1(t)(-b_1 + \eta \varepsilon + \Delta \mu), \quad t \neq (n + l - 1)T, \quad t \neq nT, \\
\Delta y_1(t) &= -py_1(t), \quad t = (n + l - 1)T, \\
\Delta y_1(t) &= q, \quad t = nT, \\
y_1(0^+) &= 0 > 0,
\end{align*}
\]

respectively; and

\[
\begin{align*}
\dot{\hat{y}}_1(t) &= \frac{\exp((-b_1 + \eta \varepsilon + \Delta \mu)(t - (n + l - 1)T))}{\exp(-b_1 \hat{y}_1(t - (n + l - 1)T))}, \quad (n + l - 1)T < t \leq (n + l - 1)T, \\
\hat{y}_1(t) &= \frac{q(1 - p)(-b_1 + \eta \varepsilon + \Delta \mu)(t - (n + l - 1)T)}{\exp(-b_1 \hat{y}_1(t - (n + l - 1)T))}, \quad (n + l - 1) < t \leq nT.
\end{align*}
\]

Therefore, there exists a \( \varepsilon' > 0 \) such that \( \hat{z}(t) - \varepsilon' < z(t) \leq \hat{y}_1(t) + \varepsilon' \) for \( t \) large enough. Let \( \varepsilon \to 0 \), we get \( \hat{y}_1(t) \to \hat{z}(t) \). Hence, \( z(t) \to \hat{z}(t) \) as \( t \to \infty \). This completes the proof. \( \square \)

5. Permanence

First, from Theorem 5.1 in [20], we know that all solutions of (2.3) are uniformly ultimately bounded.

**Theorem 5.1.** For each positive solution \( x(t) = (x_1(t), x_2(t), z(t)) \) of (2.3) there exists a constant \( M > 0 \), such that \( x_1(t) \leq M(i = 1, 2) \), and \( z(t) \leq M \) with \( t \) large enough.

In the following, we investigate the permanence of the system (2.3).

**Theorem 5.2.** System (2.3) is permanent if \( \alpha < \frac{b_1}{\eta \varepsilon}, \mu < \frac{b_2}{\varepsilon^2}, \beta < \frac{b_3}{\eta \varepsilon}, \eta < \frac{\alpha}{\beta}, \eta < \frac{\beta}{\alpha}, \text{ and } f_3(T) = (b_1 - x_2)T - \frac{\eta q(1 - p)T}{\exp(-b_1 \hat{y}_1(T - (n + l)T))} - \ln \left( \frac{1}{1 - p} \right) > 0, \]

\[ f_4(T) = (b_2 - \beta \hat{y}_1)T - \frac{\eta q(1 - p)T}{\exp(-b_1 \hat{y}_1(T - (n + l)T))} - \ln \left( \frac{1}{1 - p} \right) > 0. \]

**Proof.** Suppose that \( x(t) \) is a solution of (2.3) with \( x(0^+) > 0 \), from Theorem 5.1, we may assume that \( z(t) \leq M \) and \( M > \max \left\{ \frac{b_1}{\alpha}, \frac{b_2}{\beta} \right\} \) hold for \( t > 0 \). From the proof of Theorem 4.1 we can assume that \( x_1(t) < b_1 + \varepsilon_1 \) and \( x_2(t) < b_2 + \varepsilon_2 \) hold for \( t > 0 \). Let \( m = \min \left\{ \frac{b_1}{\eta \varepsilon}, \frac{b_2}{\varepsilon^2}, \frac{b_3}{\eta \varepsilon} \right\} \) and \( m_2 \) such that \( x_1(t) > m_1 \) and \( x_2(t) > m_2 \) for \( t > 0 \). We will do it in the following two steps for convenience.

**Step I:** Let \( 0 < m_1 < \frac{b_1 - \beta m_2 + \varepsilon_2}{\eta \varepsilon}, 0 < m_2 < \frac{b_2 - \alpha m_1 + \varepsilon_1}{\varepsilon^2} \), and \( d\varepsilon_1 + d\mu m_2 < b_3 \). We will prove that there exist \( t_1, t_2 \in (0, \infty) \), such that \( x_1(t_1) \geq m_1 \) and \( x_2(t_2) \geq m_2 \). Otherwise, there will be three cases:

(i) There exists a \( t_2 > 0 \), such that \( x_2(t_2) \geq m_2 \) and \( x_1(t) < m_1 \), for all \( t > 0 \);

(ii) There exists a \( t_3 > 0 \), such that \( x_1(t_3) > m_1 \) and \( x_2(t) < m_2 \), for all \( t > 0 \);

(iii) \( x_1(t) > m_1 \), \( x_2(t) < m_2 \), for all \( t > 0 \).

We first consider case (i). Let \( \varepsilon_1 > 0 \) small enough such that

\[
\sigma_1 = (1 - p_1) \exp \left( \frac{b_1 - m_1 - \alpha(b_2 + \varepsilon_2) - \eta \varepsilon_1)T - \eta q(1 - p)T - \frac{\eta q(1 - p)T}{\exp(-b_1 \hat{y}_1(T - (n + l)T))} - \ln \left( \frac{1}{1 - p} \right)}{(-\theta)(1 - (1 - p) \exp(-b_1 \hat{y}_1(T - (n + l)T)))} \right) > 1,
\]

where \( \theta = -b_3 + d\varepsilon_1 + d\mu(b_2 + \varepsilon_2) \). According to the above assumption, we get \( \hat{z}(t) \leq \theta z(t) \). By Lemmas 3.2 and 3.3, we have \( z(t) \leq y_2(t) \) and \( y_2(t) \to y_2(t) \) as \( t \to \infty \), where \( y_2(t) \) is the solution of

\[
\begin{align*}
\dot{y}_2(t) &= \theta y_2(t), \quad t \neq (n + l - 1)T, \quad t \neq nT, \\
\Delta y_2(t) &= -py_2(t), \quad t = (n + l - 1)T, \\
\Delta y_2(t) &= q, \quad t = nT, \\
y_2(0^+) &= 0 > 0,
\end{align*}
\]

This completes the proof. \( \square \)
\[
\tilde{y}_2(t) = \begin{cases} 
\exp((t-t_n(T))/\tau) & \frac{n-1}{T} < t \leq (n+1-1)T, \\
\exp((t-t_{n+1}(T))/\tau) & (n+1-1)T < t \leq nT.
\end{cases}
\]

Therefore, there exists a \( T_1 > 0 \) such that \( z(t) \leq y_2(t) < \tilde{y}_2(t) + \epsilon_1 \). Then we have
\[
\begin{cases}
\dot{x}_1(t) \geq x_1(t)(b_1 - m_1 - \alpha(b_2 + \epsilon_2) - \eta(\tilde{y}_2(t) + \epsilon_1)), \quad t \neq (n+1-1)T, \\
\dot{x}_1(t^+) = (1 - p_1)x_1(t), \quad t = (n+1-1)T,
\end{cases}
\]
for \( t > T_1 \). Let \( N \in \mathbb{Z}_+ \) and \((n+1-1)T \geq T_1 \). Integrating (5.2) on \((n+1-1)T, (n+1)T\), we have
\[
x_1((n+1)T) \geq x_1((n+1-1)T)(1 - p_1) \exp \left( \int_{(n+1-1)T}^{(n+1)T} (b_1 - m_1 - \alpha(b_2 + \epsilon_2) - \eta(\tilde{y}_2(t) + \epsilon_1)) \, dt \right) = x_1((n+1-1)T)\sigma_1.
\]
Then \( x_1((n+1-1)T) \geq x_1((n+1-1)T)\sigma_1 \to \infty \) as \( n \to \infty \), which is a contradiction to the boundedness of \( x_1(t) \). Cases (ii) and (iii) can be analyzed by the same method as case (i), so we omit it. From the above results, we conclude that there exist \( t_1 > 0, t_2 > 0 \) such that \( x_1(t_1) \geq m_1, x_1(t_2) \geq m_2 \).

Step II: If \( x_1(t) \geq m_1 \) for all \( t > t_1 \), then our aim is obtained. Otherwise, we have \( x_1(t) < m_1 \) for some \( t > t_1 \). Setting \( t^* = \inf_{t_1 < t} \{ x_1(t) < m_1 \} \) then \( t^* \) can be divided into two cases, \( t^* \) is impulsive point or not impulsive point.

Case (a): If \( t^* = (n+1-1)T \) for \( n_1 \in \mathbb{Z}_+ \). Then \( x_1(t) \geq m_1 \) and \((1 - p_1)m_1 \leq x_1(t^{n+1}) = (1 - p_1)x_1(t) < m_1 \) hold for \( t \in [t_1, t^*] \). Select \( n_2, n_3 \in \mathbb{Z}_+ \) such that
\[
(n_2 - 1)T > \frac{1}{b_1 + n_3} \ln \left( \frac{\epsilon_1}{M + q} \right),
\]
and
\[
(1 - p_1)^{n_2} \sigma_1^3 \exp((n_2 - 1)\delta T) > (1 - p_1)^{n_2} \sigma_1^3 \exp((n_2 + 1)\delta T) > 1,
\]
where \( \delta = b_1 - m_1 - \alpha(b_2 + \epsilon_2) - \eta M < 0 \). Set \( T = (n_2 + n_3)T \). We claim that there must be exist a \( t_2 \in (t^*, t^* + T) \) such that \( x_1(t_2) > m_1 \). Otherwise, we have \( x_1(t) < m_1 \) for \( t \in (t^*, t^* + T) \). Consider (5.2) with \( y_2(n_2T^*) = z(n_1T^*) \), we have
\[
y_2(t) = \begin{cases} 
(1 - p)^{-(n+1)} \left( y_2(n_2T^*) - \frac{q}{k_{(1-p)\exp}(\partial)} \right) \exp(\delta(t - n_2T)) + \tilde{y}_2(t), \quad (n_2T < t < (n_2 + n_3)T), \\
(1 - p)^{-(n+1)} \left( y_2(n_2T^*) - \frac{q}{k_{(1-p)\exp}(\partial)} \right) \exp(\delta(t - n_2T)) + \tilde{y}_2(t), \quad (n_2 + n_3>T < t \leq (n_2 + n_3)T),
\end{cases}
\]
for \( t \in ((n_2T), n_2T], n_2 + 1 < n \leq n_1 + n_2 + n_3 \). Then \( \int_{2T} \tilde{y}_2(t) < (M + q) \exp(\int_{T}^{t} (1 - p_1)^{n_2} \exp((n_2 + 1)\delta T) > 1, \)
and \( z(t) \leq y_2(t) < \tilde{y}_2(t) + \epsilon_1 \). So we have \( x_1(t^* + T) > x_1(t^* + n_2T) \sigma_1^3 \). Note that
\[
\begin{cases} 
\dot{x}_1(t) = x_1(t)(b_1 - m_1 - \alpha(b_2 + \epsilon_2) - \eta M), \quad t \neq (n_2 + n_3)T, \\
\dot{x}_1(t^*) = (1 - p_1)x_1(t^*), \quad t = (n_2 + n_3)T.
\end{cases}
\]
Integrating (5.3) on \([t^*, t^* + n_2T] \), we have \( x_1(t^* + n_2T) > m_1 \exp((n_2 + n_3)\delta T) \). Thus \( x_1(t^* + T) > m_1 \exp((n_2 + n_3)\delta T) > m_1 \), which is a contradiction. Let \( \bar{t} = \inf_{t^* < t} \{ x_1(t) \geq m_1 \} \). Then \( x_1(t) < m_1 \) holds for \( t \in \bar{t}, t^* \). Since \( x_1(t) \) is left continuous and \( x_1(t^*) = \lim_{t \to t^*^-} x_1(t) \leq x_1(t) \) as \( t = (n_2 + n_3)T \), so \( t^* \) cannot be impulsive point. Suppose \( t \in (t^* + (n_2 + n_3)T, t^* + T) \), then \( x_1(t) \geq m_1 \) holds for \( t \in \bar{t}, t^* \). Integrating (5.3) on \([t^*, \bar{t}] \) one yields
\[
x_1(t) \geq x_1(t^*)(1 - p_1)^{n_2} \exp((n_2 + 1)\delta T) > m_1 \exp((n_2 + 1)\delta T).
\]
Let \( m'_{1} = m_1(1 - p_1)^{n_2} \exp((n_2 + 1)\delta T) \). So we have \( x_1(t) \geq m'_{1} \) for \( t \in \bar{t}, t^* \). For \( t > t^* \), the same argument can be continued since \( x_1(t) \geq m_1 \).

Case (b): If \( t^* < (n_2 + 1)T \), then \( x_1(t) \geq m_1 \) for \( t \in [t_1, t^*] \) and \( x_1(t^*) = m_1 \). We can assume \( t^* \in ((n_2 + 1)T, (n_2 + 1)T] \). There are two possible cases for \( x_1(t) \).

If \( x_1(t < m_1) \) holds for \( t \in (t^*, (n_2 + 1)T) \). Similar to case (a), we can prove that there must exist a \( t_2 \in [(n_2 + 1)T, (n_2 + 1)T + T) \) such that \( x_1(t_2) > m_1 \). Here, we omit it. Let \( \bar{t} = \inf_{t^* < t} \{ x_1(t) > m_1 \} \), then \( x_1(t) = m_1 \) and \( x_1(t < m_1) \) hold for \( t \in \bar{t}, t^* \). For \( t \in (t^*, \bar{t}] \), we have \( x_1(t) > m_1 \exp((1 - p_1)\delta T) \geq m_1 \exp((1 - p_1)^{n_2} \exp((n_2 + 1)\delta T) > m_1 \exp((n_2 + 1)\delta T) \geq m_1 \exp((n_2 + 1)\delta T) \geq m_1 \exp((n_2 + 1)\delta T) \). Therefore, \( x_1(t) \geq m_1 \) holds for \( t \in (t^*, \bar{t}] \). For \( t > \bar{t} \), the same argument can be continued since \( x_1(t) \geq m_1 \). Hence, \( x_1(t) \geq m_1 \) for all \( t > t_1 \).

If there exists a \( t \in ((n_2 + 1)T, (n_2 + 1)T + T) \) such that \( x_1(t) > m_1 \). Let \( \bar{t} = \inf_{t^* < t} \{ x_1(t) > m_1 \} \), then \( x_1(t) < m_1 \) holds for \( t \in [t^*, \bar{t}] \) and \( x_1(t) = m_1 \). For \( t > \bar{t} \), we have \( x_1(t) \geq \max(\{ x_1(t^*) \exp((\delta(t - t^*)) \} \) \). This process can be continued since \( x_1(t) > m_1 \) holds for all \( t > t_2 \). The proof is complete. \( \square \)
From the proof of Theorems 4.1 and 5.2, we can derive the following results.

**Corollary 5.1.** Let \((x_1(t), x_2(t), z(t))\) be any solution of system (2.3), then \(x_2\) and \(z\) are permanent and \(x_1(t) \to 0\) as \(t \to \infty\) provided that \(b_2 < b_1, b_3 < b_2, \eta < b_4 d b_1, f_1(T) < 0, f_4(T) > 0\).

**Corollary 5.2.** Let \((x_1(t), x_2(t), z(t))\) be any solution of system (2.3), then \(x_1\) and \(z\) are permanent and \(x_2(t) \to 0\) as \(t \to \infty\) provided that \(a < b_1, b_2, \mu < b_2 d b_2, f_2(T) < 0, f_3(T) > 0\).

**Remark 5.1.** Since \(f_i(0) = -\ln \left(\frac{1}{1-\frac{1}{r_i}}\right) < 0, f_i(T) \to +\infty\) as \(T \to +\infty\), and \(f_i''(T) > 0\), so \(f_i(T) = 0\) has a unique positive root, denoted by \(T_i(i = 1, 3)\).

Since \(f_j(0) = -\ln \left(\frac{1}{1-\frac{1}{r_j}}\right) < 0, f_j(T) \to +\infty\) as \(T \to +\infty\), and \(f_j''(T) > 0\), so \(f_j(T) = 0\) has a unique positive root, denoted by \(T_j(j = 2, 4)\) (see Fig. 1).

### 6. Discussion and numerical simulation

In this paper, we investigate the dynamics of a two-prey one-predator system with impulsive effect concerning biological and chemical control strategy. We show that there exists a two-pest eradication periodic solution which is globally asymptotically stable, and get the sufficient conditions for the permanence of the system (2.3). From Corollaries 5.1 and 5.2, we also get the sufficient conditions for one of two-prey extinction and the remaining two pieces permanence.
From Theorem 4.1, we know that the two-pest eradication periodic solution \((0, 0, \hat{z}(t))\) is globally asymptotically stable if \(T < T_1 \equiv \min(T_1, T_2)\), that is, the pest population are eradicated totally and the predator is permanent. A typical pest-eradication case is illustrated in Figure 3 for \(T = 0.9\), and Figure 4 for \(T = 0.19\).

**Figure 3.** Time-series of system (2.3) with positive periodic solution \((T = 0.9)\).

**Figure 4.** Time-series of system (2.3) with \(x_2\) eradication \((T = 0.19)\).
The periodic solution of the system (2.3) is shown in Fig. 2, where $a = 0.6, b = 0.5, b_1 = 3, b_2 = 2, b_3 = 3, \eta = 0.2, \mu = 0.7, d = 0.2, w_1 = 1, w_2 = 1, p_1 = 0.4, p_2 = 0.3, p = 0.4, q = 0.3, l = 0.5$, then we have $T_1 \approx 0.17, T_2 \approx 0.2$, then $T = 0.17$. If we let $T = 0.1$, then the conditions of Theorem 4.1 are satisfied, see Fig. 2. So, if two prey are both target pests, we can make the impulsive period smaller than 0.17 in order to derive two-pest population to extinct.

If the pest population is eradicated totally, it is not most desired for us from the biological point and saving resources. Growers only want to control the pest population under the economic threshold level, or drive the target pest population to extinct and let the non-target pest (or harmless insect) be permanent. From Theorem 5.2, we know that the system (2.3) is permanent if $T > \bar{T}_2 = \max\{T_3, T_4\}$, that is to say, three species coexist if impulsive period is larger than $\bar{T}_2$, see Fig. 3. Where $a = 0.6, \beta = 0.5, b_1 = 3, b_2 = 2, b_3 = 3, \eta = 0.2, \mu = 0.7, d = 0.2, w_1 = 1, w_2 = 1, p_1 = 0.4, p_2 = 0.3, p = 0.4, q = 0.3, l = 0.5$, then we can calculate $T_1 \approx 0.18, T_2 \approx 0.84$, let $T = 0.9$, then $T > \max\{T_3, T_4\}$. Then three species coexist if impulsive period is larger than $\bar{T}_2$. From Corollaries 5.1 and 5.2, we know that species $x_1$ extinct but $x_2$ and $z$ are permanent if $T_4 < T < T_1$; species $x_2$ extinct, $x_1$ and $z$ are permanent if $T_3 < T < T_2$. Therefore, we can drive the target pest population to extinct and let the non-target pest (or harmless insect) be permanent by choosing the impulsive period, this can be seen clearly from Fig. 4. Parameters are the same as Fig. 3, then $T_2 \approx 0.2, T_3 \approx 0.18$, let $T = 0.19$, then $T_3 < T < T_2$. So, if one prey is target pest, another is non-target pest, we can choose the impulsive period $T$ to obtain our aim. For example in Fig. 4, when $0.18 < T < 0.2$, $x_2$ rapidly reduces to zero, and $x_1$ and $z$ tend to positive periodic solution.
In the following, we will investigate the influence of bifurcation diagrams on the system (2.3). Fig. 5 shows the bifurcation diagram of system (2.3) for $b_1 = 1, b_2 = 1, b_3 = 1, \tau = 0.99, \beta = 1.5, d = 0.5, \mu = 1, p_1 = 0.5, p_2 = 0.05, p = 0.05, l = 0.05, q = 0.05, w_1 = 0.001, w_2 = 0.001, T = 125$, and $\eta$ varying from 6.8 to 8.4. Further increasing $\eta$, we can see that the dynamical behavior of system (2.3) is very complicated, including a sequence of direct and inverse cascade of periodic-dou-

Fig. 7. Spiral chaos for system (2.3) with $\eta = 7.3$. (a–c) Time-series of species $x_1, x_2$ and $z$; and (d) phase portrait.

Fig. 8. Bifurcation diagrams of system (2.3) with $65 \leq T \leq 150$ and initial value $x_{10} = 0.12, x_{20} = 0.78, z_0 = 0.042$. 
bling, chaos, and symmetry breaking bifurcation. Fig. 6 shows periodic-doubling cascade. The bifurcation diagrams show the route to chaos through cascade of periodic-doubling, which has been extensively studied by mathematicians [22,23]. A typical spiral chaos oscillation is captured when $g = 7.3$ (see Fig. 7).

Fig. 8 shows the bifurcation diagrams of system (2.3) for $b_1 = 1, b_2 = 1, b_3 = 1, \alpha = 0.99, \beta = 1.5, d = 0.5, \eta = 7.1, \mu = 1, p_1 = 0.001, p_2 = 0.001, p = 0.05, l = 0.05, q = 0.05, w_1 = 0.001, w_2 = 0.001$, and $T$ varying from 65 to 150. If we let $65 \leq T \leq 70$, there species coexist periodically, with further increasing $T$, periodic-doubling cascade occurs. Further increasing $T$ again, we can see that the dynamical behavior of system (2.3) is very complicated, including a sequence of period-doubling cascade, cycles, periodic-doubling cascade and chaos. In Fig. 8, we find that there are occurrences of sudden changes in the types of the attractors, which is a typical feature of bifurcation diagrams. For example, a $T$-periodic solution suddenly changes to $2T$-periodic solution when $T \approx 70$ and $2T$-periodic solution suddenly changes to $4T$-periodic solution when $T \approx 72.1$ and $4T$-periodic solution suddenly changes to chaos when $T \approx 73$. Fig. 9 shows chaos of system (2.3) for $T = 75$.

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